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FIRST PASSAGE TIMES OF TWO-DIMENSIONAL BROWNIAN MOTION

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Abstract

First passage times (FPTs) of two-dimensional Brownian motion have many applications in quantitative finance. However, despite various attempts since the 1960's, there are few analytical solutions available. By solving a non-homogeneous modified Helmholtz equation in an infinite wedge, we find analytical solutions for the Laplace transforms of FPTs; these Laplace transforms can be inverted numerically. The FPT problems lead to a class of bivariate exponential distributions which are absolute continuous but do not have memoryless property. We also prove that the density of the absolute difference of FPTs tends to infinity if and only if the correlation between the two Brownian motions is positive.

Keywords: first passage times, two-dimensional Brownian motion, default correlation

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1. Introduction

Consider a two-dimensional Brownian motion

$$X_i(t) = x_i + \mu_i t + \sigma_i W_i(t), \quad x_i > 0, \quad i = 1, 2,$$

where $W_i(t)$ are standard one-dimensional Brownian motions and $Cov(W_1(t), W_2(t)) = \rho t$, $-1 < \rho < 1$. Let $\tau_i = \inf_{t \geq 0}\{t : X_i(t) = 0\}$, $i = 1, 2$. We are interested in computing

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the following quantities: (i) the joint distribution of first passage times

$$P^{(x_1,x_2)}(\tau_1 \le t_1,\ \tau_2 \le t_2); \tag{1.1}$$

(ii) the distribution

$$P^{(x_1,x_2)}(|\tau_1 - \tau_2| \le t); \tag{1.2}$$

(iii) the distribution

$$P^{(x_1,x_2)}(\tau^* \le t), \quad \text{where } \ \tau^* = \min(\tau_1, \tau_2); \tag{1.3}$$

(iv) the joint moment

$$E^{(0,0)}[(-J_1(\varepsilon_p))(-J_2(\varepsilon_p))], \tag{1.4}$$

(v) the joint moment

$$E^{(0,0)}[(-J_1(\varepsilon_{p_1}))(-J_2(\varepsilon_{p_2}))]. \tag{1.5}$$

Here $J_i(t) := \min_{0\le s\le t} X_i(s),\ i = 1, 2,\ \varepsilon_p$ denotes an exponential random variable with rate p, and ε_{p_1} and ε_{p_2} are independent. Both exponential random variables are independent of the Brownian motions. Throughout the paper, $P^{(x_1,x_2)}$ and $E^{(x_1,x_2)}$ denote the conditional probability and the conditional expectation when the Brownian motion starts from (x_1, x_2), respectively.

There are numerous applications for both (1.1) and (1.3) in finance. (1) In structural models for credit risk, defaults are modeled as first passage times of (geometric) Brownian motions or other more general jump-diffusion process; see, for example, [18] among others. Structural models have been used to study default correlations and interactions among different companies; see [13] and [28] for models where investors have complete information, and [10] for an incomplete-information model. In particular, [6] characterizes the correlation structure of multiple firms given incomplete information. As pointed out by [6], such investigation may involve significant computational costs; the numerical methods proposed in this paper allow [6] to mitigate this computational problem. (2) τ^* in (1.3) is used in [14] in pricing double lookback options. For a summary of these applications, see Table 1.

Partly motivated by the applications above, existing literatures provide some solutions to (1.1)-(1.5) using different approaches; see Table 2. Note that, except for

Table 1: Summary of the various applications of correlated Brownian motions in finance

Applications	Literature
Structural Model in Credit Risk and Default Correlations	Haworth, Reisinger and Shaw (2008) Zhou (2001) Giesecke (2004) Ching, Gu and Zheng (2014)
Pricing Double Lookback	He, Keirstead and Rebholz (1998)

[14] and [24], all others essentially assume $\mu_1 = \mu_2 = 0$. The difficulty partly lies in the fact that with nonzero drifts, the two-dimensional Brownian motion ceases to be a conformal martingale (see [20]).

In this paper, we obtain the Laplace transform (or joint Laplace transform) of (1.1)–(1.3) by solving a non-homogeneous modified Helmholtz equation in an infinite wedge. To a large extent, whether a partial differential equation (PDE) is solvable analytically depends on the boundary conditions. For the above modified Helmholtz equations, if the boundary conditions are on a disk, then the analytical solution is well known. However, in the problem at hand, the boundary conditions are on an infinite wedge with an angle, rendering a PDE problem difficult to solve. Nevertheless, we give two solutions to the PDE problem, one based on the finite Fourier transform and the other Kontorovich-Lebedev transform (by extending the method used in [23] to the case of arbitrary drifts). The finite Fourier transform leads to a more efficient numerical algorithm to compute the distribution functions; see Section 3.

Apart from the contribution listed in Table 1, there are other new results in this paper. (1) We compute $P^{(x_1,x_2)}(\tau_1 \leq t_1, \tau_2 \leq t_2)$ by numerical Laplace transform inversion, based on which we extend the study of default correlation in [28] to the case of arbitrary drifts (see Section 3). (2) We prove that the density of $|\tau_1 - \tau_2|$ with $\tau_1, \tau_2 < \infty$ tends to infinity if and only if $\rho > 0$ (see Theorem 3). (3) We point out a link between the first passage times and a class of bivariate exponential distribution

Table 2: This table summarizes existing results on first passage time problem of correlated Brownian motions (except Sacerdote et al. (2015) in which several joint densities in a more general setting of diffusion processes are obtained), where "not available" is denoted as "N.A.". In the first row (i) means $P^{(x_1,x_2)}(|\tau_1 - \tau_2| \le t)$, (ii) means $E^{(0,0)}[(-J_1(\varepsilon_p))(-J_2(\varepsilon_p))]$, and (iii) means $E^{(0,0)}[(-J_1(\varepsilon_{p_1}))(-J_2(\varepsilon_{p_2}))]$. Notably, the numerical efficiency of our methods in calculating $P^{(x_1,x_2)}(\min(\tau_1,\tau_2) \le t)$ and $P^{(x_1,x_2)}(\tau_1 \le t_1, \tau_2 \le t_2)$ for the case of $\mu_1^2 + \mu_2^2 > 0$ is independently demonstrated in Ching, Gu and Zheng (2014).

	$P^{(x_1,x_2)}(\min(\tau_1,\tau_2) \le t)$		$P^{(x_1,x_2)}(\tau_1 \le t_1, \tau_2 \le t_2)$		(i)	(ii)	(iii)
	$\mu_1 = \mu_2 = 0$	$\mu_1^2 + \mu_2^2 > 0$	$\mu_1 = \mu_2 = 0$	$\mu_1^2 + \mu_2^2 > 0$			
Spitzer (1959)	integral transform	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.
Iyengar (1985)	analytical expression	N.A.	joint density	N.A.	N.A.	N.A.	N.A.
He, Keirstead and Rebholz (1998)	analytical expression with joint distribution (two-dim. integral)	analytical expression with joint distribution (two-dim. integral)	N.A.	N.A.	N.A.	N.A.	N.A.
Zhou (2001)	analytical expression	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.
Rogers and Shepp (2006)	Laplace transform	N.A.	N.A.	N.A.	N.A.	analytical expression ($\mu_1 = \mu_2 = 0$)	N.A.
Metzler (2010)	correct typos in Iyengar (1985)	N.A.	correct typos in Iyengar (1985)	Monte Carlo method	Monte Carlo method	N.A.	N.A.
Sacerdote, Tamborrino & Zucca (2015)	N.A.	N.A.	analytical expression of density	analytical expression of density	N.A.	N.A.	N.A.
This paper	Laplace transform	Laplace transform	Laplace transform	Laplace transform	Laplace transform	analytical expression (arbitrary drifts)	analytical expression

that is absolutely continuous but does not have lack of memory property; see Section 5.

The rest of the paper is organized as follows. A general PDE problem is solved in Section 2, where we also verify the existence and uniqueness of the solution. As special cases, the joint Laplace transform $E^{(x_1,x_2)}[e^{-p_1\tau_1 - p_2\tau_2}]$, the joint Laplace transform $E^{(x_1,x_2)}[e^{-q\tau^* - s|\tau_1-\tau_2|}]$ and the Laplace transform $E^{(x_1,x_2)}[e^{-p\tau^*}]$ are given in Section 2.5. We present a numerical algorithm to compute $P^{(x_1,x_2)}(\tau_1 \le t_1, \tau_2 \le t_2)$ as well as an application to study default correlation in Section 3. The Laplace transform of $|\tau_1 - \tau_2|$ is given in Section 4, where we also discuss the property of the density of $|\tau_1 - \tau_2|$ near zero. Section 5 provides a link between the first passage times and a class of bivariate exponential distributions. All proofs of lemmas and theorems can be found in the online supplement.

2. Main Results

2.1. Basic Ideas

To obtain the required quantities mentioned above, we shall solve the following non-homogeneous PDE

$$\frac{1}{2}\sigma_1^2\frac{\partial^2 u}{\partial x_1^2}+\rho\sigma_1\sigma_2\frac{\partial^2 u}{\partial x_1\partial x_2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 u}{\partial x_2^2}+\mu_1\frac{\partial u}{\partial x_1}+\mu_2\frac{\partial u}{\partial x_2}=cu,\quad x_1,x_2>0, \tag{2.1}$$

with a non-homogeneous boundary condition

$$\begin{cases} u(x_1,x_2)|_{x_1=0}=\exp(-D_2x_2),\\ u(x_1,x_2)|_{x_2=0}=\exp(-D_1x_1),\end{cases} \tag{2.2}$$

and uniform boundedness on the whole domain

$$|u|\le C\ \ \text{(for some constant } C>1 \text{ is not depending on } x_1 \text{ and } x_2). \tag{2.3}$$

The PDE (2.1), (2.2), (2.3) is a non-homogeneous, modified Helmholtz equation in the positive quadrant (or in an infinite wedge, if one switches to the polar coordinates and removes the cross correlation term in the PDE, as we shall see shortly). To a large extent, whether one can solve a PDE analytically depends on the boundary conditions. For example, for the above modified Helmholtz equations, the analytical solution is well known if the boundary conditions are on a disk. However, the main difficulty here is that the boundary conditions are on an infinite wedge with an angle, making the PDE difficult to solve analytically.

In the existing literature, there are various numerical schemes available for solving modified Helmholtz equations with some limitations. (i) All of them are done in a case by case basis. For example, [3] solve the problem when the angle of the wedge is $\pi/4$. [5] and [17] provide fast numerical algorithms, but only on bounded domains. (ii) None of the results listed above can guarantee that the numerical solution is uniformly bounded as in (2.3) and hence a bona fide Laplace transform (see Theorem 2).

We present two approaches to solve the aforementioned PDE, one by the finite Fourier transform and the other by the Kontorovich-Lebedev transform. On the one hand, the finite Fourier transform is obtained by moving the non-homogeneous boundary conditions inside the PDE itself (see Lemma 1), from which the nonhomogeneous PDE with homogeneous boundary conditions can be solved by using the finite Fourier transform. On the other hand, the Kontorovich-Lebedev transform is used to match the non-homogeneous bounary condition exactly, thanks to an algebraic identity (C.1) in the online supplement.

After solving the PDE in two ways, several issues remain. (1) Is the solution to the PDE unique? (2) Do the two approaches yield the same solution? (3) Which solution is better? For the first question, we prove that the solution is unique using a martingale argument; see Theorem 1 below. For the second question, the two approaches aforementioned yield the same unique solution; see Theorem 2 below. For the third question, the finite-Fourier-transform approach is relatively better in terms of numerical calculation (see Section 3.1) and has broader applicability in some cases (see Remark 2). The Kontorovich-Lebedev transform, on the other hand, is convenient in verifying the uniformly boundedness of solutions; see the proof of the uniform boundedness of u_2 defined in Lemma 3 in the online supplement.

2.2. Removing the Boundary Conditions

Introduce the polar coordinates r and θ:

$$r=\sqrt{z_1^2+z_2^2},\ \ \tan\theta=\frac{z_2}{z_1};\ \ z_1=\frac{1}{\sqrt{1-\rho^2}}\left(\frac{x_1}{\sigma_1}-\rho\frac{x_2}{\sigma_2}\right),\ \ z_2=\frac{x_2}{\sigma_2}, \tag{2.4}$$

and $\alpha\in[0,\pi)$ (note that $x_2>0$) defined via

$$\sin\alpha=\sqrt{1-\rho^2},\ \ \cos\alpha=-\rho. \tag{2.5}$$

Define the constants a, A and the function G as

$$a\equiv a(c)=\sqrt{2c+\gamma_1^2+\gamma_2^2},\quad \gamma_1=\frac{\mu_1/\sigma_1-\rho\mu_2/\sigma_2}{\sqrt{1-\rho^2}},\ \ \gamma_2=\mu_2/\sigma_2, \tag{2.6}$$

$$A\equiv A(c,D_1,D_2)=\sigma_1^2D_1^2+2\rho\sigma_1\sigma_2D_1D_2+\sigma_2^2D_2^2-2\mu_1D_1-2\mu_2D_2-2c,$$

$$G(\theta):=-\gamma_1\cos\theta-\gamma_2\sin\theta+D_1\sigma_1\sin(\alpha-\theta)+D_2\sigma_2\sin\theta. \tag{2.7}$$

Lemma 1. *(i) (Removing the boundary conditions) Any solution, if exists, to the following PDE*

$$\frac{1}{2}\left(\frac{\partial^2 k}{\partial r^2}+\frac{1}{r}\frac{\partial k}{\partial r}+\frac{1}{r^2}\frac{\partial^2 k}{\partial \theta^2}\right)=\frac{1}{2}a^2 k, \tag{2.8}$$

with a nonhomogeneous boundary condition on an infinite wedge

$$k(r,\theta)|_{\theta=0}=e^{-G(0)r}, \quad k(r,\theta)|_{\theta=\alpha}=e^{-G(\alpha)r}, \tag{2.9}$$

is equivalent to a solution to

$$\frac{1}{2}\left(\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}+\frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2}\right)=\frac{1}{2}a^2 h-\frac{1}{2}A\exp(-G(\theta)r), \tag{2.10}$$

with a homogeneous boundary condition on an infinite wedge

$$h(r,\theta)|_{\theta=0}=0, \quad h(r,\theta)|_{\theta=\alpha}=0, \tag{2.11}$$

via

$$h(r,\theta)=k(r,\theta)-\exp(-G(\theta)r). \tag{2.12}$$

(ii) (Change of variables) Any solution to (2.8) *and* (2.9)*, if exists, leads to a solution to* (2.1) *and* (2.2)*:*

$$u(x_1,x_2)=e^{-(\gamma_1\cos\theta+\gamma_2\sin\theta)r}k(r,\theta). \tag{2.13}$$

Equivalently, any solution, if exists, to (2.10) *and* (2.11) *leads to a solution to* (2.1) *and* (2.2)*:*

$$u(x_1,x_2)=e^{-(\gamma_1\cos\theta+\gamma_2\sin\theta)r}h(r,\theta)+\exp(-D_1x_1-D_2x_2). \tag{2.14}$$

Proof. See the online supplement.

