# Online Supplement to Jumps in Equity Index Returns Before and During the Recent Financial Crisis: A Bayesian Analysis

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In this online supplement, we first present the option pricing formula for the SV-DEJ model as well as its proof. We then give the conditional posteriors for the SV-DEJ-VG-JV model, the largest model in the paper. At last, we show some errata in the paper.

## 1 Option Pricing Formula for the SV-DEJ model

We provide the option pricing formula of the SV-DEJ model as follows. Following the assumptions in Yu, Li and Wells (2011), we let $\gamma_t^{(1)} = \eta^s \sqrt{v_t}$ and $\gamma_t^{(2)} = -\frac{1}{\sqrt{1-\rho^2}}(\rho\eta^s + \frac{\eta^v}{\sigma_v})\sqrt{v_t}$, where $\eta^s$ and $\eta^v$ are the market prices of risk. Then there exists a risk-neutral measure $\mathcal{Q}$ under which $W_t^{(1)}(\mathcal{Q})$ and $W_t^{(2)}(\mathcal{Q})$ are standard Brownian motions: $dW_t^{(i)}(\mathcal{Q}) := dW_t^{(1)} + \gamma_t^{(i)}dt, i = 1, 2$ (Equation (2.8) in Yu, Li and Wells (2011)). Under this condition, Yu, Li and Wells (2011) provide the pricing formula of vanilla call for affine-jump diffusion model given the jump compensator $\psi_J(u)$ and the initial log price $Y_0$, initial volatility $v_0$, maturity $T$ and strike $K$: (assuming constant interest rate $r$)

$$C(Y_0, v_0, T, K) = E^{\mathcal{Q}}\left[e^{-rT}(e^{y_T} - K)^+\right] = \frac{e^{-rT}}{\pi} \times Re\left(\int_0^\infty e^{-ix\log K}\frac{\phi(x-i, Y_0, v_0, T)}{-x^2+ix}dx\right).$$

The nontrivial part $\phi(u, y, v, t)$ in the integration is given by

$$\phi(u, y, v, t) = e^{iuY_0 + iu(r+\psi_J(-i))t - t\psi_J(u) - b(t)v_0 - c(t)},$$

where

$$b(t) = \frac{(iu+u^2)(1-e^{-\delta t})}{(\delta+\kappa_M)+(\delta-\kappa_M)e^{-\delta t}},\ c(t) = \frac{\kappa\theta}{\sigma_v^2}\left[2\log\left(\frac{2\delta-(\delta-\kappa_M)(1-e^{-\delta t})}{2\delta}\right)+(\delta-\kappa_M)t\right],$$

$$\kappa_M = \kappa - \eta^v - iu\sigma_v\rho, \quad \delta = \sqrt{\kappa_M^2 + (iu+u^2)\sigma_v^2}.$$

It suffices to obtain an expression for the jump compensator: $e^{-t\psi_J(u)} = E^{\mathcal{Q}}\left[e^{iuJ_t^y}\right] \iff \psi_J(u) = -\frac{1}{t}\log(E^{\mathcal{Q}}[e^{iuJ_t^y}])$. For affine-jump diffusion models,

$$\begin{aligned} E^{\mathcal{Q}}\left[e^{iuJ_t^y}\right] &= E^{\mathcal{Q}}\left[e^{iu\sum_{n=0}^{N_t} X_n}\right] \\ &= e^{-\lambda t}\sum_{n=0}^{\infty}(E^{\mathcal{Q}}(e^{iuX}))^n\frac{(\lambda t)^n}{n!} \\ &= \exp(-\lambda t(1-E^{\mathcal{Q}}(e^{iuX}))), \end{aligned}$$

where $\lambda$ is the jump rate and $X$ is the jump size distribution. Hence $\psi_J(u) = \lambda(1 - E^{\mathcal{Q}}(e^{iuX}))$. In the special case of double-exponential distributed jumps,

$$X = \begin{cases} \xi^+ \sim \exp(\eta^+), & \text{w.p. } \frac{\lambda^+}{\lambda^++\lambda^-}; \\ -\xi^- \sim -\exp(\eta^-), & \text{w.p. } \frac{\lambda^-}{\lambda^++\lambda^-}. \end{cases}$$

Therefore,

$$\begin{aligned} E^{\mathcal{Q}}(e^{iuX}) =& \frac{\lambda^+}{\lambda^++\lambda^-}\int_0^\infty \frac{1}{\eta^+}e^{-\frac{1}{\eta^+}x+iux}dx + \frac{\lambda^-}{\lambda^++\lambda^-}\int_0^\infty \frac{1}{\eta^-}e^{-\frac{1}{\eta^-}x-iux}dx \\ =& \frac{\lambda^+}{\lambda^++\lambda^-}\frac{1}{\eta^+}\frac{1}{\frac{1}{\eta^+}-iu} + \frac{\lambda^-}{\lambda^++\lambda^-}\frac{1}{\eta^-}\frac{1}{\frac{1}{\eta^-}+iu} \\ =& \frac{\lambda^+}{\lambda^++\lambda^-}\frac{1}{1-iu\eta^+} + \frac{\lambda^-}{\lambda^++\lambda^-}\frac{1}{1+iu\eta^-}. \end{aligned}$$

With $\lambda = \lambda^+ + \lambda^-$, it follows that

$$\psi_J(u) = \lambda^+ + \lambda^- - \frac{\lambda^+}{1-iu\eta^+} - \frac{\lambda^-}{1+iu\eta^-}.$$

