Electronic Companion

A Two-Sided Laplace Inversion Algorithm with Computable Error Bounds and Its Applications in Financial Engineering

Ning Cai, S. G. Kou, Zongjian Liu

HKUST and Columbia University

Appendix A. Proof of Proposition 6.1

Proof. (i) First, $F_{X_t}(x) = P(X_t \le x) \in \mathcal{C}^1$ because X_t has a continuous distribution under P. Second, $P(X_t \le x)$ is obviously increasing. Then applying Lemma 3.1, we know $F_{X_t}(x)$ satisfies Assumption 1. Moreover, the ROAC of $F_{X_t}(x)$ is $(0, \sigma_u)$ because for any $\sigma \in (0, \sigma_u)$,

$$\int_{-\infty}^{+\infty} e^{-\sigma x} F_{X_t}(x) dx = \int_{-\infty}^{+\infty} e^{-\sigma x} P\left(X_t \le x\right) dx = E\left(\int_{X_t}^{+\infty} e^{-\sigma x} dx\right) = \frac{E e^{-\sigma X_t}}{\sigma} < +\infty.$$

Therefore, $e^{-\sigma x} F_{X_t}(x)$ is of bounded variation on $\mathbb{R}$ for any $\sigma \in (0, \sigma_u)$.

(ii) EuC(k) can be rewritten as

$$\begin{aligned}\text{EuC}(k) =& S_0 E\left[e^{X_t - rt} I_{\{S_t \ge e^{-k}\}}\right] - e^{-rt} e^{-k} P\left(S_t \ge e^{-k}\right) \\ =& S_0 \widetilde{P}\left(S_t \ge e^{-k}\right) - e^{-rt} e^{-k} P\left(S_t \ge e^{-k}\right) := f_1(k) - f_2(k).\end{aligned} \tag{32}$$

where the second equality holds because of the change of measure with $\frac{\mathrm{d}\widetilde{P}}{\mathrm{d}P}|_{\mathcal{F}_t} = e^{X_t - rt}$.

Because S_t has a continuous distribution under both P and $\widetilde{P}$, then $f_1(k)$ and $f_2(k)$ are in $\mathcal{C}^1$. Besides, $\widetilde{P}\left(S_t \ge e^{-k}\right)$ and $P\left(S_t \ge e^{-k}\right)$ are both increasing in k. Therefore, applying Lemma 3.1 yields that both $f_1(k)$ and $f_2(k)$ satisfy Assumption 1. Note that for any $\sigma \in (-1, -\sigma_l - 1)$,

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-\sigma k} f_2(k) dk =& e^{-rt} \int_{-\infty}^{+\infty} e^{-(\sigma+1)k} P\left(S_t \ge e^{-k}\right) dk \\ =& e^{-rt} E\left[\int_{-\log S_t}^{+\infty} e^{-(\sigma+1)k} dk\right] = \frac{e^{-rt} S_0^{\sigma+1} E e^{(\sigma+1) X_t}}{\sigma + 1} < +\infty,\end{aligned}$$

and for any $\sigma \in (0, -\sigma_l - 1)$,

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-\sigma k} f_1(k) dk =& S_0 \int_{-\infty}^{+\infty} e^{-\sigma k} \widetilde{P}\left(S_t \ge e^{-k}\right) dk = S_0 \widetilde{E}\left[\int_{-\log S_t}^{+\infty} e^{-\sigma k} dk\right] \\ =& \frac{S_0^{\sigma+1} \widetilde{E} e^{\sigma X_t}}{\sigma} = \frac{e^{-rt} S_0^{\sigma+1} E e^{(\sigma+1) X_t}}{\sigma} < +\infty.\end{aligned}$$

It follows that both $e^{-\sigma k} f_1(k)$ and $e^{-\sigma k} f_2(k)$ are of bounded variation on $\mathbb{R}$ for any $\sigma \in (0, -\sigma_l - 1)$, and so is $e^{-\sigma k}$EuC(k). □