Remark 1. Though simple, equations (2.12) and (2.7) are among the key steps in this paper, from which the removing of the boundary conditions in (2.11) is made possible, partly because the two boundary conditions in (2.9) and (2.2) are both exponential functions and the derivatives of exponential functions are still exponential functions. Not surprisingly, it took the authors some time to find these simple equations.

2.3. Uniqueness and Stochastic Representation

To find an expression for the unique solution to the PDE problem, we need the following definition and conditions.

Definition 1. Introduce p_1, p_2, and v as $p_1 \equiv p_1(c, D_1, D_2) := \frac{1}{4}\sigma_1^2 D_1^2 - \frac{1}{2}\mu_1 D_1 - \frac{1}{4}\sigma_2^2 D_2^2 + \frac{1}{2}\mu_2 D_2 + \frac{1}{2}c$, $p_2 \equiv p_2(c, D_1, D_2) := \frac{1}{4}\sigma_2^2 D_2^2 - \frac{1}{2}\mu_2 D_2 - \frac{1}{4}\sigma_1^2 D_1^2 + \frac{1}{2}\mu_1 D_1 + \frac{1}{2}c$ and $v \equiv v(c, D_1, D_2) := \frac{1}{4}\sigma_1^2 D_1^2 - \frac{1}{2}\mu_1 D_1 + \frac{1}{4}\sigma_2^2 D_2^2 - \frac{1}{2}\mu_2 D_2 - \frac{1}{2}c$; or equivalently,

$$c(p_1, p_2) = p_1 + p_2, \quad D_j(p_j, v) = \frac{\sqrt{\mu_j^2 + 2(p_j + v)\sigma_j^2} + \mu_j}{\sigma_j^2}, \quad j = 1, 2. \tag{2.15}$$

Note that, for the one-dimensional first passage time we have the Laplace transform $E^x(e^{-p\tau}) = \exp\left(-\frac{1}{\sigma^2}(\sqrt{\mu^2 + 2\sigma^2 p} + \mu)x\right)$, $x > 0$. This is the motivation for D_i.

Condition 1. *(i)* $p_1 + p_2 > 0$ *and (ii)* $\min(p_1, p_2) + v > 0$ *(so that c, D_1 and D_2 are positive).*

Condition 2. *(i)* $p_1 + p_2 > 0$ *and (ii)* $\min(p_1, p_2) + v \geq M$ *where* $M = \max(0, \frac{1}{2} \times [(\frac{2(|\gamma_1| + |\gamma_2|)}{\sin\alpha} + 1 - \frac{\mu_1}{\sigma_1})^2 - (\frac{\mu_1}{\sigma_1})^2], \frac{1}{2}[(\frac{2(|\gamma_1| + |\gamma_2|)}{\sin\alpha} + 1 - \frac{\mu_2}{\sigma_2})^2 - (\frac{\mu_2}{\sigma_2})^2])$ *and α, γ_1, γ_2 are defined in* (2.5) *and* (2.6).

There are cases in which Condition 1 holds but Condition 2 does not; for example, take $p_1 = p_2 = v = \frac{M-1}{2}$. There are also many cases of interest where (p_1, p_2, v) satisfies Condition 2; see Section 2.5.

Theorem 1. *(Uniqueness and Stochastic Representation) Suppose Condition 1 holds. Then any solution to* (2.1), (2.2) *and* (2.3)*, if exists, is unique and has the following stochastic representation:*

$$u(x_1, x_2) = E^{(x_1, x_2)}\left[e^{-p_1\tau_1 - p_2\tau_2 - v|\tau_2 - \tau_1|}\right].$$

Proof. See the online supplement.

2.4. Existence and Analytical Solutions

Theorem 2. *(Existence and the Analytical Solution) Suppose Condition 2 holds (and hence Condition 1 also holds). Then the unique solution u_1 to the PDE problem (2.1),* *(2.2) and (2.3) is given by*

$$u_1(x_1,x_2) := e^{-(\gamma_1\cos\theta+\gamma_2\sin\theta)r}\left(\sum_{n=1}^{\infty}\sqrt{\frac{2}{\alpha}}\sin(\nu_n\theta)U_n(r)\right) + \exp(-D_1x_1 - D_2x_2).$$

In addition, another representation of $u_1(x_1,x_2)$ *is given by*

$$u_1(x_1,x_2) = u_2(x_1,x_2) := e^{-(\gamma_1\cos\theta+\gamma_2\sin\theta)r}k(r,\theta).$$

Here

$$\nu_n = \frac{n\pi}{\alpha} \geq n,\ H_1 = G(0) = -\gamma_1 + D_1\sigma_1\sin\alpha,$$
$$H_2 = G(\alpha) = -\gamma_1\cos\alpha - \gamma_2\sin\alpha + D_2\sigma_2\sin\alpha,$$

$$\begin{aligned} U_n(r) =& \frac{1}{2}A(c,D_1,D_2)\int_{\eta=0}^{\alpha}\sqrt{\frac{2}{\alpha}}\sin(\nu_n\eta)\left[K_{\nu_n}(ar)\int_{l=0}^{r}\exp(-G(\eta)l)lI_{\nu_n}(al)dl\right.\\ &\left.+I_{\nu_n}(ar)\int_{l=r}^{\infty}\exp(-G(\eta)l)lK_{\nu_n}(al)dl\right]d\eta,\\ k(r,\theta) =& \frac{2}{\pi}\int_0^{\infty}\frac{K_{i\nu}(ar)}{\sinh(\alpha\nu)}\left[\cosh(\beta_1\nu)\sinh((\alpha-\theta)\nu) + \cosh(\beta_2\nu)\sinh(\theta\nu)\right]d\nu, \end{aligned}$$

$$\beta_j(c) = \arccos(H_j/a(c)) = -i\log\left(\frac{H_j}{a(c)} + i\sqrt{1-\frac{H_j^2}{a(c)^2}}\right),\quad j=1,2, \tag{2.16}$$

and $I_\nu(\cdot)$ *and* $K_\nu(\cdot)$ *are modified Bessel functions with order* ν *of the first kind and the second kind, respectively. Note that* H_i *depends on* D_i*, which in turns depends on* p_1*,* p_2*, and* v *via* (2.15)*.*

Proof. See the online supplement.

For the two representations in Theorem 2, u_2 is easier for proving certain theoretical properties; for example it is easier to show the uniform boundedness of u_2 (see Lemma 3 in the online supplement), while u_1 is better in numerical calculation. More precisely, for u_1 by finite Fourier transform, we can compute the double integrals in $U_n(r)$ easily by using of the matlab function "quadgk", based on an adaptive Gauss-Kronrod quadrature; see [25]. However, for u_2 we need to be able to calculate the modified Bessel function of second kind with both degrees of freedom and argument being complex, because the Laplace inversion algorithm requires the function evaluated at complex values.

This leads to difficulties in implementing the expression u_2. (i) Although there are ways of computing $K_{ix}(ar)$ or the Kontorovich-Lebedev transform directly (see, for example, [9]), it seems none of them has dealt with the case when the argument of $K_{ix}(ar)$ is complex. (ii) There are also a few asymptotic expansions of $K_{ix}(ar)$, but numerical procedures based on these formulas do not seem stable in our numerical experiments. We find that it is better to use the following definition of $K_{i\nu}(\cdot)$, which is well-defined and numerically stable when the argument of $K_{i\nu}(ar)$ is complex

$$K_{i\nu}(ar) = \int_0^\infty \exp(-ar\cosh(t))\cos(\nu t)dt.$$

Hence the expression in u_2 again becomes a double infinite integral, one with t and the other with ν. The evaluation of the double integral in u_2 takes longer than that in u_1, partly due to the combination of exp, cosh, and oscillation function cos in the integrand of $K_{i\nu}(ar)$.

2.5. Special Cases

Theorem 2 above reduces to several special cases with different choices of parameters.

(1) When $v = 0$ and $\min(p_1, p_2) \geq M$. Then $p_1 + p_2 > 0$, $\min(p_1, p_2) + v \geq M$ (Condition 2 holds) and the special case is

$$L(x_1, x_2) = E^{(x_1,x_2)}\left[e^{-p_1\tau_1 - p_2\tau_2}\right]. \tag{2.17}$$

In what follows we let L_1 be the expression by u_1 in Theorem 2 (using the finite Fourier transform) and L_2 the expression by u_2 (using the Kontorovich-Lebedev transform).

(2) When $p_1 = p_2 = \frac{q}{2}$, $v = s - \frac{q}{2}$ and $q > 0$, $s \geq M$. Then $p_1 + p_2 = q > 0$, $\min(p_1, p_2) + v = s \geq M$ (Condition 2. holds) and $p_1\tau_1 + p_2\tau_2 + v|\tau_1 - \tau_2| = \frac{q}{2}(2\tau^* + |\tau_1 - \tau_2|) + (s - \frac{q}{2})|\tau_1 - \tau_2| = q\tau^* + s|\tau_1 - \tau_2|$, and the special case is

$$T(x_1, x_2) = E^{(x_1,x_2)}\left[e^{-q\tau^* - s|\tau_2 - \tau_1|}\right]. \tag{2.18}$$

In what follows we let T_1 be the expression by u_1 in Theorem 2 (using the finite Fourier transform) and T_2 the expression by u_2 (using the Kontorovich-Lebedev transform).

(3) Let $p_1 = p_2 = v = \frac{p}{2}$ and $p \geq M$. Then $p_1 + p_2 = p > 0$, $\min(p_1, p_2) + v = p \geq M$ (Condition 2 holds) and $p_1\tau_1 + p_2\tau_2 + v|\tau_1 - \tau_2| = \frac{p}{2}(2\tau^* + |\tau_1 - \tau_2|) + \frac{p}{2}|\tau_1 - \tau_2| = \frac{p}{2}(2\tilde{\tau})$

$= p\tilde{\tau}$, where $\tilde{\tau} := \max(\tau_1, \tau_2)$. In particular,

$$E^{(x_1,x_2)}[e^{-p\tilde{\tau}}] = \frac{2}{\pi} e^{-(\gamma_1 \cos\theta + \gamma_2 \sin\theta) r} \times \int_{\nu=0}^{\infty} \frac{K_{i\nu}(ar)}{\sinh(\alpha\nu)} [\cosh(\beta_1 \nu)\sinh((\alpha-\eta)\nu) + \cosh(\beta_2\nu)\sinh(\eta\nu)] d\nu. \quad (2.19)$$

In the special case of $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$, the expression for $E^{(x_1,x_2)}(e^{-p\tilde{\tau}})$ simplifies to

$$\begin{aligned} &E^{(x_1,x_2)}(e^{-p\tilde{\tau}}) \\ =&\frac{2}{\pi}\int_0^\infty \frac{K_{i\nu}(\sqrt{2p}r)}{\sinh(\alpha\nu)} \left[\cosh\left(\left(\frac{\pi}{2}-\alpha\right)\nu\right)(\sinh((\alpha-\theta)\nu)+\sinh(\theta\nu))\right] d\nu \\ =&\frac{2}{\pi}\int_0^\infty \frac{K_{i\nu}(\sqrt{2p}r)}{\cosh(\frac{\alpha\nu}{2})} \left[\cosh\left(\left(\frac{\pi}{2}-\alpha\right)\nu\right)\cosh\left(\frac{\alpha\nu}{2}-\theta\nu\right)\right] d\nu, \end{aligned} \quad (2.20)$$

where in the second equality we have made use of the following two identities: $\sinh((\alpha-\theta)\nu) + \sinh(\theta\nu) = 2\sin\frac{\alpha\nu}{2}\cosh(\frac{\alpha\nu}{2} - \theta\nu)$ and $\sinh(\alpha\nu) = 2\sinh\frac{\alpha\nu}{2}\cosh\frac{\alpha\nu}{2}$.

(4) The special case $\tilde{L}(x_1, x_2) = E^{(x_1,x_2)}(e^{-p\tau^*})$ is then given by

$$E^{(x_1,x_2)}(e^{-p\tau^*}) = E^{x_1}(e^{-p\tau_1}) + E^{x_2}(e^{-p\tau_2}) - E^{(x_1,x_2)}(e^{-p\tilde{\tau}}). \quad (2.21)$$

In what follows we let $\tilde{L}_1$ be the expression by u_1 in Theorem 2 (using the finite Fourier transform) and $\tilde{L}_2$ the expression by u_2 (using the Kontorovich-Lebedev transform).

Remark 2. An advantage of finite Fourier transform can also be seen by comparing the expressions of T_1 and T_2 in (2.18). In particular, T_2 is not well-defined when $\mu_1 = \mu_2 = 0$ and $q \downarrow 0$. To see that, note that by definition $a(c) = a(q) = \sqrt{2q + \gamma_1^2 + \gamma_2^2} \downarrow 0$ in this case, which implies β_1, β_2 become $i \cdot \infty$ while $\cosh(i \cdot \infty)$ is not well-defined. To the contrary, T_1 by finite Fourier transform is well defined for arbitrary γ_1 and γ_2 when $q \downarrow 0$.

3. Numerical Computation of $P(\tau_1 \le t_1, \tau_2 \le t_2)$ and Application to Default Correlation

3.1. Numerical Results

To obtain the joint probability distribution of τ_1 and τ_2, numerical inversion of thel Laplace transform is applied on both the formulas of $L_1(x_1, x_2)$ and $L_2(x_1, x_2)$ in

(2.17), which are special cases of u_1 and u_2 in Theorem 2, respectively. The double Laplace inversion formula is given in [4]. Note that here the technical condition in Section 2.5 that p_1 and p_2 are sufficiently large (related to Condition 2 and Lemma 2 in the online supplement) does not impose obstacles in devising the inversion algorithm. This is because the double inversion algorithm evaluates the joint Laplace transform at complex values with large real parts; see [4] for more detail in this regard.

To obtain a benchmark, the joint probability distribution function is calculated using Monte Carlo simulation. We use 2000, 4000 and 8000 time-discretization grids, and perform the Richardson extrapolation of order $\frac{1}{2}$ to reduce discretization bias. For each of the probability, 100,000 two-dimensional Brownian paths are simulated.

One can see from Table 3 that probabilities obtained from inverting joint Laplace transforms align well with the benchmark for various combinations of correlations ($\rho = 0.2, 0.5, 0.8$), drifts ($\mu_1 = 0.2, -0.2, \mu_2 = 0.15, -0.15$) and volatilities ($\sigma_1 = \sigma_2 = 0.55, 0.2$). However, the differences for CPU time are significant. Given a set of parameters, it typically takes less than 2 minutes for algorithm using L_1 to compute one probability and 7-9 minutes for algorithm using L_2, while Monte Carlo simulation with the Richardson extrapolation usually takes 4-6 hours.

