Electronic Companion

Option Pricing under a Mixed-exponential Jump Diffusion Model

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A Proof of Theorem 3.1

Proof. Without loss of generality, we assume that $p_u > 0$, $q_d > 0$, $p_i \neq 0$ for $i = 1, \cdots, m$, and $q_j \neq 0$ for $j = 1, \cdots, n$. First of all, it is easily seen that $G(x) - \alpha$ has the same roots as $(G(x)-\alpha)\prod_{i=1}^{m}(x-\eta_i)\prod_{j=1}^{n}(x+\theta_j)$, which is a polynomial with order $m+n+2$. This implies that for any $\alpha \in \mathbb{R}$, the function $G(x) = \alpha$ has at most $(m+n+2)$ real roots. From now on we shall show that for sufficiently large $\alpha > 0$, the function has exactly $(m+n+2)$ real roots, among which $m+1$ are positive and $n+1$ are negative. Due to the symmetry, we will focus only on arguing that for sufficiently large $\alpha > 0$, the function has $(m+1)$ positive roots.

Note that there exist m positive singularities $\eta_1, \cdots, \eta_m$ for the function $G(x)$, which divide the positive real line into $(m+1)$ disjoint intervals: $(0, \eta_1)$, (η_1, η_2), $\cdots$, (η_{m-1}, η_m), $(\eta_m, +\infty)$. Since $G(\eta_1-) = +\infty$ (because $p_1 > 0$) and $G(0) - \alpha = -\alpha$, we know that for any $\alpha > 0$, $G(x) = \alpha$ has at least one real root on the interval $(0, \eta_1)$. We plan to show there exist m real roots on the other m intervals for sufficiently large $\alpha > 0$.

For convenience, we define $p_{m+1} := +\infty$, $\eta_{m+1} = +\infty$, and two sets $\mathbb{S}^+$ and $\mathbb{S}^-$ as follows

$$\mathbb{S}^+ := \{i \in \{1, \cdots, m\} : p_i > 0 \text{ and } p_{i+1} < 0\},$$

$$\mathbb{S}^- := \{i \in \{1, \cdots, m\} : p_i < 0 \text{ and } p_{i+1} > 0\}.$$

Noting that $p_1 > 0$ and $p_{m+1} = +\infty > 0$, we can easily see that the number of elements in $\mathbb{S}^+$ and that in $\mathbb{S}^-$ are identical. Moreover, if the number of elements is $k > 0$ and if we assume

$$\mathbb{S}^+ = \{i_1^+, \cdots, i_k^+\} \qquad \text{and} \qquad \mathbb{S}^- = \{i_1^-, \cdots, i_k^-\},$$

we must have $i_1^+ < i_1^- < i_2^+ < i_2^- < \cdots < i_k^+ < i_k^-$. In other words, the elements in S^+ and those in S^- are arranged alternatingly.

We categorize the m intervals, (η_1, η_2), $\cdots$, (η_{m-1}, η_m), (η_m, η_{m+1}), into three types.

Type I: (η_i, η_{i+1}) with $i \in \mathbb{S}^+$. Type II: (η_i, η_{i+1}) with $i \in \mathbb{S}^-$. Type III: (η_i, η_{i+1}) with $i \notin \mathbb{S}^+ \bigcup \mathbb{S}^-$.

Then we will show that

(1) If $\mathbb{S}^+ \neq \emptyset$, then for sufficiently large $\alpha > 0$, $G(x) = \alpha$ has no real roots on any interval of Type I;

(2) If $\mathbb{S}^- \neq \emptyset$, then for sufficiently large $\alpha > 0$, $G(x) = \alpha$ has at least two real roots on any interval of Type II;

(3) If $\mathbb{S}^+ \bigcup \mathbb{S}^- \neq \{1, \cdots, m\}$, then for any $\alpha > 0$, $G(x) = \alpha$ has at least one real root on any interval of Type III.