## 2 MCMC Subroutines and Proofs for the SV-DEJ-VG-JV model

Bayesian MCMC inferences are done on the parameter space $\Theta$ which, under the SV-DEJ-VG-JV model, is $\Theta = \{\mu, \kappa, \theta, \rho, \sigma_v, \eta^+, \eta^-, \lambda^+, \lambda^-, \mu_v, \rho_J, \gamma, \sigma, \nu\}$, latent variables $N_{1:T}$ (the DE jump times), $V_{0:T}$ (the volatilities), $\xi_{1:T}^{y+}$ (positive jump sizes), $\xi_{1:T}^{y-}$ (negative-jump sizes), $\xi_{1:T}^{v}$ (volatility jump sizes), $J_{VG,1:T}^{y}$ (VG jumps) and $G_{1:T}$ (latent variables for VG jumps), given the returns data $Y_{0:T}$. Below are details of all conditional posteriors.

## 2.1 Conditional Posteriors Shared with the SV-MJ-JV Model

First, we introduce the conditional posteriors for parameters and latent variables of the SV-DEJ-VG-JV model that are shared with the SV-MJ-JV model. One can easily discover the similarity by comparing the results here and those in A.2 of Li, Wells and Yu (2008). Thus the proofs are omitted for the following conditional posteriors.

1. **Posterior for $\mu$** The posterior of $\mu$ follows a normal distribution $\mu \sim N(\frac{S^{(\mu)}}{W^{(\mu)}}, \frac{1}{W^{(\mu)}})$, where $W^{(\mu)} = \frac{\Delta}{(1-\rho^2)}\left(\sum_{t=0}^{T-1} \frac{1}{v_t}\right) + \frac{1}{M^{(\mu)^2}}$, $S^{(\mu)} = \frac{1}{(1-\rho^2)} \sum_{t=0}^{T-1} \frac{1}{v_t}\left(C_{t+1}^{(\mu)} - \rho \frac{D_{t+1}^{(\mu)}}{\sigma_v}\right) + \frac{m^{(\mu)}}{M^{(\mu)^2}}$, $C_{t+1}^{(\mu)} = y_{t+1} - y_t - (1(N_{t+1} = 1)\xi_{t+1}^{y+} - 1(N_{t+1} = -1)\xi_{t+1}^{y-} + \rho_J 1(N_{t+1} \neq 0)\xi_{t+1}^{v} + J_{VG,t+1}^{y})$, $D_{t+1}^{(\mu)} = v_{t+1} + (\kappa\Delta - 1)v_t - \kappa\theta\Delta - 1(N_{t+1} \neq 0)\xi_{t+1}^{v}$. Here the prior for $\mu$ is $N(m^{(\mu)}, M^{(\mu)^2})$, where $m^{(\mu)} = 0$, $M^{(\mu)} = 1$ in our study.

2. **Posterior for $\theta$** The posterior of $\theta$ follows a truncated normal distribution $\theta \sim N\left(\frac{S^{(\theta)}}{W^{(\theta)}}, \frac{1}{W^{(\theta)}}\right) 1_{(\theta>0)}$, where $W^{(\theta)} = \frac{\kappa^2\Delta}{\sigma_v^2(1-\rho^2)}\left(\sum_{t=0}^{T-1} \frac{1}{v_t}\right) + \frac{1}{M^{(\theta)^2}}$, $S^{(\theta)} = \frac{\kappa}{(1-\rho^2)\sigma_v} \sum_{t=0}^{T-1}\left(\frac{D_{t+1}^{(\theta)}/\sigma_v - \rho C_{t+1}^{(\theta)}}{v_t}\right) + \frac{m^{(\theta)}}{M^{(\theta)^2}}$, $C_{t+1}^{(\theta)} = y_{t+1} - y_t - \mu\Delta - (1(N_{t+1} = 1)\xi_{t+1}^{y+} - 1(N_{t+1} = -1)\xi_{t+1}^{y-} + \rho_J 1(N_{t+1} \neq 0)\xi_{t+1}^{v} + J_{VG,t+1}^{y})$, $D_{t+1}^{(\theta)} = v_{t+1} + (\kappa\Delta - 1)v_t - 1(N_{t+1} \neq 0)\xi_{t+1}^{v}$. Here the prior for $\theta$ is $N(m^{(\theta)}, M^{(\theta)^2})1_{(\theta>0)}$, where $m^{(\theta)} = 0$, $M^{(\theta)} = 1$ in our study.

3. **Posterior for $\kappa$** The posterior of $\kappa$ follows a truncated normal distribution $\kappa \sim N\left(\frac{S^{(\kappa)}}{W^{(\kappa)}}, \frac{1}{W^{(\kappa)}}\right) 1_{(\kappa>0)}$, where $W^{(\kappa)} = \frac{\Delta}{\sigma_v^2(1-\rho^2)}\left(\sum_{t=0}^{T-1} \frac{(\theta - v_t)^2}{v_t}\right) + \frac{1}{M^{(\kappa)^2}}$, $S^{(\kappa)} = \frac{1}{(1-\rho^2)\sigma_v} \sum_{t=0}^{T-1} \frac{(\theta - v_t)(D_{t+1}^{(\kappa)}/\sigma_v - \rho C_{t+1}^{(\kappa)})}{v_t} + \frac{m^{(\kappa)}}{M^{(\kappa)^2}}$, $C_{t+1}^{(\kappa)} = y_{t+1} - y_t - \mu\Delta - (1(N_{t+1} = 1)\xi_{t+1}^{y+} - 1(N_{t+1} = -1)\xi_{t+1}^{y-} + \rho_J 1(N_{t+1} \neq 0)\xi_{t+1}^{v} + J_{VG,t+1}^{y})$, $D_{t+1}^{(\kappa)} = v_{t+1} - v_t - 1(N_{t+1} \neq 0)\xi_{t+1}^{v}$. Here the prior for $\kappa$ is $N(m^{(\kappa)}, M^{(\kappa)^2})1_{(\kappa>0)}$, where $m^{(\kappa)} = 0$, $M^{(\kappa)} = 1$ in our study.