We also compare the accuracy and efficiency of our method to the results in [24], where the joint density function of (τ_1, τ_2) (Equation (22) therein) is obtained via an interesting conditional density argument and a solution of a two-dimensional Kolmogorov forward equation. To get the joint probability function from the joint density in [24], one can in principle perform a double numerical summation:

$$\begin{aligned} P(\tau_1 \le t_1, \tau_2 \le t_2) &= \int_{s_1=0}^{t_1} \int_{s_2=0}^{t_2} f_{\tau_1,\tau_2}(s_1, s_2) ds_1 ds_2 \\ &\approx \sum_{m=1}^{N} \sum_{n=1}^{N} f_{\tau_1,\tau_2}\left(\frac{mt_1}{N}, \frac{nt_2}{N}\right) \times \frac{t_1 t_2}{N^2}, \end{aligned}$$

where f_{τ_1,τ_2} is the joint density function. However, one challenge of implementing the double summation is that the joint density function $f_{\tau_1,\tau_2}(s_1, s_2) \to \infty$ when $s_1 \to s_2$ and $\rho > 0$. Therefore, whenever $\frac{mt_1}{N} = \frac{nt_2}{N}$, we replace $f_{\tau_1,\tau_2}\left(\frac{mt_1}{N}, \frac{nt_2}{N}\right)$ by $f_{\tau_1,\tau_2}\left(\frac{mt_1}{N}, \frac{nt_2}{N} + \delta\right)$, where δ is 10^{-5} in the numerical experiment. To achieve comparable accuracy, $N = 120$ is used (and $n = 10$ is used to compute the infinite sum for G_{ij} in Equation (22) in [24]). The last two columns in Table 3 shows that

TABLE 3: $P^{(x_1,x_2)}(\tau_1 \le t_1,\ \tau_2 \le t_2)$ obtained via the inversion of the joint Laplace transform (Column L_1 (FT) and by finite Fourier transform and Column L_2 (KL) by Kontorovich-Lebedev transform), compared with those obtained by the Monte Carlo simulation (Column MC) and by the equation (22) in Sacerdote, Tamborrino and Zucca (2015) (Column STZ), when $t_1 = 0.3, t_2 = 0.5$ with various combinations of correlations ($\rho = 0.2,\ 0.5,\ 0.8$), drifts ($\mu_1 = 0.2,\ -0.2,\ \mu_2 = 0.15,\ -0.15$) and volatilities ($\sigma_1 = \sigma_2 = 0.55,\ 0.2$). Here the probabilities in MC are computed using 100000 Brownian paths with 2000, 4000 and 8000 time-discretization grids. The Richardson extrapolation of order $\frac{1}{2}$ is exploited to reduce dicretization bias in the simulation. Starting points are set to be $x_1 = x_2 = \ln(1.2) = 0.1823$. The CPU time is in seconds. Each number in parenthesis in the column of MC is the standard deviation of Monte Carlo simulation.

	High Volatilities: $\sigma_1 = \sigma_2 = 0.55$									
	μ_1	μ_2	L_1 (FT)	CPU time	L_2 (KL)	CPU time	MC (std)	CPU time	STZ	CPU time
	0.2	0.15	0.3055	115.6	0.3056	519.5	0.3053(0.0015)	22240.7	0.3053	3089.2
	-0.2	0.15	0.3804	115.3	0.3809	514.6	0.3823(0.0015)	21847.3	0.3805	3107.1
$\rho = 0.2$	0.2	-0.15	0.3583	115.3	0.3676	512.2	0.3579(0.0016)	21344.4	0.3572	3087.4
	-0.2	-0.15	0.4480	115.1	0.4483	515.8	0.4486(0.0016)	21397.7	0.4479	3092.9
	0.2	0.15	0.3489	112.5	0.3498	515.5	0.3485(0.0015)	21381.4	0.3496	3056.1
	-0.2	0.15	0.4243	112.2	0.4246	513.7	0.4238(0.0015)	21592.5	0.4239	3065.5
$\rho = 0.5$	0.2	-0.15	0.3978	112.0	0.3958	515.4	0.3966(0.0016)	21626.3	0.3999	3062.0
	-0.2	-0.15	0.4860	112.0	0.4875	512.8	0.4870(0.0015)	20387.5	0.4885	3060.4
	0.2	0.15	0.4055	116.1	0.4054	520.5	0.4041(0.0016)	21580.6	0.4036	3599.8
	-0.2	0.15	0.4789	115.0	0.4801	520.7	0.4796(0.0016)	21775.4	0.4783	3585.5
$\rho = 0.8$	0.2	-0.15	0.4409	117.6	0.4408	519.5	0.4408(0.0016)	20526.6	0.4405	3576.3
	-0.2	-0.15	0.5353	115.3	0.5362	527.3	0.5354(0.0016)	21487.6	0.5355	3597.8
	Low Volatilities: $\sigma_1 = \sigma_2 = 0.2$									
	μ_1	μ_2	L_1 (FT)	CPU time	L_2 (KL)	CPU time	MC (std)	CPU time	STZ	CPU time
	0.2	0.15	0.0059	115.9	0.0058	472.5	0.0061(0.0004)	18100.5	0.0057	3134.0
	-0.2	0.15	0.0288	115.0	0.0287	490.5	0.0285(0.0006)	17101.2	0.0283	3116.1
$\rho = 0.2$	0.2	-0.15	0.0195	114.9	0.0198	464.4	0.0192(0.0007)	17350.2	0.0191	3115.8
	-0.2	-0.15	0.0960	114.8	0.0954	490.4	0.0951(0.0005)	15830.4	0.0944	3122.3
	0.2	0.15	0.0122	127.8	0.0125	495.8	0.0122(0.0007)	17223.5	0.0119	3061.1
	-0.2	0.15	0.0462	127.6	0.0463	487.1	0.0460(0.0008)	16235.3	0.0450	3063.8
$\rho = 0.5$	0.2	-0.15	0.0258	124.5	0.0261	488.6	0.0255(0.0005)	16485.0	0.0250	3061.5
	-0.2	-0.15	0.1297	122.5	0.1298	480.6	0.1292(0.0006)	14997.9	0.1296	3081.1
	0.2	0.15	0.0224	122.6	0.0224	491.3	0.0226(0.0007)	15840.5	0.0216	3645.6
	-0.2	0.15	0.0679	122.7	0.0685	498.3	0.0679(0.0006)	14967.5	0.0670	3769.9
$\rho = 0.8$	0.2	-0.15	0.0330	123.1	0.0335	493.5	0.0334(0.0007)	15204.0	0.0338	3803.6
	-0.2	-0.15	0.1667	122.2	0.1668	503.9	0.1660(0.0008)	13601.6	0.1666	3588.6

our numerical results match those produced by the formula in [24], while our method (via either the finite Fourier transform or the Kontorovich-Lebedev transform) is much faster, partly because our formula computes the probability via the Laplace transform, thus avoiding the singularity of the density.

Furthermore, the steps that lead to the solution in [24] are not entirely rigorous. For example, the PDE problem involves the Dirac delta function in the initial condition and thus it is not immediately clear whether the equation has a unique solution via bounded martingale arguments. Even when there is a unique solution it is not clear whether that unique solution admits the required stochastic representation. The current paper attempts to deal with these issues rigorously. For example, after showing the uniqueness for uniformly bounded solutions, via the stochastic representation using a martingale argument, in Theorem 1, we put significant effort into proving the two solutions u_1 and u_2 in Theorem 2 are all uniformly bounded and are the same.

3.2. Application to Default Correlation

In [28], default correlation is defined as

$$\begin{aligned}
&Corr(1(\tau_1 \le t), 1(\tau_2 \le t)) \\
:=&\frac{E^{(x_1,x_2)}\left[1(\tau_1 \le t)1(\tau_2 \le t)\right] - E^{x_1}\left[1(\tau_1 \le t)\right]E^{x_2}\left[1(\tau_2 \le t)\right]}{\sqrt{Var(1(\tau_1 \le t)) \cdot Var(1(\tau_2 \le t))}} \\
=&\frac{P^{(x_1,x_2)}(\tau_1 \le t \ \ and \ \ \tau_2 \le t) - P^{x_1}(\tau_1 \le t)P^{x_2}(\tau_2 \le t)}{\sqrt{P^{x_1}(\tau_1 \le t)(1 - P^{x_1}(\tau_1 \le t))P^{x_2}(\tau_2 \le t)(1 - P^{x_2}(\tau_2 \le t))}}.
\end{aligned}$$

The above is computed in [28] in the special case of $\mu_1 = \mu_2 = 0$. In this section, we extend this study by checking whether the correlation will change when μ_1 and μ_2 are non-zero. Also, we consider default correlation for τ_1, τ_2 in different horizons, where $t_1 \neq t_2$ in the following:

$$\begin{aligned}
&Corr(1(\tau_1 \le t_1), 1(\tau_2 \le t_2)) \\
=&\frac{P^{(x_1,x_2)}(\tau_1 \le t_1 \ \ and \ \ \tau_2 \le t_2) - P^{x_1}(\tau_1 \le t_1)P^{x_2}(\tau_2 \le t_2)}{\sqrt{P^{x_1}(\tau_1 \le t_1)(1 - P^{x_1}(\tau_1 \le t_2))P^{x_2}(\tau_2 \le t_2)(1 - P^{x_2}(\tau_2 \le t_2))}}.
\end{aligned}$$

The results are summarized in Table 4. It appears that non-zero drifts can have a significant impact on default correlations.

Table 4: Default correlation $Corr(1(\tau_1 \leq t), 1(\tau_2 \leq t))$ (in percentage) with different combinations of drifts and maturities. Parameters used in Panal A are the same as in [28]. The numbers in bold are also computed in [28].

Panel A: $x_1 = x_2 = \ln(5)$, $\sigma_1 = \sigma_2 = 30\%$ and $\rho = 40\%$					
			$t_1 = 2$		
	$t_2 = 1$	$t_2 = 2$	$t_2 = 3$	$t_3 = 4$	$t_2 = 5$
$\mu_1 = \mu_2 = 0$	0.0198%	1.1723 %	1.3466%	1.2564%	1.0924%
$\mu_1 = \mu_2 = 30\%$	0.0012%	0.0683%	0.0643%	0.0548%	0.0482%
$\mu_1 = \mu_2 = -30\%$	0.1699%	8.4898%	9.2264%	7.1381%	4.4529%
			$t_1 = 4$		
	$t_2 = 1$	$t_2 = 2$	$t_2 = 3$	$t_3 = 4$	$t_2 = 5$
$\mu_1 = \mu_2 = 0$	0.0026%	1.2564%	4.0316%	**6.5837%**	7.0248%
$\mu_1 = \mu_2 = 30\%$	0.0003%	0.0548%	0.1783%	0.2715%	0.2748%
$\mu_1 = \mu_2 = -30\%$	-0.1495%	7.1381%	18.2286%	24.6885%	22.9000%
			$t_1 = 5$		
	$t_2 = 1$	$t_2 = 2$	$t_2 = 3$	$t_3 = 4$	$t_2 = 5$
$\mu_1 = \mu_2 = 0$	0.0001%	1.0924%	3.9462%	7.0248%	**9.1974%**
$\mu_1 = \mu_2 = 30\%$	0.0003%	0.0482%	0.1643%	0.2748%	0.3328%
$\mu_1 = \mu_2 = -30\%$	-0.2713%	4.4529%	15.8324%	22.9000%	25.8568%
Panel B: $x_1 = x_2 = \ln(1.2), \sigma_1 = \sigma_2 = 55\%$ and $\rho = 50\%$					
			$t_1 = 2$		
	$t_2 = 1$	$t_2 = 2$	$t_2 = 3$	$t_3 = 4$	$t_2 = 5$
$\mu_1 = \mu_2 = 0$	24.8%	25.5%	24.0%	22.7%	21.7%
$\mu_1 = \mu_2 = 30\%$	28.2%	28.6%	28.1%	27.7%	27.5%
$\mu_1 = \mu_2 = -30\%$	19.0%	19.8%	16.7%	14.3%	12.6%
			$t_1 = 4$		
	$t_2 = 1$	$t_2 = 2$	$t_2 = 3$	$t_3 = 4$	$t_2 = 5$
$\mu_1 = \mu_2 = 0$	20.8%	22.7%	23.2%	22.9%	22.3%
$\mu_1 = \mu_2 = 30\%$	26.8%	27.7%	27.9%	27.8%	27.7%
$\mu_1 = \mu_2 = -30\%$	11.9%	14.3%	14.9%	14.3%	12.8%
			$t_1 = 5$		
	$t_2 = 1$	$t_2 = 2$	$t_2 = 3$	$t_3 = 4$	$t_2 = 5$
$\mu_1 = \mu_2 = 0$	19.7%	21.7%	22.1%	22.3%	22.1%
$\mu_1 = \mu_2 = 30\%$	26.5%	27.5%	27.7%	27.7%	27.6%
$\mu_1 = \mu_2 = -30\%$	10.0%	12.6%	12.5%	12.8%	12.5%

4. The Density of $|\tau_2 - \tau_1|$

Suppose $s \geq M$ such that (2.18) holds. Then $q \downarrow 0$ yields the Laplace transform of $|\tau_1 - \tau_2|$ with $\tau_1, \tau_2 < \infty$:

$$
\begin{aligned}
&E^{(x_1,x_2)}(e^{-s|\tau_2-\tau_1|}1(\tau_1,\tau_2<\infty)) \\
=&e^{-(\gamma_1\cos\theta+\gamma_2\sin\theta)r}\left(\sum_{n=1}^{\infty}\sqrt{\frac{2}{\alpha}}\sin(\nu_n\theta)V_n(r)\right)+\exp(-D_1(0,s)x_1-D_2(0,s)x_2),
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{aligned}
V_n(r)=&(2s+\rho\sigma_1\sigma_2 D_1(0,s)D_2(0,s))\times \\
&\int_{\eta=0}^{\alpha}\sqrt{\frac{2}{\alpha}}\sin(\nu_n\eta)\left[K_{\nu_n}(a(0)r)\int_{l=0}^{r}\exp(-G(\eta)l)lI_{\nu_n}(a(0)l)dl\right. \\
&\left.+I_{\nu_n}(a(0)r)\int_{l=r}^{\infty}\exp(-G(\eta)l)lK_{\nu_n}(a(0)l)dl\right]d\eta,
\end{aligned}
$$

and all other parameters and functions are defined in Definition 1, (2.5), and (2.6). When $\mu_1 = \mu_2 = 0$, $a(0) = 0$ and the modified Helmholtz equation degenerates to a Helmholtz equation; as a result, in this case the expression of $V_n(r)$ takes a similar form but no special function is involved:

$$
\begin{aligned}
V_n(r)=&\frac{2s+2\rho s}{\nu_n}\int_{\eta=0}^{\alpha}\sqrt{\frac{2}{\alpha}}\sin(\nu_n\eta)\times \\
&\left[\int_{l=0}^{r}\exp(-G(\eta)l)l\left(\frac{l}{r}\right)^{\nu_n}dl+\int_{l=r}^{\infty}\exp(-G(\eta)l)l\left(\frac{r}{l}\right)^{\nu_n}dl\right]d\eta.
\end{aligned}
$$

Numerical inversion of (4.1) yields the distribution function $P^{(x_1,x_2)}(|\tau_1 - \tau_2| \leq t, \tau_1, \tau_2 < \infty)$ and the density functions $f_{|\tau_1-\tau_2|}(t)dt := P^{(x_1,x_2)}(|\tau_1 - \tau_2| \in dt, \tau_1, \tau_2 < \infty)$; see Figure 1.