Appendix B. Proof of Proposition 6.2

Proof. Replacing s in (17) by $\sigma + i\omega$ with $\sigma \in (-\hat{M}, \hat{G})$ yields

$$
\begin{aligned}
|L_{f_{X_t}}(\sigma+i\omega)| =& \left|e^{-\mu t\sigma - i\mu t\omega}\right| \cdot \exp\left\{-t\hat{C}\cdot\Gamma(-\hat{Y})\left(\hat{M}^{\hat{Y}}+\hat{G}^{\hat{Y}}\right)\right\} \\
&\cdot \left|\exp\left\{t\hat{C}\cdot\Gamma(-\hat{Y})\left[(\hat{M}+\sigma+i\omega)^{\hat{Y}}+(\hat{G}-\sigma-i\omega)^{\hat{Y}}\right]\right\}\right| \\
=& \exp\left\{-\mu t\sigma - t\hat{C}\cdot\Gamma(-\hat{Y})\left(\hat{M}^{\hat{Y}}+\hat{G}^{\hat{Y}}\right)\right\} \\
&\cdot \left|\exp\left\{t\hat{C}\cdot\Gamma(-\hat{Y})\left[e^{\hat{Y}[\ln|Z_1|+i\arg(Z_1)]}+e^{\hat{Y}[\ln|Z_2|+i\arg(Z_2)]}\right]\right\}\right| \\
=& \exp\left\{-\mu t\sigma - t\hat{C}\cdot\Gamma(-\hat{Y})\left(\hat{M}^{\hat{Y}}+\hat{G}^{\hat{Y}}\right)\right\} \\
&\cdot \exp\left\{t\hat{C}\cdot\Gamma(-\hat{Y})\cdot\left[|Z_1|^{\hat{Y}}\cdot\cos(\hat{Y}\arg(Z_1))+|Z_2|^{\hat{Y}}\cdot\cos(\hat{Y}\arg(Z_2))\right]\right\} \\
=& \exp\left\{-\mu t\sigma - t\hat{C}\cdot\Gamma(-\hat{Y})\left(\hat{M}^{\hat{Y}}+\hat{G}^{\hat{Y}}\right)\right\}\cdot Q(\hat{Y},\sigma,\omega),
\end{aligned} \tag{33}
$$

where $\ln(\cdot)$ is the real natural logarithm function, $\arg(Z_j) \in (-\pi, \pi)$ for $j = 1$ and 2,

$$
\begin{aligned}
Z_1 \equiv& Z_1(\sigma,\omega) := \hat{M}+\sigma+i\omega, \quad Z_2 \equiv Z_2(\sigma,\omega) := \hat{G}-\sigma-i\omega, \\
Q(\hat{Y},\sigma,\omega) :=& \exp\left\{t\hat{C}\cdot\Gamma(-\hat{Y})\cdot\left[|Z_1|^{\hat{Y}}\cdot\cos(\hat{Y}\arg(Z_1))+|Z_2|^{\hat{Y}}\cdot\cos(\hat{Y}\arg(Z_2))\right]\right\} \\
=& \exp\left\{t\hat{C}\cdot\Gamma(-\hat{Y})\cos(\hat{Y}\pi/2)|\omega|^{\hat{Y}}\cdot[f_1(|\omega|)+f_2(|\omega|)]\right\}, \quad \text{and} \\
f_j(|\omega|) :=& \frac{|Z_j|^{\hat{Y}}}{|\omega|^{\hat{Y}}}\cdot\frac{\cos(\hat{Y}\arg(Z_j))}{\cos(\hat{Y}\pi/2)}, \quad \text{for } j=1 \text{ and } 2.
\end{aligned}
$$

Note that $f_1(|\omega|)$ can be rewritten as

$$
f_1(|\omega|) = \left[1+\frac{(\hat{M}+\sigma)^2}{|\omega|^2}\right]^{\hat{Y}/2}\cdot\frac{\cos\left(\hat{Y}\cdot\arctan\left(\frac{|\omega|}{\hat{M}+\sigma}\right)\right)}{\cos(\hat{Y}\pi/2)}.
$$