4. **Posterior for $\sigma_v$ and $\rho$** Following Jacquier, Polson and Rossi (1994), we transform $(\rho, \sigma_v)$ one-to-one into $(\phi_v, w_v)$, where $\phi_v = \sigma_v\rho$ and $w_v = \sigma_v^2(1-\rho^2)$. The priors are a normal-inverse-gamma distribution: $\phi_v | w_v \sim N(0, \frac{1}{2}w_v)$ and $w_v \sim IG(m^{(RS)}, M^{(RS)})$, where $m^{(RS)} = 2$, $M^{(RS)} = 200$ in our study. Given this re-parameterization, the joint posteriors of $(\phi_v, w_v)$ are the conjugate of the priors

$$w_v \sim IG\left(\frac{T}{2} + m^{(RS)}, \frac{1}{\frac{1}{2}\sum_{t=0}^{T-1}(D_{t+1}^{(RS)})^2 + \frac{1}{M^{(RS)}} - \frac{S^{(RS)^2}}{2W^{(RS)}}}\right)$$

and

$$\phi_v | w_v \sim N\left(\frac{S^{(RS)}}{W^{(RS)}}, \frac{w_v}{W^{(RS)}}\right),$$

where $W^{(RS)} = \sum_{t=0}^{T-1}(C_{t+1}^{(RS)})^2 + 2$, $S^{(RS)} = \sum_{t=0}^{T-1} C_{t+1}^{(RS)} D_{t+1}^{(RS)}$, $C_{t+1}^{(RS)} = (y_{t+1} - y_t - \mu\Delta - (1(N_{t+1} = 1)\xi_{t+1}^{y+} - 1(N_{t+1} = -1)\xi_{t+1}^{y-} + \rho_J 1(N_{t+1} \neq 0)\xi_{t+1}^v + J_{VG,t+1}^y))/\sqrt{v_t\Delta}$, $D_{t+1}^{(RS)} = (v_{t+1} - v_t - \kappa(\theta - v_t)\Delta - 1(N_{t+1} \neq 0)\xi_{t+1}^v)/\sqrt{v_t\Delta}$.

5. **Posterior for the latent variable** $v_{t+1}$ For $0 < t+1 < T$, the posterior of $v_{t+1}$ is

$$p(v_{t+1}|\cdot) \propto \exp\left[-\frac{-2\rho\epsilon_{t+1}^y\epsilon_{t+1}^v + (\epsilon_{t+1}^v)^2}{2(1-\rho^2)}\right] \times \frac{1}{v_{t+1}}$$
$$\times \exp\left[-\frac{(\epsilon_{t+2}^y)^2 - 2\rho\epsilon_{t+2}^y\epsilon_{t+2}^v + (\epsilon_{t+2}^v)^2}{2(1-\rho^2)}\right]$$

where $\epsilon_{t+1}^y = (y_{t+1} - y_t - \mu\Delta - (1(N_{t+1} = 1)\xi_{t+1}^{y+} - 1(N_{t+1} = -1)\xi_{t+1}^{y-} + \rho_J 1(N_{t+1} \neq 0)\xi_{t+1}^v + J_{VG,t+1}^y))/\sqrt{v_t\Delta}$, $\epsilon_{t+1}^v = (v_{t+1} - v_t - \kappa(\theta - v_t)\Delta - 1(N_{t+1} \neq 0)\xi_{t+1}^v)/(\sigma_v\sqrt{v_t\Delta})$. For $t+1 = T$, the above posterior only has the first exponential part because $v_T$ depends only on $v_{T-1}$. Similarly, the posterior of $p(v_0|\cdot)$ depends on $\frac{1}{v_0}$ and the second exponential part. Following Li, Wells and Yu (2008), we make use of the Adaptive Rejection Metropolitan Sampling (ARMS) method of Gilks, Best and Tan (1994) to draw from this conditional posterior.

6. **Posterior for** $\mu_v$ The posterior of $\mu_v$ follows an inverse gamma distribution

$$\mu_v \sim IG\left(T + m^{(mv)}, \frac{1}{\sum_{t=0}^{T-1} \xi_{t+1}^v + \frac{1}{M^{(mv)}}}\right),$$

The prior for $\mu_v$ is $IG(m^{(mv)}, M^{(mv)})$ where $m^{(mv)} = 10$ and $M^{(mv)} = 0.1$.

## 2.2 Conditional Posteriors Shared with the SV-VG Model

We now turn to parameters and variables that are shared with the SV-VG model. Proofs are again omitted for their similarity and we defer interested readers to Appendix A.4 in Li, Wells and Yu (2008).

1. **Posterior for** $\gamma$ The posterior of $\gamma$ is $\gamma \sim N\left(\frac{S^{(\gamma)}}{W^{(\gamma)}}, \frac{1}{W^{(\gamma)}}\right)$, where $W^{(\gamma)} = \frac{1}{\sigma^2}\sum_{t=0}^{T-1} G_{t+1} + \frac{1}{(M^{(\gamma)})^2}$, $S^{(\gamma)} = \frac{1}{\sigma^2}\sum_{t=0}^{T-1} J_{VG,t+1}^y + \frac{(m^{(\gamma)})^2}{(M^{(\gamma)})^2}$. Here the prior for $\gamma$ is $N(m^{(\gamma)}, {M^{(\gamma)}}^2)$, where $m^{(\gamma)} = 0$, $M^{(\gamma)} = 1$ in our study.