Theorem 3. *For arbitrary drifts μ_1 and μ_2, (1) $P^{(x_1,x_2)}(\tau_1 = \tau_2, \tau_1, \tau_2 < \infty) = 0$ for any $\rho \in (-1, 1)$; (2) $f_{|\tau_1-\tau_2|}(0+) = \infty$ if and only if $\rho > 0$. Thus, near zero, the distribution function tends to zero and the density function tends to infinity.*

Proof. See the online supplement.

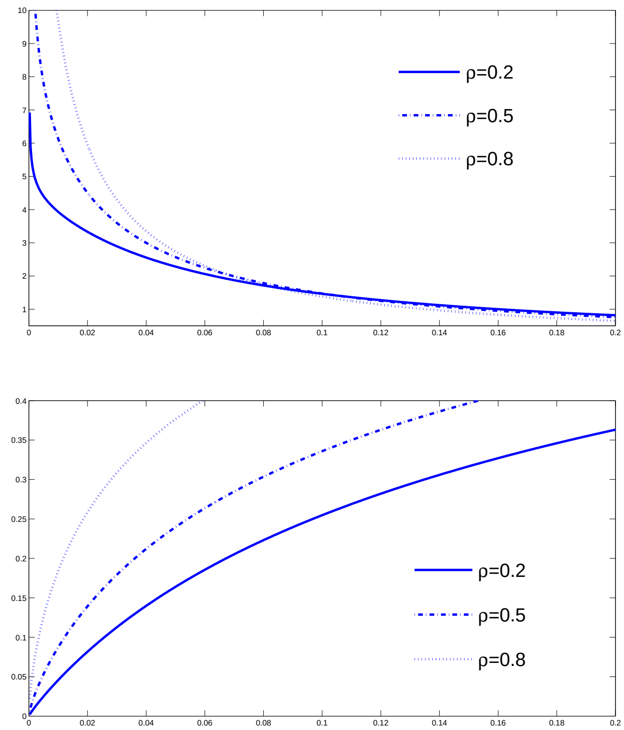

FIGURE 1: The distribution function (top) and density (bottom) of $|\tau_1 - \tau_2|$ when $\rho = 0.2, 0.5$ and 0.8. We perform numerical Laplace inversion on (4.1). The correlated zero-drifted Brownian motion starts at $x_1 = x_2 = \log(1.2) = 0.1823$. It can be seen that numerically all three distribution functions tend to zero as $t \to 0$, while all three density functions seem to tend to infinity as $t \to 0$. This is proved rigorously in Theorem 3.

5. A Class of Bivariate Exponential Distributions

In this section we point out that special cases of solutions in Section 2.5 lead to a class of bivariate exponential distributions. To obtain some insight, we first review the first passage time of Brownian motion problem in the one-dimensional case. Let ε_p be an exponential random variable with rate p, independent of the Brownian motion $X(t)$. Further let $J(t) = \min_{0 \le s \le t} X(s)$. Then for $x > 0$ the Laplace transform of the one-dimensional first passage time is

$$\begin{aligned}E^x(e^{-p\tau}) &= \int_0^\infty e^{-pt} dP^x(\tau \le t) \\ &= \left([e^{-pt} P^x(\tau \le t)]|_{t=0}^\infty + \int_0^\infty p e^{-pt} P^x(\tau \le t) dt \right) \\ &= \int_0^\infty p e^{-pt} P^x(-J(t) \ge 0) dt \\ &= \int_0^\infty p e^{-pt} P^0(-J(t) \ge x) dt = P^0(-J(\varepsilon_p) \ge x).\end{aligned}$$

That is, using the standard result of one-dimensional first-passage time problem, we are able to show that

$$P^0(-J(\varepsilon_p) \ge x) = E^x(e^{-p\tau}) = \exp\left(-\frac{1}{\sigma^2}(\sqrt{\mu^2 + 2\sigma^2 p} + \mu)x \right), \quad x > 0.$$

The above implies that starting from 0, $-J(\varepsilon_p)$ is exponentially distributed. Thus, the two-dimensional minimums lead to bivariate exponential distributions (BVE).

5.1. Two Joint Probabilities

There are various definitions of BVE's, depending on features that are desired. The following three features receive most attention:

(a) marginal distributions are exponential;

(b) absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^2$;

(c) the memoryless property holds, namely, for any $x_0, x, y_0, y > 0$,

$$P(\mathcal{E}_1 > x_0 + x, \mathcal{E}_2 > y_0 + y) = P(\mathcal{E}_1 > x, \mathcal{E}_2 > y) P(\mathcal{E}_1 > x_0, \mathcal{E}_2 > y_0).$$

For any BVE with all (a), (b) and (c), the two exponential random variables must be independent ([19]). For more discussion of BVEs, see, e.g. [7], [8], [11], and [19].

Example 1. For $x_1, x_2 > 0$ and independent exponential random variables $\varepsilon_{p_1}, \varepsilon_{p_2}$ independent of the Brownian motions,

$$\begin{aligned}&L(x_1, x_2) = E^{(x_1,x_2)}(e^{-p_1\tau_1 - p_2\tau_2}) \\ =& \int_0^\infty \int_0^\infty p_1 p_2 e^{-p_1 t_1 - p_2 t_2} P^{(x_1,x_2)}(\tau_1 \le t_1, \tau_2 \le t_2) dt_1 dt_2 \\ =& \int_0^\infty \int_0^\infty p_1 p_2 e^{-p_1 t_1 - p_2 t_2} P^{(0,0)}(-J_1(t_1) \ge x_1, -J_2(t_2) \ge x_2) dt_1 dt_2 \\ =& P^{(0,0)}(-J_1(\varepsilon_{p_1}) \ge x_1, -J_2(\varepsilon_{p_2}) \ge x_2), \qquad (5.1)\end{aligned}$$

which can be computed via (2.17).

Since for each $i = 1, 2$, $-J_i(\varepsilon_{p_i})$ is an exponential random variable, the last term in (5.1) is then the joint survival function of some BVE. Clearly, $(-J_1(\varepsilon_{p_1}), -J_2(\varepsilon_{p_2}))$ satisfies (a). Moreover, (b) is also satisfied, since $L(x_1, x_2)$ is by definition twice differentiable and hence continuous in (x_1, x_2). Thus, $(-J_1(\varepsilon_{p_1}), -J_2(\varepsilon_{p_2}))$ does not have memoryless property of (c), unless $\rho = 0$ and $J_1(\epsilon_{p_1})$ is independent of $J_2(\epsilon_{p_2})$.

Example 2. Stopping at the same exponential variable ε_p independent of the Brownian motions, $(-J_1(\varepsilon_p), -J_2(\varepsilon_p))$ is also a BVE. From (2.21), the joint survival function of $(-J_1(\varepsilon_p), -J_2(\varepsilon_p))$ is given by

$$\begin{aligned} E^{(x_1,x_2)}(e^{-p\tilde{\tau}}) &= \int_0^\infty pe^{-pt} P^{(x_1,x_2)}(\tau_1 \le t, \tau_2 \le t)dt \\ &= \int_0^\infty pe^{-pt} P^{(0,0)}(-J_1(t) \ge x_1, -J_2(t) \ge x_2)dt \\ &= P^{(0,0)}(-J_1(\varepsilon_p) \ge x_1, -J_2(\varepsilon_p) \ge x_2), \end{aligned} \tag{5.2}$$

which can be computed via (2.19).

Remark 3. Even when $\rho = 0$, $-J_1(\varepsilon_p)$ is not independent of $-J_2(\varepsilon_p)$, because the same ε_p is shared. Hence the joint distribution of $(-J_1(\varepsilon_p), -J_2(\varepsilon_p))$ is different from that of $(-J_1(\varepsilon_{p_1}), -J_2(\varepsilon_{p_2}))$.

5.2. Two Joint Moments

A closed form expression for $E^{(0,0)}\ [(-J_1(\varepsilon_p))(-J_2(\varepsilon_p))]$ is given in [23] when the underlying correlated Brownian motions have zero drifts. In this subsection, their approach is first generalized to the case when $(X_1(t), X_2(t))$ has arbitrary drifts; see the first part of Theorem 4. Then we provide a characterization of $E^{(0,0)}[(-J_1(\varepsilon_{p_1}))(-J_2(\varepsilon_{p_2}))]$ in the case of drifted Brownian motion, where p_1 and p_2 may not be the same; see the second part of Theorem 4.

We first introduce some notations. Let $\Gamma(\cdot)$ be the gamma function, $P_\nu^{-\mu}(\cdot)$ the Legendre function (for definition of Legendre function, see, for example, Chapter 8 of

[22]), and

$$b(\eta, c) := \frac{\gamma_1 \cos(\eta) + \gamma_2 \sin(\eta)}{a(c)}$$

$$Q(\eta, c) := \begin{cases} \log(b(\eta, c) + \sqrt{b(\eta, c)^2 - 1}), & \text{when } b(\eta, c) \in [1, \infty); \\ i \cdot \arccos(b(\eta, c)), & \text{when } b(\eta, c) \in [-1, 1). \end{cases}$$

Since $a(c) = \sqrt{2c + \gamma_1^2 + \gamma_2^2} \uparrow \infty$ and $b(\eta, c) \to 0$ as $c \uparrow \infty$. Hence there exists a constant $N > 0$, such that when $c \geq N$, $b(\eta, c) > -1$ uniformly for any $\eta \in [0, \alpha]$ and $Q(\eta, c)$ is well defined. Note that by this definition, for $b(\eta, c) > -1$,

$$\gamma_1 \cos \eta + \gamma_2 \sin \eta = \cosh(Q(\eta, c)) \cdot a(c). \tag{5.3}$$

Moreover, it can be seen from the definition that $Q(\eta, c)$ is continuous. Given the notations above, we are now in position to introduce Theorem 4.

Theorem 4. *(1) When* $p \geq \max(M, N)$ *(where* M *is defined in Condition 2) such that* $b(\eta, c) > -1$ *holds and* $Q(\eta, p)$ *is well defined, the joint moment of* $-J_1(\varepsilon_p)$ *and* $-J_2(\varepsilon_p)$ *is given by the following double integral*

$$\begin{aligned} &E^{(0,0)}[(-J_1(\varepsilon_p))(-J_2(\varepsilon_p))] \\ =&\sqrt{\frac{2}{\pi}} \frac{\sigma_1 \sigma_2 \sin \alpha}{(a(p))^2} \left(\int_{\eta=0}^{\alpha} \int_{\nu=0}^{\infty} \frac{\Gamma(2 - i\nu)\Gamma(2 + i\nu)}{\sinh(\alpha\nu)} \times \right. \\ &[\cosh(\beta_1(c)\nu) \sinh((\alpha - \eta)\nu) + \cosh(\beta_2(c)\nu) \sinh(\eta\nu)] \\ &\left. \frac{P_{i\nu - 1/2}^{-3/2}(\cosh(Q(\eta, p)))}{(\sinh Q(\eta, p))^{3/2}} d\eta d\nu \right), \end{aligned}$$

with $p_1 = p_2 = v = p/2$ *in* (2.15).

In particular, when $\mu_1 = \mu_2 = 0, \sigma_1 = \sigma_2 = 1$*, the expression reduces to a result in [23] (the second to last equation therein).*

$$E^{(0,0)}[(-J_1(\varepsilon_p))(-J_2(\varepsilon_p))] = \frac{\sin \alpha}{p} \int_0^{\infty} \frac{\cosh((\frac{\pi}{2} - \alpha)\nu)}{\sinh(\nu\pi/2)} \tanh \frac{\alpha\nu}{2} d\nu.$$

(2) When $\min(p_1, p_2) \geq \max(M, N)$ *(where* M *is defined in Condition 2.2) such that* $b(\eta, c) > -1$ *holds and* $Q(\eta, p_1 + p_2)$ *is well defined, the joint moment of* $-J_1(\varepsilon_{p_1})$ *and*

$-J_2(\varepsilon_{p_2})$ *is given by the following double integral*

$$
\begin{aligned}
&E^{(0,0)}[(-J_1(\varepsilon_{p_1}))(-J_2(\varepsilon_{p_2}))] \\
=&\sqrt{\frac{2}{\pi}}\frac{\sin\alpha\sigma_1\sigma_2}{(a(p_1+p_2))^2}\left(\int_{\eta=0}^{\alpha}\int_{\nu=0}^{\infty}\frac{\Gamma(2-i\nu)\Gamma(2+i\nu)}{\sinh(\alpha\nu)}\times\right. \\
&[\cosh(\beta_1(c)\nu)\sinh((\alpha-\eta)\nu)+\cosh(\beta_2(c)\nu)\sinh(\eta\nu)]\times \\
&\left.\frac{P_{i\nu-1/2}^{-3/2}(\cosh(Q(\eta,p_1+p_2)))}{(\sinh Q(\eta,p_1+p_2))^{3/2}}d\eta d\nu\right),
\end{aligned}
$$

with $v=0$ *in* (2.15).

Proof. See the online supplement.