Then for any $|\omega| > 0$, some algebra yields

$$
f_1'(|\omega|) = \frac{-(\hat{M}+\sigma)\hat{Y}}{\cos(\hat{Y}\pi/2)|\omega|^2}\cdot\left[1+\frac{(\hat{M}+\sigma)^2}{|\omega|^2}\right]^{\hat{Y}/2-1}\cdot R(|\omega|), \tag{34}
$$

where

$$
R(|\omega|) := \frac{\hat{M}+\sigma}{|\omega|}\cos\left(\hat{Y}\cdot\arctan\left(\frac{|\omega|}{\hat{M}+\sigma}\right)\right)+\sin\left(\hat{Y}\cdot\arctan\left(\frac{|\omega|}{\hat{M}+\sigma}\right)\right),
$$

Defining $z = \arctan\left(\frac{|\omega|}{\hat{M}+\sigma}\right) \in (0, \pi/2)$, we obtain

$$
R(|\omega|) = \frac{1}{\tan(z)}\cos(\hat{Y}z)+\sin(\hat{Y}z) = \frac{1}{\sin(z)}\left[\cos(\hat{Y}z)\cos(z)+\sin(\hat{Y}z)\sin(z)\right] = \frac{1}{\sin(z)}\cos((\hat{Y}-1)z).
$$

Since $z \in (0, \pi/2)$, we have $R(|\omega|) > 0$ for any $\hat{Y} \in (0,1)\cup(1,2)$. Then by (34), we obtain $f_1'(|\omega|) < 0$ if $Y \in (0,1)$, and $f_1'(|\omega|) > 0$ if $\hat{Y} \in (1,2)$. Similarly, we can prove $f_2'(|\omega|) < 0$ if $\hat{Y} \in (0,1)$, and $f_2'(|\omega|) > 0$ if $\hat{Y} \in (1,2)$. It follows that if $\hat{Y} \in (0,1)$, $f_1(|\omega|)$ and $f_2(|\omega|)$ are both decreasing functions of $|\omega|$ in $(0,+\infty)$. This implies that $f_1(|\omega|)+f_2(|\omega|) \geq \lim_{|\omega|\to+\infty}[f_1(|\omega|)+f_2(|\omega|)] = 2$ for any $|\omega| > 0$. Hence for any $|\omega| > \omega^* := 0$, we have $Q(\hat{Y},\sigma,\omega) \leq \exp\left\{2t\hat{C}\cdot\Gamma(-\hat{Y})\cos(\hat{Y}\pi/2)|\omega|^{\hat{Y}}\right\}$, which along with (33) completes the proof for the case of $\hat{Y} \in (0,1)$.

If $\hat{Y} \in (1,2)$, $f_1(|\omega|)$ and $f_2(|\omega|)$ are both increasing functions of $|\omega|$ in $(0,+\infty)$. Then for any $|\omega| > \omega^* := \max\{\hat{M}+\sigma, \hat{G}-\sigma\}\tan((\pi/2+\epsilon)/\hat{Y}) > 0$ with $\epsilon \in (0, \pi(\hat{Y}-1)/2)$, we have $f_j(|\omega|) \geq \frac{\cos(\pi/2+\epsilon)}{\cos(\hat{Y}\pi/2)}$ for $j=1$ and 2. As a result, for any $|\omega| > \omega^*$, we have $Q(\hat{Y},\sigma,\omega) \leq \exp\left\{2t\hat{C}\cdot\Gamma(-\hat{Y})\cos(\pi/2+\epsilon)|\omega|^{\hat{Y}}\right\}$, which along with (33) completes the proof of (18). □

Appendix C. Proof of Proposition 6.3

Proof. Substituting $\sigma + i\omega$ for s in (19) yields that for any $|\omega| > \omega^* := 0$,