2. **Posterior for $\sigma$** The posterior of $\sigma$ is $\sigma^2 \sim IG\left(\frac{T}{2} + m^{(\sigma)}, \frac{1}{\frac{1}{2}\sum_{t=0}^{T-1}\frac{(J^y_{VG,t+1}-\gamma G_{t+1})^2}{G_{t+1}} + \frac{1}{M^{(\sigma)}}}\right)$. Here the prior for $\sigma^2$ is $\sigma^2 \sim IG(m^{(\sigma)}, M^{(\sigma)})$ with $m^{(\sigma)} = 2.5, M^{(\sigma)} = 5$.

3. **Posterior for $\nu$** The posterior of $\nu$ is

$$p(\nu|\cdot) \propto \left(\frac{1}{\nu^{\frac{\Delta}{\nu}}\Gamma(\frac{\Delta}{\nu})}\right)^T \left(\prod_{t=0}^{T-1} G_t\right)^{\frac{\Delta}{\nu}} \exp\left(-\frac{1}{\nu}\left(\sum_{t=0}^{T-1} G_t + \frac{1}{M^{(\nu)}}\right)\right)\left(\frac{1}{\nu}\right)^{m^{(\nu)}+1}.$$

Here the prior for $\nu$ is $\nu \sim IG(m^{(\nu)}, M^{(\nu)})$ with $m^{(\nu)} = 10, M^{(\nu)} = 0.1$.

4. **Posterior for $J^y_{VG,t+1}$** The posterior of $J^y_{VG,t+1}$ is $J^y_{VG,t+1} \sim N\left(\frac{S^{(J)}}{W^{(J)}}, \frac{1}{W^{(J)}}\right)$, where $W^{(J)} = \frac{1}{(1-\rho^2)v_t\Delta} + \frac{1}{\sigma^2 G_{t+1}}$, $S^{(J)} = \frac{1}{(1-\rho^2)v_t\Delta}\left(C^{(J)}_{t+1} - \frac{\rho D^{(J)}_{t+1}}{\sigma_v}\right) + \frac{\gamma}{\sigma^2}$. Here $C^{(J)}_{t+1} = y_{t+1} - y_t - \mu\Delta - J^y_{DE,t+1}$, $D^{(J)}_{t+1} = v_{t+1} + (\kappa\Delta - 1)v_t - \kappa\theta\Delta - 1(N_{t+1} \neq 0)\xi^v_{t+1}$.

5. **Posterior for $G_{t+1}$** The posterior of $G_{t+1}$ is

$$p(G_{t+1}|\cdot) \propto G_{t+1}^{\frac{\Delta}{\nu}-\frac{3}{2}} \exp\left(-\frac{(J^y_{VG,t})^2}{2\sigma^2}\frac{1}{G_{t+1}}\right)\exp\left(-\left(\frac{\gamma^2}{2\sigma^2} + \frac{1}{\gamma}\right)G_{t+1}\right)$$

Again ARMS has a very good performance in updating this posterior distribution. In the ARMS algorithm, a lower bound is set to be zero for $G_{t+1}$, since it is a Gamma process.

## 2.3 Conditional Posteriors Unique to the SV-DEJ-VG-JV Model

The following parameters and variables are new (i.e. never appeared in the literature) and hence results are presented with proofs (unless the proof is identical to that for the SV-DEJ model outlined in the appendix of the paper).

1. **Posterior for $\eta^+$, $\eta^-$** The posterior of $\eta^+$ follows an inverse-gamma distribution

$$\eta^+ \sim IG\left(T + m^{(EP)}, \frac{1}{\sum_{t=0}^{T-1}\xi^{y+}_{t+1} + 1/M^{(EP)}}\right),$$

where the prior of $\eta^+$ is $IG(m^{(EP)}, M^{(EP)})$ and $m^{(EP)} = 20$, $M^{(EP)} = 10$ in our study. Similarly, the posterior of $\eta^-$ also follows an inverse-gamma distribution

$$\eta^- \sim IG\left(T + m^{(EM)}, \frac{1}{\sum_{t=0}^{T-1}\xi^{y-}_{t+1} + 1/M^{(EM)}}\right),$$

where the prior of $\eta^-$ is $IG(m^{(EM)}, M^{(EM)})$ and $m^{(EM)} = 20$, $M^{(EM)} = 10$ in our study.

2. **Posterior for the latent variables $\xi^{y+}_{t+1}$, $\xi^{y-}_{t+1}$** For $1 \leq t+1 \leq T$, the posterior of $\xi^{y+}_{t+1}$ follows a truncated normal distribution when $N_{t+1} = 1$

$$\xi^{y+}_{t+1} \sim N\left(\frac{S^{(XP)}}{W^{(XP)}}, \frac{1}{W^{(XP)}}\right) 1_{(\xi^{y+}_{t+1}>0)},$$