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FIRST PASSAGE TIMES OF TWO-DIMENSIONAL BROWNIAN MOTION: ONLINE SUPPLEMENT

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Appendix A. Proof of Lemma 1

Proof. Part (i) follows because

$$\left(\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)-\frac{1}{2}a^2\right)\exp(-G(\theta)r)=\frac{1}{2}A\exp(-G(\theta)r).$$

Now we shall prove part (ii). Suppose k is a solution to (2.8). It follows that k is a solution to

$$\frac{1}{2}\frac{\partial^2 k}{\partial z_1^2}+\frac{1}{2}\frac{\partial^2 k}{\partial z_2^2}=\left(c+\frac{1}{2}\gamma_1^2+\frac{1}{2}\gamma_2^2\right)k.$$

Consider u defined in (2.13). Changing of variables yields $u(z_1,z_2)=e^{-\gamma_1 z_1-\gamma_2 z_2}$ $k(z_1,z_2)$, from which we have

$$\frac{1}{2}\frac{\partial^2 u}{\partial z_1^2}+\frac{1}{2}\frac{\partial^2 u}{\partial z_2^2}+\gamma_1\frac{\partial u}{\partial z_1}+\gamma_2\frac{\partial u}{\partial z_2}=cu.$$

Changing of the variables again further leads to

$$\frac{1}{2}\sigma_1^2\frac{\partial^2 u}{\partial x_1^2}+\rho\sigma_1\sigma_2\frac{\partial^2 u}{\partial x_1\partial x_2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 u}{\partial x_1^2}+\mu_1\frac{\partial u}{\partial x_1}+\mu_2\frac{\partial u}{\partial x_2}=cu,$$

which is (2.1) in the main text. Next, we shall check the boundary conditions. Note that

$$r\sin(\alpha-\theta)=r\sin(\alpha)\cos(\theta)-r\cos(\alpha)\sin(\theta)=\sqrt{1-\rho^2}z_1+\rho z_2=\frac{x_1}{\sigma_1},$$

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from which we have

$$x_1 = \sigma_1 r \sin(\alpha - \theta), \quad x_2 = \sigma_2 z_2 = \sigma_2 r \sin(\theta).$$

Thus,

$$e^{-(\gamma_1 \cos\theta + \gamma_2 \sin\theta) r} \exp(-G(\theta) r) = \exp(-D_1 x_1 - D_2 x_2). \tag{A.1}$$

Hence, the boundary conditions (2.9) and (2.2) are the same, thanks to (A.1) and the fact that

$$x_2 = 0 \Leftrightarrow \theta = 0, \quad x_1 = 0 \Leftrightarrow \theta = \alpha.$$

Finally, (2.12) and (2.13) imply (2.14), due also to (A.1). The proof is thus completed.

Appendix B. Proof of Theorem 1

Proof. Suppose U is a solution to the aforementioned PDE Problem. Consider stopping times $\tau_*^{(n)} := \inf_{t\ge 0}\{X_1(t) \le \frac{1}{n} \text{ or } X_2(t) \le \frac{1}{n}\}$. Then $\lim_{n\to\infty} \tau_*^{(n)} = \tau^*$. Consider $M_t^{(n)} := \exp[-(p_1+p_2)(t \wedge \tau_*^{(n)})] U\left(\vec{X}(t \wedge \tau_*^{(n)})\right)$, where $\vec{X} = (X_1, X_2)$. It follows from Ito's formula that

$$\begin{aligned} d(M_t^{(n)}) &= e^{-(p_1+p_2)(t\wedge\tau_*^{(n)})} \Bigg[\Bigg(\frac{1}{2}\sigma_1^2 \frac{\partial^2 U}{\partial x_1^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 U}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 U}{\partial x_2^2} \\ &\quad + \mu_1 \frac{\partial U}{\partial x_1} + \mu_2 \frac{\partial U}{\partial x_2} - (p_1+p_2) U \Bigg) dt \\ &\quad + \sigma_1 \frac{\partial U}{\partial x_1} dW_1 + \sigma_2 \frac{\partial U}{\partial x_2} dW_2 \Bigg] \\ &= \exp[-(p_1+p_2)(t\wedge\tau_*^{(n)})] \left(\sigma_1 \frac{\partial U}{\partial x_1} dW_1 + \sigma_2 \frac{\partial U}{\partial x_2} dW_2 \right), \end{aligned}$$

where the last equality comes from the fact that U solves the PDE (2.1) in the main text and $c = p_1 + p_2$. It follows that $M_t^{(n)}$ is a local martingale. Since $|U| \le C$, as $p_1 + p_2 > 0$, we further have $|M_t^{(n)}| \le C$. Thus $M_t^{(n)}$ is a martingale.

Letting $n \to \infty$ and using the bounded convergence theorem, we have $M_t := \exp[-(p_1+p_2)(t \wedge \tau^*)] U\left(\vec{X}(t\wedge\tau^*)\right)$ is a martingale. Again, since

M_t is uniformly bounded: $|M_t| \leq C$, the optimal sampling theorem yield that $E^{(x_1,x_2)}(M_0) = E^{(x_1,x_2)}(M_\infty)$, where

$$\begin{aligned}
&E^{(x_1,x_2)}(M_0) = E^{(x_1,x_2)}\left[U(\vec{X}(0))\right] = U(x_1,x_2),\\
&E^{(x_1,x_2)}(M_\infty) = E^{(x_1,x_2)}\left[e^{-(p_1+p_2)\tau^*}U(\vec{X}(\tau^*))1(\tau^*<\infty)\right] \quad (p_1+p_2>0)\\
=&E^{(x_1,x_2)}\left[e^{-(p_1+p_2)\tau^*}U(\vec{X}(\tau^*))\right]\\
=&E^{(x_1,x_2)}\left[e^{-(p_1+p_2)\tau_1}U(\vec{X}(\tau_1))1(\tau_1<\tau_2)\right]\\
&+E^{(x_1,x_2)}\left[e^{-(p_1+p_2)\tau_2}U(\vec{X}(\tau_2))1(\tau_1\geq\tau_2)\right].
\end{aligned} \tag{B.1}$$

It suffices to prove that the two parts in the expression above equals to

$$E^{(x_1,x_2)}(e^{-p_1\tau_1-p_2\tau_2-v|\tau_1-\tau_2|}).$$

Splitting in two cases, one has

$$\begin{aligned}
&E^{(x_1,x_2)}[e^{-p_1\tau_1-p_2\tau_2-v|\tau_1-\tau_2|}]\\
=&E^{(x_1,x_2)}[e^{-p_1\tau_1-p_2\tau_2-v|\tau_1-\tau_2|}1(\tau_1<\tau_2)]\\
&+E^{(x_1,x_2)}[e^{-p_1\tau_1-p_2\tau_2-v|\tau_1-\tau_2|}1(\tau_1\geq\tau_2)].
\end{aligned} \tag{B.2}$$

We shall prove the two components in (B.1) equal to those in (B.2). Indeed, by iterated expectations,

$$\begin{aligned}
&E^{(x_1,x_2)}[e^{-p_1\tau_1-p_2\tau_2-v|\tau_1-\tau_2|}1(\tau_1<\tau_2)]\\
=&E^{(x_1,x_2)}[e^{-(p_1+p_2)\tau_1}1(\tau_1<\tau_2)E^{(x_1,x_2)}[e^{-(p_2+v)(\tau_2-\tau_1)}|\tau_1,\tau_1<\tau_2,X_2(\tau_1)]]\\
=&E^{(x_1,x_2)}[e^{-(p_1+p_2)\tau_1}1(\tau_1<\tau_2)E^{X_2(\tau_1)}[e^{-(p_2+v)\tau_2}]],
\end{aligned}$$

where the last equality follows from strong Markov property of Brownian motions. From the classical results of FPT of one-dimensional Brownian motion, it follows that

$$\begin{aligned}
&E^{(x_1,x_2)}[e^{-(p_1+p_2)\tau_1}1(\tau_1<\tau_2)E^{X_2(\tau_1)}[e^{-(p_2+v)\tau_2}]]\\
=&E^{(x_1,x_2)}[e^{-(p_1+p_2)\tau_1}1(\tau_1<\tau_2)e^{-D_2X_2(\tau_1)}]\\
=&E^{(x_1,x_2)}[e^{-(p_1+p_2)\tau_1}1(\tau_1<\tau_2)U(\vec{X}(\tau_1))].
\end{aligned}$$

Similarly, it can also be proved that

$$\begin{aligned}&E^{(x_1,x_2)}[e^{-p_1\tau_1-p_2\tau_2-v|\tau_1-\tau_2|}1(\tau_1\geq\tau_2)]\\=&E^{(x_1,x_2)}[e^{-(p_1+p_2)\tau_2}1(\tau_1\geq\tau_2)U(\vec{X}(\tau_2))].\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}&U(x_1,x_2)=E^{(x_1,x_2)}(M_0)=E^{(x_1,x_2)}(M_\infty)\\=&E^{(x_1,x_2)}[e^{-p_1\tau_1-p_2\tau_2-v|\tau_1-\tau_2|}1(\tau_1<\tau_2)]\\&+E^{(x_1,x_2)}[e^{-p_1\tau_1-p_2\tau_2-v|\tau_1-\tau_2|}1(\tau_1\geq\tau_2)]\\=&E^{(x_1,x_2)}\left[e^{-p_1\tau_1-p_2\tau_2-v|\tau_2-\tau_1|}\right],\end{aligned}$$

as required.

Appendix C. Proof of Theorem 2

Lemma 2. *Condition 2 in the main text implies $G(\eta)>|\gamma_1|+|\gamma_2|\geq 0$, for any $\eta\in[0,\alpha]$, in particular, $G(0),G(\alpha)>0$.*

Proof. Since $\sin(\alpha-\eta),\sin\eta\geq 0$ and by definition $p_1+v,p_2+v\geq M$, we have

$$\begin{aligned}&D_1\sigma_1\sin(\alpha-\eta)+D_2\sigma_2\sin\eta\\=&\left(\sqrt{\frac{\mu_1^2}{\sigma_1^2}+2(p_1+v)}+\frac{\mu_1}{\sigma_1}\right)\sin(\alpha-\eta)+\left(\sqrt{\frac{\mu_2^2}{\sigma_2^2}+2(p_2+v)}+\frac{\mu_2}{\sigma_2}\right)\sin\eta\\\geq&\left(\sqrt{\frac{\mu_1^2}{\sigma_1^2}+2M}+\frac{\mu_1}{\sigma_1}\right)\sin(\alpha-\eta)+\left(\sqrt{\frac{\mu_2^2}{\sigma_2^2}+2M}+\frac{\mu_2}{\sigma_2}\right)\sin\eta\\\geq&\left(\left|\frac{2(|\gamma_1|+|\gamma_2|)}{\sin\alpha}+1-\frac{\mu_1}{\sigma_1}\right|+\frac{\mu_1}{\sigma_1}\right)\sin(\alpha-\eta)\\&+\left(\left|\frac{|\gamma_1|+|\gamma_2|}{\sin\alpha}+1-\frac{\mu_2}{\sigma_2}\right|+\frac{\mu_2}{\sigma_2}\right)\sin\eta\\\geq&\left(\frac{2(|\gamma_1|+|\gamma_2|)}{\sin\alpha}+1\right)(\sin(\alpha-\eta)+\sin\eta)\\\geq&2(|\gamma_1|+|\gamma_2|)+(\sin(\alpha-\eta)+\sin\eta),\end{aligned}$$

because $\frac{\sin(\alpha-\eta)+\sin\eta}{\sin\alpha} \geq 1$ as $\frac{1-\cos(\eta)}{\sin(\eta)} = \tan(\eta/2)$ is an increasing function for $\eta \in [0, \alpha]$. Therefore,

$$\begin{aligned} G(\eta) \geq & -\gamma_1 \cos\eta - \gamma_2 \sin\eta + 2(|\gamma_1| + |\gamma_2|) + (\sin(\alpha-\eta) + \sin\eta) \\ \geq & |\gamma_1| + |\gamma_2| + \sin(\alpha-\eta) + \sin\eta > |\gamma_1| + |\gamma_2| \geq 0, \end{aligned}$$

as required.

Now we start the proof of Theorem 2.

Proof. The proof can be split into two parts. (I) Prove that $k(r,\theta)$ satisfies (2.8) and (2.9) and u_2 is uniformly bounded in the sense of (2.3) in the main text. Therefore, by the second part of Lemma 1, u_2 is the unique solution to (2.1), (2.2) and (2.3) in the main text. (II) Prove that $u_1 = u_2$, so that u_1 also represents the unique solution to (2.1), (2.2) and (2.3) in the main text.

Start with Part (I). Note that by separation of variable, for arbitrary constants C_1, C_2, $K_{iv}(ar)(C_1\cosh(\nu\theta) + C_2\sinh(\nu\theta))$ is a solution to the partial differential equation (2.8). Hence by Principle of Superposition for integrals (see, for example, p. 439 of [12]), $k(r,\theta)$ is also a solution to (2.8). To check boundary conditions (2.9) for $k(r,\theta)$, note that

$$\begin{aligned} k(r,\theta)|_{\theta=0} &= \frac{2}{\pi}\int_0^\infty \cosh(\beta_1\nu)K_{i\nu}(ar)d\nu = \exp(-ar\cos\beta_1) = \exp(-G(0)r), \\ k(r,\theta)|_{\theta=\alpha} &= \frac{2}{\pi}\int_0^\infty \cosh(\beta_2\nu)K_{i\nu}(ar)d\nu = \exp(-ar\cos\beta_2) = \exp(-G(\alpha)r). \end{aligned} \tag{C.1}$$

In both of the second equalities of the two lines above, we have appealed to the identities of (inverse) Kontorovich-Lebedev transform from [21] (pp.244) (note that $G(0), G(\alpha) > 0$ when Condition 2 in the main text holds, as proven in Lemma 2) and the last equalities are due to definition of β_1 and β_2. The proof that u_2 is uniformly bounded can be found in Lemma 3 below, which completes the proof of Part (I).

We proceed to prove Part (II). Let $h(r,\theta) := k(r,\theta) - \exp(-G(\theta)r)$. By part (i) of Lemma 1, h satisfies (2.10) and (2.11). The finite Fourier transform of h, $W_n(r) := \int_0^\alpha \sqrt{\frac{2}{\alpha}} \sin(\nu_n \eta) h(r,\eta) d\eta$, is then a solution to the following (nonhomogeneous modified Bessel) ODE:

$$\frac{1}{2}\left(\frac{d^2 W_n}{dr^2} + \frac{1}{r}\frac{dW_n}{dr} - \frac{\nu_n^2}{r^2} W_n - \frac{\nu_n}{r^2}\sqrt{\frac{2}{\alpha}}(h(r,\alpha) - h(r,0))\right)$$
$$= \frac{1}{2} a^2 W_n - \frac{A}{2}\int_0^\alpha \sqrt{\frac{2}{\alpha}} \sin(\nu_n \eta) \exp(-G(\eta) r) d\eta.$$

Here the exchange of derivatives with respect to r and the integral in W_n is justified by Leibniz's rule, since h is a continuous function. By (2.11), $h(r,\alpha) = h(r,0) = 0$ and the ODE above becomes

$$\frac{d^2 W_n}{dr^2} + \frac{1}{r}\frac{dW_n}{dr} - \frac{\nu_n^2}{r^2} W_n = a^2 W_n - A\int_0^\alpha \sqrt{\frac{2}{\alpha}} \sin(\nu_n \eta) \exp(-G(\eta) r) d\eta. \quad \text{(C.2)}$$

By Lemma 2, $G(\eta) > 0$ for any $\eta \in [0,\alpha]$. In addition, by part (ii) in Lemma 3, $k(r,\theta)$ is uniformly bounded for any $r > 0$ and $\theta \in [0,\alpha]$. It then follows from definition that $h(r,\theta) = k(r,\theta) - e^{-G(\theta)r}$ is also uniformly bounded for any $r > 0$ and $\theta \in [0,\alpha]$. Hence by definition, W_n satisfies the following boundary conditions

$$W_n(r) < \infty, \text{ as } r \to 0 \textit{ or } \infty \text{ for any fix } n \geq 1. \quad \text{(C.3)}$$

On the other hand, the standard Wronskian theory about the modified Bessel equation tells us that any solution of (C.2) is given by $c_1 K_\nu(ar) + c_2 I_\nu(ar) + U_n(r)$, where U_n is given in the statement of Theorem 2 and c_1 and c_2 are arbitrary constants; see Section 0.2.1-6 in [22]. Since $K_\nu(ar) \uparrow \infty$ as $r \downarrow 0$ and $I_\nu(ar) \uparrow \infty$ as $r \uparrow \infty$, boundary condition (C.3) prescribes that $c_1 = c_2 = 0$ and the unique solution is $U_n(r)$, thanks to part (i) in Lemma 3. Therefore, the uniqueness implies that $W_n(r) = U_n(r)$. Since $U_n(r)$ and $W_n(r)$ are both continuous functions in r, it follows from taking inverse finite Fourier transform that $h(r,\theta) = \sum_{n=1}^\infty \sqrt{\frac{2}{\alpha}} \sin(\nu_n \theta) U_n(r)$. Thus, the proof is completed by part (ii) of Lemma 1.