$$
\begin{aligned}
|L_{f_{X_t}}(\sigma+i\omega)| =& \left|\exp\left\{\frac{t\bar{\sigma}^2(-\sigma-i\omega)^2}{2} - \mu t(\sigma+i\omega) + \lambda t\left(p_u\sum_{l=1}^m \frac{p_l\eta_l}{\eta_l+\sigma+i\omega} + q_d\sum_{j=1}^n \frac{q_j\theta_j}{\theta_j-\sigma-i\omega} - 1\right)\right\}\right| \\
=& \exp\left\{t\left(\frac{\bar{\sigma}^2\sigma^2}{2} - \mu\sigma\right)\right\}\cdot\exp\left\{-t\frac{\bar{\sigma}^2}{2}\omega^2\right\} \\
&\cdot\exp\left\{\lambda t\left(p_u\sum_{l=1}^m \frac{p_l\eta_l(\eta_l+\sigma)}{(\eta_l+\sigma)^2+\omega^2} + q_d\sum_{j=1}^n \frac{q_j\theta_j(\theta_j-\sigma)}{(\theta_j-\sigma)^2+\omega^2} - 1\right)\right\} \\
\leq& \exp\left\{t\left(\frac{\bar{\sigma}^2\sigma^2}{2} - \mu\sigma\right)\right\}\cdot\exp\left\{-t\frac{\bar{\sigma}^2}{2}\omega^2\right\} \\
&\cdot\exp\left\{\lambda t\left(p_u\sum_{l=1}^m \frac{|p_l|\eta_l(\eta_l+\sigma)}{(\eta_l+\sigma)^2+\omega^2} + q_d\sum_{j=1}^n \frac{|q_j|\theta_j(\theta_j-\sigma)}{(\theta_j-\sigma)^2+\omega^2} - 1\right)\right\} \\
\leq& \exp\left\{t\left(\frac{\bar{\sigma}^2\sigma^2}{2} - \mu\sigma\right)\right\}\cdot\exp\left\{-t\frac{\bar{\sigma}^2}{2}\omega^2\right\}\cdot\exp\left\{\lambda t\left(p_u\sum_{l=1}^m \frac{|p_l|\eta_l}{\eta_l+\sigma} + q_d\sum_{j=1}^n \frac{|q_j|\theta_j}{\theta_j-\sigma} - 1\right)\right\} \\
=& \exp\left\{t\left[\frac{\bar{\sigma}^2\sigma^2}{2} - \mu\sigma + \lambda\left(p_u\sum_{l=1}^m \frac{|p_l|\eta_l}{\eta_l+\sigma} + q_d\sum_{j=1}^n \frac{|q_j|\theta_j}{\theta_j-\sigma} - 1\right)\right]\right\}\cdot\exp\left\{-t\frac{\bar{\sigma}^2}{2}\omega^2\right\},
\end{aligned}
$$

which completes the proof. □

Appendix D. Proof of Propositions 7.1–7.3

We shall defer the proof of Proposition 7.1 to the end because it uses the results of Propositions 7.3.

Proof of Proposition 7.2. For any $|\omega| > \omega^*$, we know from Lemma D.2 in Cai and Shi [5] that

$$|\beta_{1,s+r} - \eta| \leq \frac{Y_3}{|\omega|\theta} < \frac{Y_3}{\omega^*\theta} \quad \text{and} \quad |\beta_{2,s+r} - z_\omega| \leq \frac{2|\mu|}{\bar{\sigma}^2}. \tag{35}$$

It follows that

$$0 \leq \eta - \frac{Y_3}{\omega^*\theta} < |\beta_{1,s+r}| < \frac{Y_3}{\omega^*\theta} + \eta \quad \text{and} \quad |\beta_{2,s+r}| \geq |z_\omega| - \frac{2|\mu|}{\bar{\sigma}^2} = \frac{\sqrt{2|\omega|}}{\bar{\sigma}} - \frac{2|\mu|}{\bar{\sigma}^2}. \tag{36}$$

where the first inequality holds also because $\omega^* \geq Y(\sigma+r) \geq Y_4 = \frac{Y_3}{\eta\theta}$. From the second inequality of (36) we further obtain

$$|\beta_{2,s+r} - 1| \geq |\beta_{2,s+r}| - 1 \geq \frac{\sqrt{2|\omega|}}{\bar{\sigma}} - \frac{2|\mu|}{\bar{\sigma}^2} - 1 > \frac{\sqrt{2|\omega|}}{2\bar{\sigma}} + \frac{\sqrt{2\omega^*}}{2\bar{\sigma}} - \frac{2|\mu|}{\bar{\sigma}^2} - 1 > \frac{\sqrt{2|\omega|}}{2\bar{\sigma}}, \tag{37}$$