where $S^{(XP)} = \frac{C^{(X)}_{t+1} - \rho D^{(X)}_{t+1}/\sigma_v}{v_t\Delta(1-\rho^2)} - \frac{1}{\eta^+}$, $W^{(XP)} = \frac{1}{v_t\Delta(1-\rho^2)}$, $C^{(X)}_{t+1} = y_{t+1} - y_t - \mu\Delta - J^y_{VG,t+1} - \rho_J \xi^v_{t+1}$ and $D^{(X)}_{t+1} = v_{t+1} - v_t - \kappa(\theta - v_t)\Delta - 1(N_{t+1} \neq 0)\xi^v_{t+1}$. When $N_{t+1} \neq 1$, the posterior is the same as its prior $\xi^{y+}_{t+1} \sim \exp(\eta^+)$. Similarly, for $1 \leq t+1 \leq T$, the posterior of $\xi^{y-}_{t+1}$ follows a truncated normal distribution when $N_{t+1} = -1$

$$\xi^{y-}_{t+1} \sim N\left(\frac{S^{(XM)}}{W^{(XM)}}, \frac{1}{W^{(XM)}}\right) 1_{(\xi^{y-}_{t+1}>0)},$$

where $S^{(XM)} = -\frac{C^{(X)}_{t+1} - \rho D^{(X)}_{t+1}/\sigma_v}{v_t\Delta(1-\rho^2)} - \frac{1}{\eta^-}$, $W^{(XM)} = \frac{1}{v_t\Delta(1-\rho^2)}$. And when $N_{t+1} \neq -1$, the posterior is the same as its prior $\xi^{y-}_{t+1} \sim \exp(\eta^-)$.

*Proof.* Since the cases for $N_{t+1} = 1$ and $N_{t+1} = -1$ are similar, we focus on the case where $N_{t+1} = 1$. To ease notation, in this proof, we denote $m = m^{(XP)}$, $M = M^{(XP)}$, $C_{t+1} = C^{(XP)}_{t+1}$, $D_{t+1} = D^{(XP)}_{t+1}$, $S = S^{(XP)}$ and $W = W^{(XP)}$. It is easy to see the posterior is $\xi^{y+}_{t+1} \sim \exp(\eta^+)$ when $N_{t+1} \neq 1$ since the data provides no information. By Bayes' rule, when $N_{t+1} = 1$,

$$\begin{aligned}
&p(\xi^{y+}_{t+1}|\cdot) = p(\xi^{y+}_{t+1}|y_{t+1}, y_t, v_{t+1}, v_t, N_{t+1} = 1, \Theta, \xi^v_{t+1}, J^y_{VG,t+1}) \\
\propto& p(y_{t+1}|y_t, \Theta, N_{t+1} = 1, v_t, v_{t+1}, \xi^{y+}_{t+1}, \xi^v_{t+1}, J^y_{VG,t+1}) p(\xi^{y+}_{t+1}|\eta^+) \\
\propto& \exp\left[-\frac{1}{2v_t\Delta(1-\rho^2)}(C_{t+1} - \xi^{y+}_{t+1} - \rho D_{t+1}/\sigma_v)^2\right] \exp\left(-\frac{\xi^{y+}_{t+1}}{\eta^+}\right) 1_{(\xi^{y+}_{t+1}>0)} \\
\propto& \exp\left(-\frac{1}{2}W(\xi^{y+}_{t+1})^2 + S\xi^{y+}_{t+1}\right) 1_{(\xi^{y+}_{t+1}>0)}.
\end{aligned}$$

The required conclusion readily follows by completing the squares and comparing the parameters. □

3. **Posterior for $(\lambda^+, \lambda_0, \lambda^-)$** The posterior of $(\lambda^+, \lambda_0, \lambda^-)$ follows a Dirichlet distribution

$$p((\lambda^+, \lambda_0, \lambda^-)|\cdot) \sim D\left(\alpha_1 + \sum_{t=0}^{T-1} 1_{(N_{t+1}=1)}, \alpha_0 + \sum_{t=0}^{T-1} 1_{(N_{t+1}=0)}, \alpha_{-1} + \sum_{t=0}^{T-1} 1_{(N_{t+1}=-1)}\right),$$

where the prior of $(\lambda^+, \lambda_0, \lambda^-)$ is $D(\alpha_1, \alpha_0, \alpha_{-1})$ and $\alpha_1 = \alpha_{-1} = 2$ and $\alpha_0 = 40$ in our study.

4. **Posterior for the latent variable $N_{t+1}$** For $1 \leq t+1 \leq T$, the posterior of $N_{t+1}$ follows a trinomial distribution

$$p(N_{t+1} = i|\cdot) = \begin{cases} \frac{\lambda^+ \exp(U_1)}{S^{(N)}}, & \text{when } i = 1; \\ \frac{\lambda_0 \exp(U_0)}{S^{(N)}}, & \text{when } i = 0; \\ \frac{\lambda^- \exp(U_{-1})}{S^{(N)}}, & \text{when } i = -1, \end{cases}$$