The proof of the above theorem requires the following lemma.

Lemma 3. *Suppose Condition 2 in the main text holds. (i) For any fixed n, $\lim_{r\uparrow\infty} U_n(r) < \infty$, $\lim_{r\downarrow 0} U_n(r) < \infty$. (ii) $|u_2(x_1, x_2)| \leq C$ and $|k(r,\theta)| \leq C$ for some $C > 1$, for any $x_1, x_2 > 0$ and equivalently any $r > 0$ and $\theta \in (0, \alpha)$, where u_2 and k are defined in Theorem 2.*

Proof. (i) To prove that $U_n(r)$ satisfies boundary condition (C.3), we need the following results in 9.7.1 and 9.7.2 in [2]: For any fixed constant $\nu > 0$, as $r \to \infty$ $K_\nu(ar) \sim \sqrt{1/ar}\exp(-ar)$ and $I_\nu(ar) \sim \sqrt{1/ar}\exp(ar)$; (c) when $\nu > 0$ is fixed and $r \to 0$, $I_\nu(ar) \sim (\frac{1}{2}ar)^\nu/\Gamma(\nu+1)$ and $K_\nu(ar) \sim \Gamma(\nu)/(\frac{1}{2}ar)^\nu$. Here "$A \sim B$" means $\lim(A/B) = 1$. In what follows, we shall denote ν_n as ν for the sake of notation simplification, since n is fixed. We now prove $U_n(r)$ satisfies boundary condition (C.3) in two steps.

(1) For any fixed n, we show that $\lim_{r\uparrow\infty} U_n(r) < \infty$, by proving a stronger result: $\lim_{r\uparrow\infty} U_n(r) = 0$. Indeed, there exist constants C_ν and r_ν (both depending on ν) such that $K_\nu(ar) \leq C_\nu\sqrt{1/ar}\exp(-ar)$ and $I_\nu(ar) \leq C_\nu\sqrt{1/ar}\exp(ar)$ for any $r \geq r_\nu$. When r is sufficiently large, let

$$
\begin{aligned}
C_1(r) &= K_\nu(ar)\int_{l=0}^{r_\nu} \exp(-G(\eta^*)l)lI_\nu(al)dl \\
C_2(r) &= K_\nu(ar)\int_{l=r_\nu}^{r} \exp(-G(\eta^*)l)lI_\nu(al)dl \\
C_3(r) &= I_\nu(ar)\int_{l=r}^{\infty} \exp(-G(\eta^*)l)lK_\nu(al)dl,
\end{aligned}
$$

where $\eta^* \in [0, \alpha]$ is chosen such that $0 < G(\eta^*) \leq G(\eta)$ and $\exp(-G(\eta)l) \leq \exp(-G(\eta^*)l)$ for any $\eta \in [0, \alpha]$. By definition, $|U_n(r)| \leq \frac{A}{2}\int_0^\alpha \sqrt{\frac{2}{\alpha}}(C_1(r) + C_2(r) + C_3(r))d\eta = \frac{A}{2}\sqrt{2\alpha}(C_1(r) + C_2(r) + C_3(r))$. Hence it suffices to prove that $C_1(r), C_2(r), C_3(r) \to 0$ as $r \uparrow \infty$. For C_1, $C_1(r) \leq (C_\nu/\sqrt{ar})\exp(-ar)\cdot \left[\int_0^{r_\nu} \exp(-G(\eta^*)l)lI_\nu(al)dl\right]$. Since the term in the bracket does not rely on r, the above tends to 0 as $r \uparrow \infty$. For C_2,

$$\begin{aligned} C_2(r) \leq & (C_\nu^2/a\sqrt{r}) \exp(-ar) \int_{l=r_\nu}^{r} \exp(-(G(\eta^*) - a)l)\sqrt{l}dl \\ \leq & (C_\nu^2/a) \exp(-ar) \int_{l=r_\nu}^{r} \exp(-(G(\eta^*) - a)l)dl \\ \leq & (C_\nu^2/a) \exp(-ar) \int_{l=r_\nu}^{r} 1dl \\ = & (C_\nu^2/a) \exp(-ar)(r - r_\nu). \end{aligned}$$

Note that the above tends to 0 as $r \uparrow \infty$ since r_ν is fixed and $a > 0$. As for C_3, since $I_\nu(ar) \leq I_\nu(al)$ for any $r \leq l$, it follows that

$$C_3(r) \leq \int_{l=r}^{\infty} \exp(-G(\eta^*)l)lI_\nu(al)K_\nu(al)dl \leq (C_\nu^2/a) \int_{l=r}^{\infty} \exp(-G(\eta^*)l)dl.$$

Since $\exp(-G(\eta^*)l)$ is integrable for $l > r$, the integral above tends to 0 as $r \uparrow \infty$, i.e. $C_3(r) \to 0$ as $r \uparrow \infty$.

(2) For any fixed n, we shall prove that $\lim_{r \downarrow 0} U_n(r) < \infty$. Indeed, there exist constants $\bar{C}_\nu$ and $\bar{r}_\nu$ (both depending on ν) such that $K_\nu(ar) \leq \bar{C}_\nu/(\frac{1}{2}ar)^\nu$ and $I_\nu(ar) \leq \bar{C}_\nu(\frac{1}{2}ar)^\nu$ for any $r \leq \bar{r}_\nu$. When r is sufficiently small, let

$$\begin{aligned} \bar{C}_1(r) &= K_\nu(ar) \int_{l=0}^{r} \exp(-G(\eta^*)l)lI_\nu(al)dl \\ \bar{C}_2(r) &= I_\nu(ar) \int_{l=r}^{\bar{r}_\nu} \exp(-G(\eta^*)l)lK_\nu(al)dl \\ \bar{C}_3(r) &= I_\nu(ar) \int_{l=\bar{r}_\nu}^{\infty} \exp(-G(\eta^*)l)lK_\nu(al)dl. \end{aligned}$$

By definition, $|U_n(r)| \leq \frac{A}{2} \int_0^\alpha \sqrt{\frac{2}{\alpha}} (\bar{C}_1(r) + \bar{C}_2(r) + \bar{C}_3(r))d\eta = \frac{A}{2}\sqrt{2\alpha}(\bar{C}_1(r) + \bar{C}_2(r) + \bar{C}_3(r))$. Hence it suffices to prove that $\bar{C}_1(r), \bar{C}_2(r), \bar{C}_3(r) < \infty$ as $r \downarrow 0$. For $\bar{C}_1$, since $I_\nu(al) \leq I_\nu(ar)$ for $l \leq r$, it follows that

$$\bar{C}_1(r) \leq K_\nu(ar)I_\nu(ar) \int_{l=0}^{r} \exp(-G(\eta^*)l)ldl \leq \bar{C}_\nu^2 \int_{l=0}^{r} \exp(-G(\eta^*)l)ldl.$$

Since $\exp(-G(\eta^*)l)l$ is integrable in $[0, r]$ for any finite $r > 0$, the above is finite as $r \downarrow 0$. For $\bar{C}_2$, since $K_{\nu_n}(al) \leq K_{\nu_n}(ar)$ for $l \geq r$, it follows that

$$\bar{C}_2(r) \leq K_\nu(ar)I_\nu(ar) \int_{l=r}^{\bar{r}_\nu} \exp(-G(\eta^*)l)ldl \leq \bar{C}_\nu^2 \int_0^{\bar{r}_\nu} \exp(-G(\eta^*)l)ldl$$

Since the integral does not depend on r, the above is finite as $r \downarrow 0$. As for $\bar{C}_3$, since $I_\nu(ar) \downarrow 0$ as $r \downarrow 0$ and $\int_{l=\bar{r}_\nu}^{\infty} \exp(-G(\eta^*)l) l K_\nu(al) dl < \infty$ (as $K_\nu(al) \sim \sqrt{1/al}\exp(-al)$ when $l \uparrow \infty$), it follows that $\bar{C}_3(r) < \infty$ as $r \downarrow 0$.

(ii) It suffices to prove that $\exp(-(\gamma_1 \cos\theta + \gamma_2 \sin\theta)r) | \int_0^\infty K_{i\nu}(ar) \cosh(\beta_1 \nu) \cdot \frac{\sinh((\alpha-\theta)\nu)}{\sinh \alpha\nu} d\nu| \le C$ for any $r > m$ for some fixed $m > 0$ and any $\eta \in [0, \alpha]$, since all involving functions are continuous. The required conclusion for u_2 then follows from definition and symmetry, while the case for $k(r, \theta)$ is similar and hence omitted. We shall split in two cases.

(1) If $\frac{G(0)}{a} \in (0, 1]$, by definition $\beta_1 \in [0, \frac{\pi}{2})$ and $\cosh(\beta_1 \nu) \ge 0$. Therefore,

$$\begin{aligned}
&\left|\int_0^\infty K_{i\nu}(ar)\cosh(\beta_1\nu)\frac{\sinh((\alpha-\theta)\nu)}{\sinh\alpha\nu}d\nu\right| \\
=&\left|\int_0^{ar} K_{i\nu}(ar)\cosh(\beta_1\nu)\frac{\sinh((\alpha-\theta)\nu)}{\sinh\alpha\nu}d\nu\right. \\
&\left.+\int_{ar}^\infty K_{i\nu}(ar)\cosh(\beta_1\nu)\frac{\sinh((\alpha-\theta)\nu)}{\sinh\alpha\nu}d\nu\right| \\
\le&\int_0^{ar} K_{i\nu}(ar)\cosh(\beta_1\nu)d\nu + \int_{ar}^\infty |K_{i\nu}(ar)|\cosh(\beta_1\nu)d\nu,
\end{aligned}$$

where the inequality is due to the facts that $\frac{\sinh(\alpha-\theta)\nu}{\sinh\alpha\nu} \le 1$ when $\theta \in [0, \alpha]$ and $K_{iv}(ar) > 0$ when $\nu \le ar$ (see, for example, [9]). Furthermore,

$$\begin{aligned}
&\int_0^{ar} K_{i\nu}(ar)\cosh(\beta_1\nu)d\nu + \int_{ar}^\infty |K_{i\nu}(ar)|\cosh(\beta_1\nu)d\nu \\
=&\int_0^{\infty} K_{i\nu}(ar)\cosh(\beta_1\nu)d\nu - \int_{ar}^\infty K_{i\nu}(ar)\cosh(\beta_1\nu)d\nu \\
&+\int_{ar}^\infty |K_{i\nu}(ar)|\cosh(\beta_1\nu)d\nu \\
\le&\int_0^{\infty} K_{i\nu}(ar)\cosh(\beta_1\nu)d\nu + 2\int_{ar}^\infty |K_{i\nu}(ar)|\cosh(\beta_1\nu)d\nu.
\end{aligned}$$

By (C.1),

$$\int_0^\infty K_{i\nu}(ar)\cosh(\beta_1\nu)d\nu = \frac{2}{\pi}\exp(-G(0)r),$$

and by Lemma 2,

$$\exp(-(\gamma_1\cos\theta+\gamma_2\sin\theta)r)\cdot\exp(-G(0)r) \le \exp(-(\gamma_1\cos\theta+\gamma_2\sin\theta+|\gamma_1|+|\gamma_2|)r) \le 1.$$

Hence it remains to prove that

$$\exp(-(\gamma_1\cos\theta+\gamma_2\sin\theta)r)\int_{ar}^{\infty}|K_{i\nu}(ar)|\cosh(\beta_1\nu)d\nu$$

is uniformly bounded. Indeed,

$$\int_{ar}^{\infty}|K_{i\nu}(ar)|\cosh(\beta_1\nu)d\nu\le c\cdot(am)^{-\frac{1}{4}}\int_{ar}^{\infty}e^{-(\frac{\pi}{2}-\beta_1)\nu}d\nu=\frac{c\cdot(am)^{-\frac{1}{4}}}{\frac{\pi}{2}-\beta_1}e^{-(\frac{\pi}{2}-\beta_1)ar},$$

where the inequality is due to (5.5) of [9]: $|K_{i\nu}(ar)|\le c\cdot(ar)^{-\frac{1}{4}}e^{-\frac{\pi}{2}\nu}\le c\cdot(am)^{-\frac{1}{4}}e^{-\frac{\pi}{2}\nu}$ and the fact that $\cosh(\beta_1\nu)\le\exp(\beta_1\nu)$. By definition of β_1, $\frac{\pi}{2}-\beta_1=\arcsin\frac{G(0)}{a}\ge\frac{G(0)}{a}$ when $\frac{G(0)}{a}\in(0,1]$. Hence

$$\frac{c\cdot(am)^{-\frac{1}{4}}}{\frac{\pi}{2}-\beta_1}e^{-(\frac{\pi}{2}-\beta_1)ar}\le\frac{c\cdot(am)^{-\frac{1}{4}}}{\frac{\pi}{2}-\beta_1}e^{-G(0)r}$$

and $e^{-(\gamma_1\cos\theta+\gamma_2\sin\theta)r}\int_{ar}^{\infty}|K_{i\nu}(ar)|\cdot\cosh(\beta_1\nu)d\nu\le\frac{c\cdot(am)^{-\frac{1}{4}}}{\frac{\pi}{2}-\beta_1}e^{-(\gamma_1\cos\theta+\gamma_2\sin\theta+G(0))r}$ $\le\frac{c\cdot(am)^{-\frac{1}{4}}}{\frac{\pi}{2}-\beta_1}$, which is uniformly bounded.