thanks to the fact that $|\omega| > \omega^* \geq Y(\sigma+r) \geq Y_1 = 2(\bar{\sigma}\eta + \frac{2|\mu|}{\bar{\sigma}})^2$ and $\eta > 1$. Moreover, note that

$$\mathrm{Re}(z_\omega) - \frac{2|\mu|}{\bar{\sigma}^2} > \mathrm{Re}(z_{\omega^*}) - \frac{2|\mu|}{\bar{\sigma}^2} = \frac{\sqrt{\omega^*}}{\bar{\sigma}} - \frac{2|\mu|}{\bar{\sigma}^2} \geq \frac{\sqrt{Y_1}}{\bar{\sigma}} - \frac{2|\mu|}{\bar{\sigma}^2} = \sqrt{2}\eta + \frac{2(\sqrt{2}-1)|\mu|}{\bar{\sigma}^2} > \eta,$$

This implies that $\mathrm{Re}(z_\omega) - \eta > \mathrm{Re}(z_{\omega^*}) - \eta > \frac{2|\mu|}{\bar{\sigma}^2} \geq 0$ and hence $|z_\omega - \eta| \geq |z_{\omega^*} - \eta|$. Then by (35) we obtain that for any $|\omega| > \omega^*$,

$$\begin{aligned}|\beta_{2,s+r} - \beta_{1,s+r}| &\geq |z_\omega - \eta| - |z_\omega - \beta_{2,s+r}| - |\beta_{1,s+r} - \eta| \geq |z_{\omega^*} - \eta| - |z_\omega - \beta_{2,s+r}| - |\beta_{1,s+r} - \eta| \\ &\geq |z_{\omega^*} - \eta| - \frac{2|\mu|}{\bar{\sigma}^2} - \frac{Y_3}{\omega^*\theta} > 0.\end{aligned}$$

where the last inequality holds because $|z_{\omega^*} - \eta| - \frac{2|\mu|}{\bar{\sigma}^2} - \frac{Y_3}{\omega^*\theta} \geq \mathrm{Im}(z_{\omega^*}) - \frac{2|\mu|}{\bar{\sigma}^2} - \eta = \mathrm{Re}(z_{\omega^*}) - \frac{2|\mu|}{\bar{\sigma}^2} - \eta > 0$. According to (35), (36) and the equality above, we have that for any $|\omega| > \omega^*$,

$$\begin{aligned}I_1(\omega) :=& \left|\frac{\beta_{2,s+r} - \eta}{\beta_{2,s+r} - \beta_{1,s+r}}\right| \leq 1 + \left|\frac{\beta_{1,s+r} - \eta}{\beta_{2,s+r} - \beta_{1,s+r}}\right| \\ \leq& 1 + \frac{\frac{Y_3}{\omega^*\theta}}{|z_{\omega^*} - \eta| - \frac{2|\mu|}{\bar{\sigma}^2} - \frac{Y_3}{\omega^*\theta}} = \frac{|z_{\omega^*} - \eta| - \frac{2|\mu|}{\bar{\sigma}^2}}{|z_{\omega^*} - \eta| - \frac{2|\mu|}{\bar{\sigma}^2} - \frac{Y_3}{\omega^*\theta}}, \qquad (38)\\ I_2(\omega) :=& \left|\frac{\beta_{2,s+r}}{\beta_{2,s+r} - \beta_{1,s+r}}\right| \leq 1 + \left|\frac{\beta_{1,s+r}}{\beta_{2,s+r} - \beta_{1,s+r}}\right| \\ \leq& 1 + \frac{\frac{Y_3}{\omega^*\theta} + \eta}{|z_{\omega^*} - \eta| - \frac{2|\mu|}{\bar{\sigma}^2} - \frac{Y_3}{\omega^*\theta}} = \frac{|z_{\omega^*} - \eta| - \frac{2|\mu|}{\bar{\sigma}^2} + \eta}{|z_{\omega^*} - \eta| - \frac{2|\mu|}{\bar{\sigma}^2} - \frac{Y_3}{\omega^*\theta}}, \qquad (39)\end{aligned}$$