where

$$U_1 = -\frac{1}{2v_t\sigma_v^2\Delta(1-\rho^2)} \times$$
$$\left[\sigma_v^2(C_{t+1}^{(N)} - (\xi_{t+1}^{y+} + \rho_J\xi_{t+1}^v))^2 - 2\rho\sigma_v(C_{t+1}^{(N)} - (\xi_{t+1}^{y+} + \rho_J\xi_{t+1}^v))(D_{t+1}^{(N)} - \xi_{t+1}^v) + (D_{t+1}^{(N)} - \xi_{t+1}^v)^2\right],$$
$$U_0 = -\frac{1}{2v_t\sigma_v^2\Delta(1-\rho^2)}\left[\sigma_v^2(C_{t+1}^{(N)})^2 - 2\rho\sigma_v C_{t+1}^{(N)} D_{t+1}^{(N)} + (D_{t+1}^{(N)})^2\right],$$
$$U_{-1} = -\frac{1}{2v_t\sigma_v^2\Delta(1-\rho^2)} \times$$
$$\left[\sigma_v^2(C_{t+1}^{(N)} - (-\xi_{t+1}^{y-} + \rho_J\xi_{t+1}^v))^2 - 2\rho\sigma_v(C_{t+1}^{(N)} - (-\xi_{t+1}^{y-} + \rho_J\xi_{t+1}^v))(D_{t+1}^{(N)} - \xi_{t+1}^v) + (D_{t+1}^{(N)} - \xi_{t+1}^v)^2\right],$$
$$S^{(N)} = \lambda_1 \exp(U_1) + \lambda_0 \exp(U_0) + \lambda^- \exp(U_{-1}),$$
$$C_{t+1}^{(N)} = y_{t+1} - y_t - \mu\Delta - J_{VG,t+1}^y,$$
$$D_{t+1}^{(N)} = v_{t+1} - v_t - \kappa(\theta - v_t)\Delta.$$

*Proof.* To ease notations, in this proof, define $p_1 := \lambda^+$, $p_0 := \lambda_0$ and $p_{-1} := \lambda^-$. By Bayes' rule, we have for $i = -1, 0, 1$,

$$\begin{aligned} & p(N_{t+1} = i|\cdot) = p(N_{t+1} = i|\Theta, y_t, y_{t+1}, v_t, v_{t+1}, \xi_{t+1}^{y+}, \xi_{t+1}^{y-}, \xi_{t+1}^v) \\ =& \frac{p(y_{t+1}, v_{t+1}|y_t, \Theta, N_{t+1} = i, v_t, \xi_{t+1}^{y+}, \xi_{t+1}^{y-}, \xi_{t+1}^v)p(N_{t+1} = i|\Theta)}{\sum_j p(y_{t+1}, v_{t+1}|y_t, \Theta, N_{t+1} = i, v_t, \xi_{t+1}^{y+}, \xi_{t+1}^{y-}, \xi_{t+1}^v)p(N_{t+1} = j|\Theta)} \\ =& \frac{p(y_{t+1}, v_{t+1}|y_t, \Theta, N_{t+1} = i, v_t, \xi_{t+1}^{y+}, \xi_{t+1}^{y-}, \xi_{t+1}^v)p_i}{\sum_j p(y_{t+1}, v_{t+1}|y_t, \Theta, N_{t+1} = i, v_t, \xi_{t+1}^{y+}, \xi_{t+1}^{y-}, \xi_{t+1}^v)p_j}, \end{aligned}$$

Since all three cases are similar, only the case of $i = 1$ is considered here. When $i = 1$,

$$\begin{aligned} & p(y_{t+1}, v_{t+1}|y_t, \Theta, N_{t+1} = i, v_t, \xi_{t+1}^{y+}, \xi_{t+1}^{y-}, \xi_{t+1}^v) \\ =& p(y_{t+1}, v_{t+1}|y_t, \Theta, N_{t+1} = 1, v_t, \xi_{t+1}^{y+}, \xi_{t+1}^v) \\ =& \frac{1}{2\pi\sqrt{v_t\Delta(1-\rho^2)\sigma_v^2}} \exp(U_1). \end{aligned}$$

The required conclusion readily follows when all common factors are canceled out. □

5. **Posterior for the latent variables $\xi^v_{t+1}$** For $1 \le t+1 \le T$, the posterior of $\xi^v_{t+1}$ follows a truncated normal distribution when $N_{t+1} \neq 0$

$$\xi^v_{t+1} \sim N\left(\frac{S^{(X)}}{W^{(X)}}, \frac{1}{W^{(X)}}\right) 1_{(\xi^v_{t+1}>0)},$$

where $S^{(X)} = \frac{\sigma_v^2 C^{(X)}_{t+1}\rho_J - 2\rho\sigma_v(C^{(X)}_{t+1}+\rho_J D^{(X)}_{t+1}) + D^{(X)}_{t+1}}{v_t\Delta(1-\rho^2)\sigma_v^2} - \frac{1}{\mu_v}$, $W^{(X)} = \frac{\sigma_v^2\rho_J^2 - 2\sigma_v\rho\rho_J + 1}{v_t\Delta(1-\rho^2)\sigma_2}$, $C^{(X)}_{t+1} = y_{t+1} - y_t - \mu\Delta - J^y_{VG,t+1} - (1(N_{t+1}=1)\xi^{y+}_{t+1} - 1(N_{t+1}=-1)\xi^{y-}_{t+1})$ and $D^{(X)}_{t+1} = v_{t+1} - v_t - \kappa(\theta - v_t)\Delta$. When $N_{t+1} = 0$, the posterior is the same as its prior $\xi^v_{t+1} \sim \exp(\mu_v)$.