(2) If $\frac{G(0)}{a}>1$, β_1 is pure imaginary and $|\cosh(\beta_1\nu)|=|\cos(-i\beta\nu)|\le1$. Following similar steps as in the previous case, we have $|\int_0^{\infty}K_{i\nu}(ar)\cosh(\beta_1\nu)\frac{\sinh((\alpha-\theta)\nu)}{\sinh\alpha\nu}d\nu|\le\int_0^{\infty}K_{i\nu}(ar)d\nu+2\int_{ar}^{\infty}|K_{i\nu}(ar)|d\nu$. On the one hand, by Equation 2 on Page 244 of [21], $\int_0^{\infty}K_{i\nu}(ar)d\nu=\frac{2}{\pi}\exp(-ar)$. On the other hand, again from (5.5) of [9] it follows that $\int_{ar}^{\infty}|K_{i\nu}(ar)|d\nu\le\frac{2c\cdot(am)^{-\frac{1}{4}}}{\pi}e^{-\frac{\pi}{2}ar}$. These two upper bounds complete the proof, since by Cauchy-Schwarz Inequality, $\frac{\pi}{2}a\ge a=\sqrt{2c+\gamma_1^2+\gamma_2^2}\ge\sqrt{\gamma_1^2+\gamma_2^2}\ge-(\gamma_1\cos\theta+\gamma_2\sin\theta)$ and $\exp(-(\gamma_1\cos\theta+\gamma_2\sin\theta+a)r)\le1$.

Appendix D. Proof of Theorem 3

Proof. We first provide a proof for Part (1) of Theorem 3. In what follows, let $P^{(x_1,x_2;\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)}$ denote the probability law when the correlated Brownian motion with drifts μ_1, μ_2, volatilities σ_1 and σ_2 and correlation ρ; we write $P_{(z_1,z_2)}$ when $\mu_1=\mu_2=0$, $\sigma_1=\sigma_2=1$ and $\rho=0$, i.e. when the underlying is the standard two dimensional Brownian motion. Expectations are denoted in the same manner. τ_1 and τ_2 are the first passage times to hit the boundaries $x_2 = 0$ and $x_1 = 0$ under $P^{(x_1,x_2;\mu_1,\mu_2,\sigma_1,\sigma_2)}$. Note that from the change of variables in Lemma 1, it follows that τ_1 and τ_2 under $P^{(x_1,x_2;\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)}$ become hitting times to L_1 and L_2 under $P_{(z_1,z_2)}$, where $L_1 = \{(z_1, z_2) : z_2 = 0\}$, $L_2 = \{(z_1, z_2) : \sqrt{1-\rho^2}z_1 + \rho z_2 = 0\}$. Further let $\vec{Z} = (Z_1, Z_2)$ be an uncorrelated Brownian motion starting from $\vec{z}$.

By Girsanov's Theorem (see, for example, the last identity in Section 4.1 in [20]; note that by definition $1(|\tau_1 - \tau_2| \le t, \tau_1 < \infty, \tau_2 < \infty) = 0$ when $\tau_1 = \tau_2 = \infty$),

$$
\begin{aligned}
&P^{(x_1,x_2;\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)}(|\tau_1 - \tau_2| \le t, \tau_1 < \infty, \tau_2 < \infty) \\
=&E^{(x_1,x_2;\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)}[1(|\tau_1 - \tau_2| \le t, \tau_1 < \infty, \tau_2 < \infty)] \qquad \text{(D.1)} \\
=&\int_{t_1=0}^{\infty}\int_{\vec{y}\in R^2} e^{\vec{\gamma}'(\vec{y}-\vec{z})-(\gamma_1^2+\gamma_2^2)t_1/2} P_{(z_1,z_2)}(\vec{Z}(t_1) = d\vec{y}, \tau_1 = dt_1, \tau_2 - \tau_1 \in [0,t]) \\
+&\int_{t_2=0}^{\infty}\int_{\vec{y}\in R^2} e^{\vec{\gamma}'(\vec{y}-\vec{z})-(\gamma_1^2+\gamma_2^2)t_2/2} P_{(z_1,z_2)}(\vec{Z}(t_2) = d\vec{y}, \tau_2 = dt_2, \tau_1 - \tau_2 \in (0,t])
\end{aligned}
$$

Here $\vec{\gamma} = (\gamma_1, \gamma_2)$ and $\vec{z} = (z_1, z_2)$ are given as in Lemma 1 and its proof. Hence when $t = 0$,

$$
\begin{aligned}
&P^{(x_1,x_2;\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)}(\tau_1 = \tau_2, \tau_1 < \infty, \tau_2 < \infty) \\
=&\int_{t=0}^{\infty}\int_{\vec{y}\in R^2} e^{\vec{\gamma}'(\vec{y}-\vec{z})-(\gamma_1^2+\gamma_2^2)t/2} P_{(z_1,z_2)}(\vec{Z}(t) = d\vec{y}, \tau_1 = dt, \tau_2 = \tau_1) \\
=&e^{-\vec{\gamma}'\vec{z}} \int_{t=0}^{\infty} e^{-(\gamma_1^2+\gamma_2^2)t/2} P_{(z_1,z_2)}(\tau_1 = dt, \tau_2 = \tau_1) \\
\le&e^{-\vec{\gamma}'\vec{z}} \int_{t=0}^{\infty} P_{(z_1,z_2)}(\tau_1 = dt, \tau_2 = \tau_1) \\
=&e^{-\vec{\gamma}'\vec{z}} P_{(z_1,z_2)}(\tau_1 = \tau_2, \tau_1 < \infty, \tau_2 < \infty) \\
=&0 \;\; (\text{under } P_{(z_1,z_2)},\ \tau_1 \text{ is independent of } \tau_2, \text{ both of them continuous}),
\end{aligned}
$$

where the second equality follows from the fact that when $\tau_1 = \tau_2 = t$, the two-dimensional Brownian motion hits the corner of the wedge formed by L_1 and L_2 (the original point $(0, 0)$), which implies $\vec{Z}(t) = (0, 0)$. The proof for Part (1) is completed.

As for Part (2), let $g(t_1,t_2)$ be the joint density function of τ_1,τ_2 for a correlated Brownian motion starting from (x_1,x_2) with $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$. Then due to [20], $g(t_1,t_2)$ is given by

$$\begin{aligned} g(t_1,t_2) =& \frac{\pi \sin\alpha}{2\alpha^2\sqrt{t_1(t_2 - t_1\cos^2\alpha)}(t_2-t_1)} \\ &\times \exp\left(-\frac{r^2}{2t_1}\frac{t_2 - t_1\cos 2\alpha}{(t_2-t_1)+(t_2-t_1\cos 2\alpha)}\right) \\ &\times \sum_{n=1}^{\infty} n\sin\left(\frac{n\pi(\alpha-\theta)}{\alpha}\right) I_{n\pi/2\alpha}\left(\frac{r^2}{2t_1}\frac{t_2-t_1}{(t_2-t_1)+(t_2-t_1\cos 2\alpha)}\right), \end{aligned} \tag{D.2}$$

when $t_1 \le t_2$ and

$$\begin{aligned} g(t_1,t_2) =& \frac{\pi \sin\alpha}{2\alpha^2\sqrt{t_2(t_1 - t_2\cos^2\alpha)}(t_1-t_2)} \\ &\times \exp\left(-\frac{r^2}{2t_2}\frac{t_1 - t_2\cos 2\alpha}{(t_1-t_2)+(t_1-t_2\cos 2\alpha)}\right) \\ &\times \sum_{n=1}^{\infty} n\sin\left(\frac{n\pi\theta}{\alpha}\right) I_{n\pi/2\alpha}\left(\frac{r^2}{2t_2}\frac{t_1-t_2}{(t_1-t_2)+(t_1-t_2\cos 2\alpha)}\right), \end{aligned} \tag{D.3}$$

when $t_1 > t_2$. Here all other parameters are given in Lemma 1 and Theorem 2. We have the following property of $g(t_1,t_2)$, which is useful in proving Part (2) of Theorem 3.

Lemma 4. *Fix any $r, t_1, t_2 > 0$ and $\theta \in [0,\alpha]$. When $\rho \le 0$, $g(t,t) \le \frac{C}{t^{\pi/\alpha+1}}\exp(-\frac{r^2}{4t})$, where C is a constant not depending on t. When $\rho > 0$, $\lim_{t\downarrow 0} g(t_1,t_1+t) = \lim_{t\downarrow 0} g(t_2+t,t_2) = \infty$ for any $t_1,t_2 > 0$.*

Proof. For the case of $\rho > 0$, see the remark following (3.2) and (3.3) in [20].

We now focus on the case of $\rho \leq 0$. Fix any $t_2 > 0$. By (D.3)

$$\begin{aligned}
g(t_2+t,t_2) =& \frac{\pi \sin\alpha}{2\alpha^2\sqrt{t_2(t+t_2\sin^2\alpha)t}} \exp\left(-\frac{r^2}{2t_2}\frac{t+2t_2\sin^2\alpha}{2t+2t_2\sin^2\alpha}\right)\times \\
& \sum_{n=1}^{\infty} n\sin\left(\frac{n\pi\theta}{\alpha}\right) I_{n\pi/2\alpha}\left(\frac{r^2}{2t_2}\frac{t}{2t+2t_2\sin^2\alpha}\right) \\
\leq & \frac{\pi}{2\alpha^2 t_2}\exp\left(-\frac{r^2}{2t_2}\frac{t+2t_2\sin^2\alpha}{2t+2t_2\sin^2\alpha}\right)\times \\
& \left[\frac{1}{t}\sum_{n=1}^{\infty} nI_{n\pi/2\alpha}\left(\frac{r^2}{2t_2}\frac{t}{2t+2t_2\sin^2\alpha}\right)\right] \\
\leq & \frac{\pi}{2\alpha^2 t_2}\exp\left(-\frac{r^2}{2t_2}\cdot\frac{1}{2}\right)\left[\frac{1}{t}\sum_{n=1}^{\infty} nI_{n\pi/2\alpha}\left(\frac{r^2}{2t_2}\frac{t}{2t+2t_2\sin^2\alpha}\right)\right],
\end{aligned} \tag{D.4}$$

where the first inequality follows from $\sqrt{t_2(t+t_2\sin^2\alpha)} \geq t_2\sin\alpha > 0$ and $\sin\left(\frac{n\pi\theta}{\alpha}\right) \leq 1$, and the last inequality follows from the observation that $\frac{t+2t_2\sin^2\alpha}{2t+2t_2\sin^2\alpha} > \frac{1}{2}$ for any $t, t_2 > 0$. To study $\lim_{t\to 0} g(t_2+t,t_2)$, we need to obtain an upper bound of (D.4). Note that by 9.6.10 of [2], it holds that $I_\nu(z) = (\frac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\Gamma(\nu+k+1)}$, where Γ denotes the gamma function and ν is real. Therefore, it follows that when $z \leq 1$,

$$\frac{I_\nu(z)}{(\frac{1}{2}z)^\nu/\Gamma(\nu+1)} = \sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{k!}\frac{\Gamma(\nu+1)}{\Gamma(\nu+k+1)} \leq \sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{k!} = \exp\left(\frac{1}{4}z^2\right) \leq e^{\frac{1}{4}}. \tag{D.5}$$

For r, t_2, α fixed, there exists a $\delta > 0$ such that as long as $t < \delta$, we have

$$\frac{r^2}{2t_2}\frac{t}{2t+2t_2\sin^2\alpha} < \frac{r^2}{2t_2}\frac{t}{2t_2\sin^2\alpha} < 1. \tag{D.6}$$

Thus an upper bound of $g(t_2+t,t_2)$ for $t < \delta$ follows from (D.4) and (D.5) as

$$\begin{aligned}
g(t_2+t,t_2) &\leq \frac{\pi}{2\alpha^2 t_2}\exp\left(-\frac{r^2}{4t_2}\right)\left[\frac{1}{t}\sum_{n=1}^{\infty}\frac{ne^{\frac{1}{4}}}{\Gamma(\frac{n\pi}{2\alpha}+1)}\left(\frac{r^2}{4t_2}\frac{t}{2t+2t_2\sin^2\alpha}\right)^{\frac{n\pi}{2\alpha}}\right] \\
&\leq \frac{\pi}{2\alpha^2 t_2}\exp\left(-\frac{r^2}{4t_2}\right)\left[\frac{1}{t}\sum_{n=1}^{\infty}\frac{n}{\Gamma(\frac{n\pi}{2\alpha}+1)}e^{\frac{1}{4}}\left(\frac{r^2 t}{8t_2^2\sin^2\alpha}\right)^{\frac{n\pi}{2\alpha}}\right].
\end{aligned}$$

Since $\lim_{n\uparrow\infty} \frac{n}{\Gamma(\frac{n\pi}{2\alpha}+1)} = 0$, there exists a constant $m > 0$, such that $\frac{n}{\Gamma(\frac{n\pi}{2\alpha}+1)} \leq m$, for any $n \geq 1$. Hence the upper bound becomes

$$g(t_2+t, t_2) \leq \frac{\pi e^{\frac{1}{4}} m}{2\alpha^2 t_2} \exp\left(-\frac{r^2}{4t_2}\right) \left[\frac{1}{t} \sum_{n=1}^{\infty} \left(\frac{r^2 t}{8t_2^2 \sin^2\alpha}\right)^{\frac{n\pi}{2\alpha}}\right]$$

$$\leq \frac{\pi e^{\frac{1}{4}} m}{2\alpha^2 t_2} \exp\left(-\frac{r^2}{4t_2}\right) \frac{1}{t} \frac{\left(\frac{r^2 t}{8t_2^2 \sin^2\alpha}\right)^{\pi/2\alpha}}{1 - \left(\frac{r^2 t}{8t_2^2 \sin^2\alpha}\right)^{\pi/2\alpha}}$$

$$\leq \frac{\pi e^{\frac{1}{4}} m}{2\alpha^2 t_2} \exp\left(-\frac{r^2}{4t_2}\right) \frac{1}{t} \frac{\left(\frac{r^2 t}{8t_2^2 \sin^2\alpha}\right)^{\pi/2\alpha}}{1 - (1/2)^{\pi/2\alpha}},$$

where the last inequality follows from (D.6) that $\frac{r^2 t}{8t_2^2 \sin^2\alpha} \leq \frac{1}{2}$ for any $t < \delta$. Therefore, $g(t_2, t_2) \leq \left(\frac{\pi e^{\frac{1}{4}} m r^{\frac{\pi}{\alpha}}}{2\alpha^2 (8\sin^2\alpha)^{\pi/2\alpha}(1-(1/2)^{\pi/2\alpha})} \lim_{t\downarrow 0} t^{\frac{\pi}{2\alpha}-1}\right) \frac{\exp(-\frac{r^2}{4t_2})}{t_2^{\pi/\alpha+1}}$. When $\rho \leq 0$, by definition $0 \leq \alpha \leq \frac{\pi}{2}$. Hence $\frac{\pi e^{\frac{1}{4}} m r^{\frac{\pi}{\alpha}}}{2\alpha^2 (8\sin^2\alpha)^{\pi/2\alpha}(1-(1/2)^{\pi/2\alpha})} \times \lim_{t\downarrow 0} t^{\frac{\pi}{2\alpha}-1}$ is bounded by some constant not relying on t_2, as required. The proof of $g(t_1, t_1+)$ follows from symmetry.