Besides, define $h(x) := xe^{-bx}$ for $x \geq 0$ with $b > 0$. It is easy to see that $h(x)$ attains its maximum at $x = \frac{1}{b}$, and decreases in x when $x > \frac{1}{b}$. Thus by (36) we have that for any $|\omega| > \omega^* \geq \frac{100}{b^2}$

$$I_3(\omega) := \sqrt{|\omega|}\left|\left(\frac{S_0}{M}\right)^{\beta_{2,s+r}}\right| \leq \sqrt{|\omega|}e^{-b\sqrt{|\omega|}}\left(\frac{S_0}{M}\right)^{-\frac{2|\mu|}{\bar{\sigma}^2}} \leq h\left(\frac{10}{b}\right)\left(\frac{S_0}{M}\right)^{-\frac{2|\mu|}{\bar{\sigma}^2}} = \frac{10}{be^{10}}\left(\frac{M}{S_0}\right)^{\frac{2|\mu|}{\bar{\sigma}^2}}. \qquad (40)$$

Note that from (21), we can obtain

$$|L_f(s)| \leq M\left(\frac{S_0}{M}\right)^{\mathrm{Re}(\beta_{1,s+r})} \frac{(|\beta_{1,s+r} - \eta|)I_2(\omega)}{(|\beta_{1,s+r} - 1|)\eta(s+r)} + M\frac{|\beta_{1,s+r}|I_1(\omega)I_3(\omega)}{\eta(s+r)(|\beta_{2,s+r} - 1|)\sqrt{|\omega|}} + \left|\frac{rM}{s(s+r)}\right|.$$

Substituting (35)–(40) into the RHS of the inequality above concludes the proof. □

Proof of Proposition 7.3. Define $\tilde{f}(T) := f(T) + M > 0$. Then $L_{\tilde{f}}(s) = L_f(s) + \frac{M}{s}$ for any $\mathrm{Re}(s) > 0$. Moreover, for any $\sigma > 0$ and $\omega \in \mathbb{R}$, we have

$$|L_{\tilde{f}}(\sigma + i\omega)| = \left|\int_0^{+\infty} e^{-(\sigma+i\omega)T}\tilde{f}(T)dT\right| \leq \int_0^{+\infty} \left|e^{-(\sigma+i\omega)T}\tilde{f}(T)\right| dT = L_{\tilde{f}}(\sigma) = L_f(\sigma) + \frac{M}{\sigma}.$$

Applying Proposition 7.2 yields

$$|L_f(\sigma + i\omega)| \leq L(\omega;\sigma) := \begin{cases} L_f(\sigma) + \dfrac{2M}{\sigma}, & \text{for all } |\omega| \leq \omega^*; \\ \zeta(\sigma)|\omega|^{-2}, & \text{for all } |\omega| > \omega^*. \end{cases} \qquad (41)$$

Then by the Bromwich contour integral we obtain

$$e^{-\sigma T}|f(T)| \leq \frac{1}{2\pi}\int_{-\infty}^{+\infty} |L_f(\sigma + i\omega)|d\omega \leq \frac{1}{2\pi}\int_{-\infty}^{+\infty} |L(\omega;\sigma)|d\omega = \delta(\sigma). \qquad \square$$

Proof of Proposition 7.1. Define a function $u(T) := E\left[\max\left\{M, \max_{0\le t\le T} S_t\right\}\right] 1_{\{T\ge 0\}}$. Then $u(T)$ is positive and increasing in T when $T \ge 0$, and $f(T) \equiv e^{-rT}u(T) - M$ when $T \ge 0$. To show $e^{-\sigma T}f(T)$ is of bounded variation on $\mathbb{R}$ for any $\sigma > 0$, it suffices to prove that for any $\sigma > 0$, the function $g_u(T) := e^{-(\sigma+r)T}u(T)$ is of bounded variation on $\mathbb{R}^+$ for any $\sigma > 0$.