In fact, (1) is implied by the fact that $G(\eta_i+) = G(\eta_{i+1}-) = -\infty$ due to $p_i > 0$ and $p_{i+1} < 0$. Similarly, (2) is correct because $G(\eta_i+) = G(\eta_{i+1}-) = +\infty$. As for (3), if $i \notin \mathbb{S}^+ \bigcup \mathbb{S}^-$, we have either $p_i > 0$ and $p_{i+1} > 0$, or $p_i < 0$ and $p_{i+1} < 0$. In the former case, we obtain $G(\eta_i+) = -\infty$ and $G(\eta_{i+1}-) = +\infty$, while in the latter case, we obtain $G(\eta_i+) = +\infty$ and $G(\eta_{i+1}-) = -\infty$. Thus (3) follows immediately. Recall that the number of elements in $\mathbb{S}^+$ and that in $\mathbb{S}^-$ are identical. Accordingly, the number of intervals of Type I and that of Type II are identical, too. So for sufficiently large $\alpha > 0$, the equation $G(x) = \alpha$ must have m roots on the m intervals, (η_1, η_2), $\cdots$, (η_{m-1}, η_m), (η_m, η_{m+1}). The proof is finished. □

B Proof of Theorem 3.2

Proof. First of all, it suffices to show that Theorem 3.2 holds when $u(x)$ solves the OIDE $(Lu)(x) = \alpha u(x)$ for any $x \in \mathbb{R}$, because all the terms involving the function g will disappear eventually. We give an example to describe this point more explicitly.

Consider a special case $m = 1$ and $n = 1$, i.e., a double exponential jump diffusion process. Then the OIDE (6) is reduced to

$$\begin{aligned}
&\frac{\sigma^2}{2}u''(x) + \mu u'(x) - (\lambda + \alpha)u(x) \\
&+\lambda \left[q_d\theta_1 \int_{-\infty}^{0} u(x+y)e^{\theta_1 y}dy + p_u\eta_1 \int_{0}^{x_0 - x} u(x+y)e^{-\eta_1 y}dy + p_u\eta_1 \int_{x_0 - x}^{+\infty} g(x+y)e^{-\eta_1 y}dy \right] = 0.
\end{aligned}$$

A transformation $z = x + y$ leads to

$$\begin{aligned}
&\frac{\sigma^2}{2}u''(x) + \mu u'(x) - (\lambda + \alpha)u(x) \\
&+\lambda q_d\theta_1 e^{-\theta_1 x} \int_{-\infty}^{x} u(z)e^{\theta_1 z}dz + \lambda p_u\eta_1 e^{\eta_1 x} \left[\int_{x}^{x_0} u(z)e^{-\eta_1 z}dz + C_g \right] = 0,
\end{aligned} \tag{28}$$

where $C_g := \int_{x_0}^{+\infty} g(z)e^{-\eta_1 z}dz$ is a constant. Then multiplying both sides by $e^{-\eta_1 x}$ gives

$$\begin{aligned}
&\frac{\sigma^2}{2}e^{-\eta_1 x}u''(x) + \mu e^{-\eta_1 x}u'(x) - (\lambda + \alpha)e^{-\eta_1 x}u(x) \\
&+\lambda q_d\theta_1 e^{-(\theta_1+\eta_1)x} \int_{-\infty}^{x} u(z)e^{\theta_1 z}dz + \lambda p_u\eta_1 \left[\int_{x}^{x_0} u(z)e^{-\eta_1 z}dz + C_g \right] = 0.
\end{aligned} \tag{29}$$

Finally taking a derivative w.r.t. x and then multiplying both sides by $e^{\eta x}$, we have

$$\frac{\sigma^2}{2}u'''(x)+(-\frac{\sigma^2}{2}\eta_1+\mu)u''(x)+(-\mu p_u\eta_1-\lambda-\alpha)u'(x)+(\alpha\eta_1+\lambda q_d\theta_1+\lambda q_d\eta_1)u(x)$$
$$-\lambda q_d\theta_1(\theta_1+\eta_1)e^{-\theta_1 x}\int_{-\infty}^{x}u(z)e^{\theta_1 z}dz=0. \tag{30}$$

Note that now the OIDE (30) does not involve the function g. Moreover, if we consider the OIDE $(Lu)(x)=\alpha u(x)$ not for $x<x_0$ but for any $x\in\mathbb{R}$, the same procedure as above will lead to the same OIDE (30), since the only difference is replacing $\int_x^{x_0}u(z)e^{-\eta_1 z}dz+C_g$ in (28) and (29) by $\int_x^{+\infty}u(z)e^{-\eta_1 z}dz$, both of which lead to the same result after differentiating w.r.t. x. Consequently, without loss of generality, we can focus only on the OIDE (6) with a slight change, i.e., $(Lu)(x)=\alpha u(x)$ holds for any $x\in\mathbb{R}$.