We now turn back to the proof of the theorem. Note that by Tonelli's Theorem, it is legitimate to integrate over the domain in arbitrary order for the expectation of (D.1), since by definition all integrands are non-negative. To facilitate analysis below and to make use of Lemma 4, we choose to first integrate over $\vec{Z}$, then within the two-dimensional domain $\{(t_1, t_2) : |t_1 - t_2| \leq t\}$. If $t_2 > t_1$, we first integrate along the line $t_2 = t_1 + u$ then over $u \in [0, t]$ and similarly first along the line $t_1 = t_2 + u$ otherwise:

$$P^{(x_1,x_2;\mu_1,\mu_2,\sigma_1,\sigma_2)}(|\tau_1 - \tau_2| \leq t, \tau_1 < \infty, \tau_2 < \infty)$$
$$= \int_{u=0}^{t} \int_{t_1=0}^{\infty} \int_{\vec{y}\in R^2} e^{\vec{\gamma}'(\vec{y}-\vec{z})-(\gamma_1^2+\gamma_2^2)t_1/2} P_{(z_1,z_2)}(\vec{Z}(t_1) = d\vec{y},$$
$$\tau_1 = dt_1, \tau_2 = d(t_1+u))$$
$$+ \int_{u=0}^{t} \int_{t_2=0}^{\infty} \int_{\vec{y}\in R^2} e^{\vec{\gamma}'(\vec{y}-\vec{z})-(\gamma_1^2+\gamma_2^2)t_2/2} P_{(z_1,z_2)}(\vec{Z}(t_2) = d\vec{y},$$
$$\tau_1 = d(t_2+u), \tau_2 = dt_2).$$

It then follows from fundamental theorem of calculus that

$$
\begin{aligned}
&f_{|\tau_1-\tau_2|}(0+) = \frac{\partial}{\partial t}\left(P^{(x_1,x_2;\mu_1,\mu_2,\sigma_1,\sigma_2)}(|\tau_1-\tau_2|\le t, \tau_1<\infty, \tau_2<\infty)\right)|_{t=0}\\
=&\int_{t_1=0}^{\infty} E^{(z_1,z_2)}(e^{\vec{\gamma}'(\vec{Z}(t_1)-\vec{z})}|\tau_1=t_1,\tau_2=t_1)e^{-(\gamma_1^2+\gamma_2^2)t_1/2}g(t_1,t_1)dt_1\\
+&\int_{t_2=0}^{\infty} E^{(z_1,z_2)}(e^{\vec{\gamma}'(\vec{Z}(t_2)-\vec{z})}|\tau_1=t_2,\tau_2=t_2)e^{-(\gamma_1^2+\gamma_2^2)t_2/2}g(t_2,t_2)dt_2\\
=&e^{-\vec{\gamma}'\vec{z}}\left(\int_{t_1=0}^{\infty} e^{-(\gamma_1^2+\gamma_2^2)t_1/2}g(t_1,t_1)dt_1+\int_{t_2=0}^{\infty} e^{-(\gamma_1^2+\gamma_2^2)t_2/2}g(t_2,t_2)dt_2\right).
\end{aligned}
$$

Note that the last equality is again due to $\vec{Z}(s)=(0,0)$ when $\tau_1=\tau_2=s$ for any $s>0$. First consider the case where $\rho>0$. In this case, according to Lemma 4,

$$
\begin{aligned}
&f_{|\tau_1-\tau_2|}(0+)\\
=&e^{-\vec{\gamma}'\vec{z}}\left(\int_{t_1=0}^{\infty} e^{-(\gamma_1^2+\gamma_2^2)t_1/2}\cdot\infty\, dt_1+\int_{t_2=0}^{\infty} e^{-(\gamma_1^2+\gamma_2^2)t_2/2}\cdot\infty\, dt_2\right)\\
=&\infty,
\end{aligned}
$$

as required. When $\rho\le 0$, we are to prove $f_{|\tau_1-\tau_2|}(0+)<\infty$. It suffices to prove that

$$
\int_{t_1=0}^{\infty} e^{-(\gamma_1^2+\gamma_2^2)t_1/2}g(t_1,t_1)dt_1\int_{t_2=0}^{\infty} e^{-(\gamma_1^2+\gamma_2^2)t_2/2}g(t_2,t_2)dt_2<\infty.
$$

Indeed,

$$
\begin{aligned}
&\int_{t_1=0}^{\infty} e^{-(\gamma_1^2+\gamma_2^2)t_1/2}g(t_1,t_1)dt_1+\int_{t_2=0}^{\infty} e^{-(\gamma_1^2+\gamma_2^2)t_2/2}g(t_2,t_2)dt_2\\
\le&\int_{t_1=0}^{\infty} g(t_1,t_1)dt_1+\int_{t_2=0}^{\infty} g(t_2,t_2)dt_2\\
\le&C\int_{t_1=0}^{\infty}\frac{1}{t_1^{\pi/\alpha+1}}\exp(-\frac{r^2}{4t_1})dt_1+C\int_{t_2=0}^{\infty}\frac{1}{t_2^{\pi/\alpha+1}}\exp(-\frac{r^2}{4t_2})dt_2\\
&\qquad\qquad\qquad\qquad\qquad\qquad(\text{letting } u_j=\tfrac{1}{t_j},\ j=1,2)\\
=&C\int_{u_1=0}^{\infty}u_1^{\pi/\alpha-1}\exp(-\frac{r^2u_1}{4})du_1+C\int_{u_2=0}^{\infty}u_2^{\pi/\alpha-1}\exp(-\frac{r^2u_2}{4})du_2,
\end{aligned}
$$

where the second inequality follows from Lemma 4. Since any finite moment of exponential distribution exists, the above is finite. The proof for Part (2) is completed.

Appendix E. Proof of Theorem 4

Proof. For Part (1), by property of joint moments, we have

$$\begin{aligned}&E^{(0,0)}[(-J_1(\varepsilon_p))(-J_2(\varepsilon_p))]\\=&\int_0^\infty\int_0^\infty P^{(0,0)}(-J_1(\varepsilon_p)\geq x_1,-J_2(\varepsilon_p)\geq x_2)dx_1dx_2\\=&\int_0^\infty\int_0^\infty E^{(x_1,x_2)}[e^{-p\tilde{\tau}}]dx_1dx_2,\end{aligned}$$

where the second equality is due to (5.2) in the main text.

To perform the integral for $E^{(0,0)}[(-J_1(\varepsilon_p))(-J_2(\varepsilon_p))]$, a change from (x_1,x_2) to (r,θ) is necessary, since $E^{(x_1,x_2)}[e^{-p\tilde{\tau}}]$ is in the notation of r and θ. The Jacobian matrix for the change-of-variable writes that $dx_1dx_2=\sigma_1\sigma_2 r\sin\alpha dr d\theta$. The corresponding domain of $(x_1,x_2)\in(0,\infty)\times(0,\infty)$ is $[r,\theta]\in(0,\infty)\times[0,\alpha]$. Combining all these steps above, we thus have

$$\begin{aligned}&E^{(0,0)}[(-J_1(\varepsilon_p))(-J_2(\varepsilon_p))]\\=&\sin\alpha\sigma_1\sigma_2\int_{r=0}^\infty\int_{\eta=0}^\alpha\frac{2r}{\pi}e^{-(\gamma_1\cos\eta+\gamma_2\sin\eta)r}\times\\&\left[\int_{\nu=0}^\infty\frac{K_{i\nu}(a(p)r)}{\sinh(\alpha\nu)}\cosh(\beta_1\nu)\sinh((\alpha-\eta)\nu)+\cosh(\beta_2\nu)\sinh(\eta\nu)]d\nu\right]d\eta dr\\=&\frac{2\sin\alpha\sigma_1\sigma_2}{\pi a(p)^2}\int_0^\alpha d\eta\int_0^\infty d\nu\left[\int_0^\infty e^{-\cosh(Q(\eta,p))\cdot ar}(ar)K_{i\nu}(a(p)r)d(a(p)r)\right]\\&\times\frac{1}{\sinh(\alpha\nu)}[\cosh(\beta_1(c)\nu)\sinh((\alpha-\eta)\nu)+\cosh(\beta_2(c)\nu)\sinh(\eta\nu)].\end{aligned}\qquad\text{(E.1)}$$

To obtain the last equality in (E.1), we have used (5.3) in the main text and applied the Tonelli's Theorem and integrated first over r and then over ν and η. For a closed form expression for the integral in the bracket in the last equality of (E.1), we appeal to the following identity in [27] (p. 388, Equation (7)):

$$\int_0^\infty e^{-t\cosh Q}K_{i\nu}(t)t^{\mu-1}dt=\sqrt{\frac{1}{2}\pi}\cdot\Gamma(\mu-i\nu)\Gamma(\mu+i\nu)\frac{P_{i\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\cosh Q)}{\sinh^{\mu-\frac{1}{2}}Q},\qquad\text{(E.2)}$$

subject to the conditions that $R(\mu)>0$ and $R(\cosh Q)>-1$, where $R(\cdot)$ denotes the real part; these conditions are satisfied because here $\mu=2>0$

and by definition $\cosh(Q(\eta,p)) = b(\eta,p) > -1$ when $p \geq N$ (see the comment proceeding this theorem in the main text).

Similarly, in the special case of $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$, using (2.20) in the main text, the formula becomes (the following steps are also documented in Rogers and Shepp (2006) with different notations):

$$\begin{aligned}
&E^{(0,0)}[(-J_1(\varepsilon_p))(-J_2(\varepsilon_p))] \\
=&\sin\alpha \int_0^\infty d\nu \int_0^\alpha d\eta \frac{2}{\pi}\cosh\left(\left(\frac{\pi}{2}-\alpha\right)\nu\right)\frac{\cosh\left(\frac{\alpha\nu}{2}-\eta\nu\right)}{\cosh\frac{\alpha\nu}{2}}\left[\int_0^\infty rdrK_{i\nu}(\sqrt{2p}r)\right] \\
=&\frac{\sin\alpha}{2p}\int_0^\infty d\nu \int_0^\alpha d\eta \frac{\cosh\left(\left(\frac{\pi}{2}-\alpha\right)\nu\right)}{\sinh\nu\pi/2}\frac{\nu\cosh\left(\frac{\alpha\nu}{2}-\eta\nu\right)}{\cosh\frac{\alpha\nu}{2}} \\
=&\frac{\sin\alpha}{p}\int_0^\infty d\nu \frac{\cosh\left(\left(\frac{\pi}{2}-\alpha\right)\nu\right)}{\sinh\nu\pi/2}\tanh\frac{\alpha\nu}{2}.
\end{aligned} \tag{E.3}$$

In the last equality of (E.3), we have made use of the following identity: $\int_0^\alpha \nu\cosh(\nu(\eta-\frac{\alpha}{2}))d\eta = 2\sinh\frac{\alpha\nu}{2}$, and in the second equality of (E.3) we have made use of the following fact

$$\begin{aligned}
&\int_0^\infty K_{i\nu}(t)tdt = \Gamma(1-\frac{i\nu}{2})\Gamma(1+\frac{i\nu}{2}) = \frac{[\Gamma(1-\frac{i\nu}{2})\Gamma(\frac{i\nu}{2})][\Gamma(1+\frac{i\nu}{2})\Gamma(-\frac{i\nu}{2})]}{\Gamma(\frac{i\nu}{2})\Gamma(-\frac{i\nu}{2})} \\
&= \frac{(\pi/\sin(i\pi\nu/2))(\pi/\sin(-i\pi\nu/2))}{\pi/\sin(i\pi\nu/2)\frac{i\nu}{2}} = \frac{\pi}{2}\frac{\nu}{\sinh(\frac{\pi\nu}{2})},
\end{aligned} \tag{E.4}$$

where the first equality in (E.4) is due to Equation (8) in [27] (p. 388), the third equality of (E.4) due to the identities $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ and $\Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)}$ (see 6.1.17 in [2]) and the last equality of (E.4) due to $\sin(iz) = i\sinh z$. The proof of Part (1) is then completed.

The proof for Part (2) is similar. Note that

$$\begin{aligned}
&E^{(0,0)}[(-J_1(\varepsilon_{p_1}))(-J_2(\varepsilon_{p_2}))] \\
&= \int_0^\infty\int_0^\infty P^{(0,0)}(-J_1(\varepsilon_{p_1}) \geq x_1, -J_2(\varepsilon_{p_2}) \geq x_2)dx_1dx_2 \\
&= \int_0^\infty\int_0^\infty E^{(x_1,x_2)}[e^{-p_1\tau_1-p_2\tau_2}]dx_1dx_2,
\end{aligned}$$

The expression for $E^{(x_1,x_2)}[e^{-p_1\tau_1-p_2\tau_2}]$ is almost identical to that of $E^{(x_1,x_2)}$ $[e^{-p\tilde{\tau}}]$; compare (1) to (3) in Section 2.5 in the main text. Now proceed in the way of (E.1). One needs $\min\{p_1, p_2\} \geq \max\{M, N\}$ for (E.2) to hold (note that now the definitions of a and Q have changed) and it follows that

$$\begin{aligned}
& E^{(0,0)}[(-J_1(\varepsilon_{p_1}))(-J_2(\varepsilon_{p_2}))] \\
= & \sin\alpha\sigma_1\sigma_2 \int_{r=0}^{\infty}\int_{\eta=0}^{\alpha} d\eta dr \frac{2r}{\pi} e^{-(\gamma_1\cos\eta+\gamma_2\sin\eta)r} \times \\
& \left[\int_{\nu=0}^{\infty} \frac{K_{i\nu}(a(p_1+p_2)r)}{\sinh(\alpha\nu)} \cosh(\beta_1\nu)\sinh((\alpha-\eta)\nu) + \cosh(\beta_2\nu)\sinh(\eta\nu)]d\nu\right] \\
= & \frac{2\sin\alpha\sigma_1\sigma_2}{\pi a(p_1+p_2)^2} \int_0^{\alpha} d\eta \int_0^{\infty} d\nu \\
& \left[\int_0^{\infty} e^{-\cosh(Q(\eta,p_1+p_2))\cdot a(p_1+p_2)r}(ar)K_{i\nu}(a(p_1+p_2)r)d(a(p_1+p_2)r)\right] \\
& \times \frac{1}{\sinh(\alpha\nu)}[\cosh(\beta_1(c)\nu)\sinh((\alpha-\eta)\nu) + \cosh(\beta_2(c)\nu)\sinh(\eta\nu)]. \qquad \text{(E.5)}
\end{aligned}$$

The proof for Part (2) is also completed.