To this end, we shall first compute the total variation of $g_u(T)$ on the interval $[0, A]$ for any $A > 0$ (denoted by $\bigvee_0^A(g_u)$), and then show that $\limsup_{\{A>0\}} \bigvee_0^A(g_u) < +\infty$. Recall that

$$\bigvee_0^A(g_u) := \lim_{||\Delta||\to 0} \sum_{i=0}^{n_\Delta - 1} |g_u(x_{i+1}) - g_u(x_i)|, \tag{42}$$

where Δ: $0 = x_0 < x_1 < \cdots < x_{n_\Delta} = A$, is any partition of the interval $[0, A]$ with the norm $||\Delta|| := \sup_{0\le i\le n_\Delta - 1} |x_{i+1} - x_i|$. For any $\sigma > 0$ and $0 < T_1 < T_2$, we have

$$\begin{aligned}
|g_u(T_2) - g_u(T_1)| &\le |e^{-(\sigma+r)T_2}u(T_1) - g_u(T_1))| + |g_u(T_2) - e^{-(\sigma+r)T_2}u(T_1)| \\
&= [e^{-(\sigma+r)T_1} - e^{-(\sigma+r)T_2}]u(T_1) + [u(T_2) - u(T_1)]e^{-(\sigma+r)T_2} \\
&\le (T_2 - T_1)(\sigma + r)e^{-(\sigma+r)T_1}u(T_1) + [u(T_2) - u(T_1)]e^{-(\sigma+r)T_2},
\end{aligned}$$

where the last inequality is obtained based on the mean value theorem. It follows that

$$\sum_{i=0}^{n_\Delta - 1} |g_u(x_{i+1}) - g_u(x_i)| \le S_1^\Delta + S_2^\Delta \tag{43}$$

where

$$S_1^\Delta := (\sigma + r) \sum_{i=0}^{n_\Delta - 1} (x_{i+1} - x_i)e^{-(\sigma+r)x_i}u(x_i) \quad \text{and} \quad S_2^\Delta := \sum_{i=0}^{n_\Delta - 1} [u(x_{i+1}) - u(x_i)]e^{-(\sigma+r)x_{i+1}}.$$

Letting $||\Delta|| \to 0$ yields

$$\begin{aligned}
\lim_{||\Delta||\to 0} S_1^\Delta &= (\sigma + r)\int_0^A e^{-(\sigma+r)T}u(T)dT = (\sigma + r)\int_0^A [e^{-\sigma T}f(T) + e^{-\sigma T}M]dT \\
&\le (\sigma + r)\int_0^{+\infty} e^{-\sigma T}|f(T)|dT + \frac{M(\sigma + r)}{\sigma} < +\infty.
\end{aligned} \tag{44}$$

As for S_2^Δ, note that

$$\begin{aligned}
S_2^\Delta &= \sum_{i=0}^{n_\Delta - 1} u(x_i)[e^{-(\sigma+r)x_i} - e^{-(\sigma+r)x_{i+1}}] - u(0) + u(A)e^{-(\sigma+r)A} \\
&\le (\sigma + r)\sum_{i=0}^{n_\Delta - 1} g_u(x_i)(x_i - x_{i-1}) + (e^{-\sigma A} - 1)M + f(A)e^{-\sigma A} \\
&\le (\sigma + r)\sum_{i=0}^{n_\Delta - 1} g_u(x_i)(x_i - x_{i-1}) + \delta(\sigma),
\end{aligned}$$

where the first and second inequalities hold due to the mean value theorem and Proposition 7.3, respectively. Then letting $||\Delta|| \to 0$, we have

$$\lim_{||\Delta||\to 0} S_2^{\Delta} \leq (\sigma + r)\int_0^A e^{-(\sigma+r)T} u(T)dT + \delta(\sigma)$$
$$\leq (\sigma + r)\int_0^{+\infty} e^{-\sigma T}|f(T)|dT + \frac{M}{\sigma}(\sigma + r) + \delta(\sigma) < +\infty. \tag{45}$$

Substituting (43), (44) and (45) into (42) yields

$$\bigvee_0^A (g_u) \leq 2(\sigma + r)\int_0^{+\infty} e^{-\sigma T}|f(T)|dT + \frac{2M}{\sigma}(\sigma + r) + \delta(\sigma) < +\infty, \quad \text{for all } A > 0.$$

Since the bound in the inequality above is independent of A, it follows that $g_u(T)$ is of bounded variation on $\mathbb{R}^+$. The proof is completed. □