*Proof.* To ease notation, in this proof, we denote $C_{t+1} = C^{(X)}_{t+1}$, $D_{t+1} = D^{(X)}_{t+1}$, $S = S^{(X)}$ and $W = W^{(X)}$. It is easy to see the posterior is $\xi^v_{t+1} \sim \exp(\mu_v)$ when $N_{t+1} = 0$ since the data provides no information. By Bayes' rule, when $N_{t+1} \neq 0$,

$$\begin{aligned}
&p(\xi^v_{t+1}|\cdot) = p(\xi^v_{t+1}|y_{t+1}, y_t, v_{t+1}, v_t, N_{t+1} \neq 0, \Theta, \xi^{y+}_{t+1}, \xi^{y-}_{t+1}, J^y_{VG,t+1}) \\
&\propto p(y_{t+1}, v_{t+1}|y_t, \Theta, N_{t+1} \neq 0, v_t, \xi^{y+}_{t+1}, \xi^{y-}_{t+1}, \xi^v_{t+1}, J^y_{VG,t+1}) p(\xi^v_{t+1}|\eta^+) \\
&\propto \exp\left[-\frac{\sigma_v^2(C_{t+1} - \rho_J\xi^v_{t+1})^2 - 2\rho\sigma_v(C_{t+1} - \rho_J\xi^v_{t+1})(D_{t+1} - \xi^v_{t+1}) + (D_{t+1} - \xi^v_{t+1})^2}{2v_t\Delta(1-\rho^2)\sigma_v^2}\right] \\
&\quad \times \exp\left(-\frac{\xi^v_{t+1}}{\mu_v}\right) 1_{(\xi^{y+}_{t+1}>0)} \\
&\propto \exp\left(-\frac{1}{2}W(\xi^v_{t+1})^2 + S\xi^v_{t+1}\right) 1_{(\xi^v_{t+1}>0)}.
\end{aligned}$$

The required conclusion readily follows by completing the squares and comparing the parameters.

□

6. **Posterior for $\rho_J$** The posterior of $\rho_J$ follows a normal distribution

$$\rho_J \sim N\left(\frac{S^{(\rho_J)}}{W^{(\rho_J)}}, \frac{1}{W^{(\rho_J)}}\right),$$

where $W^{(\rho_J)} = \sum_{t=0}^{T-1} \frac{1_{(N_{t+1}\neq 0)}(\xi^v_{t+1})^2}{(1-\rho^2)v_t\Delta} + \frac{1}{M^{(\rho_J)^2}}$, $S^{(\rho_J)} = \sum_{t=0}^{T-1} \frac{1_{(N_{t+1}\neq 0)}\xi^v_{t+1}(C^{(\rho_J)}_{t+1} - \rho D^{(\rho_J)}_{t+1}/\sigma_v)}{(1-\rho^2)v_t\Delta} + \frac{m^{(\rho_J)}}{M^{(\rho_J)^2}}$, $C^{(\rho_J)}_{t+1} = y_{t+1} - y_t - \mu\Delta - J^y_{VG,t+1} - (1_{(N_{t+1}=1)}\xi^{y+}_{t+1} - 1_{(N_{t+1}=-1)}\xi^{y-}_{t+1})$ and $D^{(\rho_J)}_{t+1} = v_{t+1} - v_t - \kappa(\theta - v_t)\Delta - 1(N_{t+1} \neq 0)\xi^v_{t+1}$. Here the prior for $\rho_J$ is $N(m^{(\rho_J)}, M^{(\rho_J)^2})$, where $m^{(\rho_J)} = 0$, $M^{(\rho_J)} = 1$ in our study.

*Proof.* The posterior distribution

$$
\begin{aligned}
& p(\rho_J|\cdot) \\
=& p(\rho_J|Y_{0:T}, V_{0:T}, \Theta_{-\rho_J}, \xi^{y+}_{t+1}, \xi^{y-}_{t+1}, \xi^{v}_{t+1}, N_{1:T}) \\
\propto& p(Y_{0:T}|V_{0:T}, \rho_J, \Theta_{-\rho_J}, \xi^{y+}_{t+1}, \xi^{y-}_{t+1}, \xi^{v}_{t+1}, N_{1:T})p(\rho_J) \\
\propto& \exp\left(-\sum_{t=0}^{T-1}\frac{1}{2(1-\rho^2)v_t\Delta}(C^{(\rho_J)}_{t+1} - \rho D^{(\rho_J)}_{t+1}/\sigma_v - 1_{(N_{t+1}\neq 0)}\rho_J\xi^v_{t+1})^2 - \frac{1}{2(M^{(\rho_J)})^2}(\rho_J - m^{(\rho_J)})^2\right) \\
=& \exp\left(-\frac{1}{2}W^{(\rho_J)}\rho_J^2 + S^{(\rho_J)}\rho_J\right)
\end{aligned}
$$

The required conclusion readily follows by completing the squares and comparing the parameters.

□

Table 1 summarizes for the prior distributions, means and standard deviations of parameters of the SV-DEJ-VG-JV model.

# 3 Jumps and Volatility in S&P 500 and Nasdaq 100 Returns

The following Figure 1 and 2 correct the erroneous unit for returns in Figure 8 and 9 in the main text. We also augment the plot of returns and jumps with the model-imputed volatility.

It seems from the graphs below the model-imputed volatility is elevated in episodes of market turmoil, such as the 2007–2008 financial crisis, the dot-com bubble and the European debt crisis. Judging by the relative magnitude, a fair portion of large movements in the returns seems to be due to volatility rather than jumps. This is consistent with the finding of Eraker, Johannes and Polson (2003), that a fair portion of return variations is explained by stochastic volatility in affine jump-diffusion models, although they study the pre-2000 data and uses the SV-MJ-JV model; see Table III in their paper.