A key point of the problem is how to deal with the integral part in the OIDE (6), which we rewrite as the following

$$\begin{aligned}(IT) &= \int_{-\infty}^{+\infty}u(x+y)f_Y(y)dy\\ &= \sum_{i=1}^{m}p_up_i\eta_i\int_0^{+\infty}u(x+y)e^{-\eta_i y}dy+\sum_{j=1}^{n}q_dq_j\theta_j\int_{-\infty}^{0}u(x+y)e^{\theta_j y}dy.\end{aligned}$$

A transformation $z=x+y$ yields

$$\begin{aligned}\eta_i\int_0^{+\infty}u(x+y)e^{-\eta_i y}dy &= -\eta_i e^{\eta_i x}\int_{+\infty}^{x}u(z)e^{-\eta_i z}dz,\\ \theta_j\int_{-\infty}^{0}u(x+y)e^{\theta_j y}dy &= \theta_j e^{-\theta_j x}\int_{-\infty}^{x}u(z)e^{\theta_j z}dz.\end{aligned}$$

So (IT) can also be expressed as follows.

$$(IT)=\sum_{i=1}^{m}p_up_i(-\eta_i)e^{\eta_i x}\int_{+\infty}^{x}u(z)e^{-\eta_i z}dz+\sum_{j=1}^{n}q_dq_j\theta_je^{-\theta_j x}\int_{-\infty}^{x}u(z)e^{\theta_j z}dz.$$

Therefore the OIDE (6) is given by

$$\begin{aligned}&\frac{\sigma^2}{2}u''(x)+\mu u'(x)-(\lambda+\alpha)u(x)\\ &+\lambda\left(\sum_{i=1}^{m}p_up_i(-\eta_i)e^{\eta_i x}\int_{+\infty}^{x}u(z)e^{-\eta_i z}dz+\sum_{j=1}^{n}q_dq_j\theta_je^{-\theta_j x}\int_{-\infty}^{x}u(z)e^{\theta_j z}dz\right)=0.\end{aligned}$$

For notation simplicity, we can rewrite the above OIDE by combining the two parts of jumping up and down together:

$$\frac{\sigma^2}{2}u''(x)+\mu u'(x)-(\lambda+\alpha)u(x)+\lambda\sum_{i=1}^{M}\left(r_i\rho_ie^{-\rho_i x}\int_{\pm\infty}^{x}u(z)e^{\rho_i z}dz\right)=0, \tag{31}$$

where

$$M = m + n, \quad r_i = p_u p_i \quad \text{and} \quad \rho_i = -\eta_i, \quad \text{for } i = 1, \cdots, m,$$

$$r_i = q_d q_{i-m} \quad \text{and} \quad \rho_i = \theta_{i-m}, \quad \text{for } i = m+1, \cdots, m+n,$$

and the lower limit of the integral $\int_{\pm\infty}^{x} u(z)e^{\rho_i z}dz$ is equal to $+\infty$ when $\rho_i < 0$ and $-\infty$ when $\rho_i > 0$. Similarly the equation $G(x) - \alpha = 0$ can be expressed as

$$G(x) - \alpha = \frac{\sigma^2}{2}x^2 + \mu x - (\alpha + \lambda) + \lambda \sum_{i=1}^{M} \frac{r_i \rho_i}{\rho_i + x} = 0. \tag{32}$$

To demonstrate our idea clearly, we consider a more general OIDE (33) by investigating its connection with the rational equation (34).