| Parameter | Prior Distribution | Prior Mean | Prior Standard Deviation |
| --- | --- | --- | --- |
| $\eta^+$ | (conjugate) Inverse Gamma | 1.0 | 1.0 |
| $\eta^-$ | (conjugate) Inverse Gamma | 1.0 | 1.0 |
| $(\lambda^+, \lambda^-)$ | (conjugate) Dirichlet | (0.0455, 0.0455) | (0.0311, 0.0311) |
| $\mu$ | (conjugate) Normal | 0 | 1 |
| $\theta$ | (conjugate) Truncated Normal | 0.7979 | 0.6028 |
| $\kappa$ | (conjugate) Truncated Normal | 0.7979 | 0.6028 |
| $(\sigma_v, \rho)$ | $\phi_v \vert w_v$ Normal,<br>$w_v$ Inverse Gamma,<br>where $\sigma_v = \sqrt{\phi_v^2 + w_v}$,<br>$\rho = \phi_v / \sqrt{\phi_v^2 + w_v}$ | 0<br>100 | $\sqrt{w_v/2}$<br>$\infty$ |
| $\gamma$ | (conjugate) Normal | 0 | 1 |
| $\sigma$ | (conjugate) Inverse Gamma | 0.1333 | 0.1086 |
| $\nu$ | Inverse Gamma | 1.1111 | 0.3704 |
| $\mu_v$ | (conjugate) Inverse Gamma | 1.1111 | 0.3704 |
| $\rho_J$ | (conjugate) Normal | 0 | 1 |

Table 1: Prior distributions, means and standard deviations for the SV-DEJ-VG-JV models

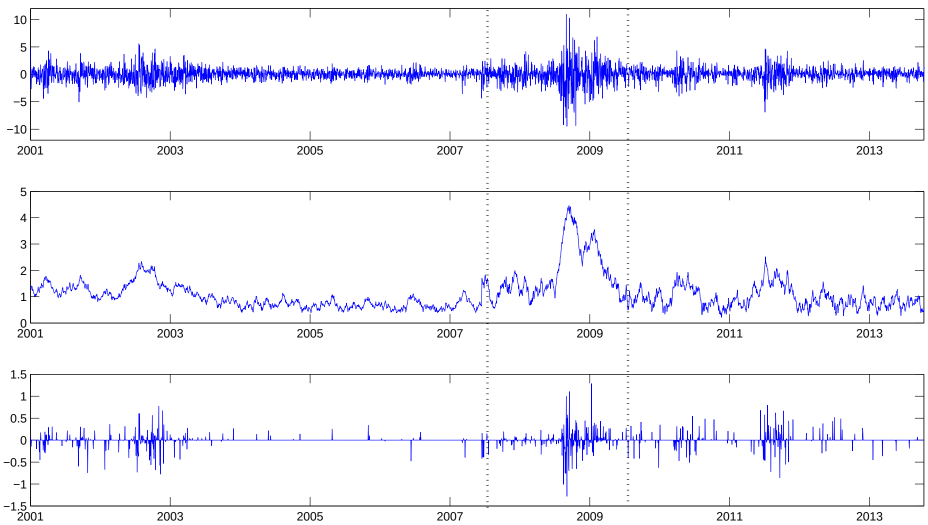


Figure 1: S&P 500 - from top to bottom: returns, volatilities and jumps (last two imputed by the SV-DEJ model). Using the notations in Section 2 (and (1) in the main text), returns in the plot are $y_{t+1} - y_t$, where $y$ are log-returns scaled by 100; volatilities in the plot are $\sqrt{v_t}$ and jumps in the plot are $1_{(N_{t+1}=1)}(\xi^+_{t+1}) + 1_{(N_{t+1}=-1)}(-\xi^-_{t+1})$. The dotted vertical lines indicate the outbreak of the financial crisis (August 2007) and its end (June 2009).

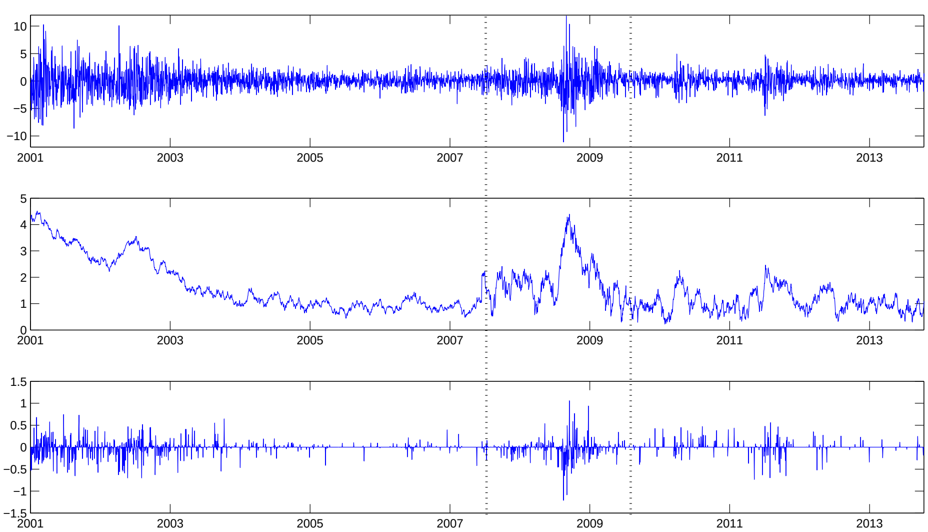


Figure 2: Nasdaq 100 - from top to bottom: returns, volatilities and jumps (last two imputed by the SV-DEJ model). Using the notations in Section 2 (and (1) in the main text), returns in the plot are $y_{t+1} - y_t$, where $y$ are log-returns scaled by 100; volatilities in the plot are $\sqrt{v_t}$ and jumps in the plot are $1_{(N_{t+1}=1)}(\xi^+_{t+1}) + 1_{(N_{t+1}=-1)}(-\xi^-_{t+1})$. The dotted vertical lines indicate the outbreak of the financial crisis (August 2007) and its end (June 2009).