$$\sum_{k=0}^{N} a_k u^{(k)}(x) + \sum_{i=1}^{M} \left(r_i \rho_i e^{-\rho_i x} \int_{\pm\infty}^{x} u(z) e^{\rho_i z} dz \right) = 0 \tag{33}$$

and

$$\sum_{k=0}^{N} a_k x^k + \sum_{i=1}^{M} \frac{r_i \rho_i}{\rho_i + x} = 0. \tag{34}$$

Definition B.1 *The characteristic vector of the OIDE (33) is defined to be*

$$(N, M, a_0, a_1, \cdots, a_N, r_1, \cdots, r_M, \rho_1, \cdots, \rho_M),$$

where N and M represent the order of the differential equation part and the number of integral terms, respectively, a_0, a_1, $\cdots$, a_N correspond to the coefficients of the differential terms, and r_1, $\cdots$, r_M, ρ_1, $\cdots$, ρ_M are associated with the integral parts.

Definition B.2 *The characteristic vector of the equation (34) is defined to be*

$$(N, M, a_0, a_1, \cdots, a_N, r_1, \cdots, r_M, \rho_1, \cdots, \rho_M),$$

where N and M represent the order of the polynomial part and the number of fraction terms, respectively, a_0, a_1, $\cdots$, a_N correspond to the coefficients of the polynomial terms, and r_1, $\cdots$, r_M, ρ_1, $\cdots$, ρ_M are associated with the fraction parts.

Since the OIDE (33) and the equation (34) can be uniquely determined by their characteristic vectors, we can simply use their characteristic vectors to represent them.

We can see that the OIDE (31) and the equation (32) are special cases of (33) and (34), respectively. Furthermore, using the above definitions, we have shown that the characteristic vectors of the OIDE (31) and the equation (32) are identical and are given by

$$(2, M, -\lambda - \alpha, \mu, \frac{\sigma^2}{2}, \lambda r_1, \cdots, \lambda r_M, \rho_1, \cdots, \rho_M).$$

In general, we define two operators: $\mathcal{A}$ and $\mathcal{B}$, which act on the OIDE (33) and the function (34) respectively to produce a new OIDE and a new function with the same types as before. Let us expatiate on the details of the two operators in turn.

The operation of the operator $\mathcal{A}$ on the OIDE (33) w.r.t. ρ_M includes the following two steps:

Step 1: Multiplying both sides of (33) by $e^{\rho_M x}$ yields

$$\sum_{k=0}^{N} a_k e^{\rho_M x} u^{(k)}(x) + r_M \rho_M \int_{\pm\infty}^{x} u(z) e^{\rho_M z} dz + \sum_{i=1}^{M-1} \left(r_i \rho_i e^{(\rho_M - \rho_i)x} \int_{\pm\infty}^{x} u(z) e^{\rho_i z} dz \right) = 0. \tag{35}$$

Step 2: Taking derivative on (35) w.r.t. x and then multiplying both sides by $e^{-\rho_M x}$ yields a new OIDE

$$a_N u^{(N+1)}(x) + \sum_{k=1}^{N} (\rho_M a_k + a_{k-1}) u^{(k)}(x) + \left(\rho_M a_0 + \sum_{i=1}^{M} r_i \rho_i \right) u(x) + \sum_{i=1}^{M-1} \left(r_i (\rho_M - \rho_i) \rho_i e^{-\rho_i x} \int_{\pm\infty}^{x} u(z) e^{\rho_i z} dz \right) = 0. \tag{36}$$

By performing the operator $\mathcal{A}$ on the OIDE (33) w.r.t. ρ_M, we decrease the number of integrals of OIDE (33) by one, and increase the order of the differentiation by one as well. Moreover, the new OIDE (36) has a characteristic vector given by: $\Big(N+1,\ M-1,\ \rho_M a_0 + \sum_{i=1}^{M} r_i \rho_i,\ \rho_M a_1 + a_0,\ \cdots,\ \rho_M a_N + a_{N-1},\ a_N, r_1(\rho_M - \rho_1),\ \cdots, r_{M-1}(\rho_M - \rho_{M-1}),\ \rho_1,\ \cdots,\ \rho_{M-1}\Big)$.

To summarize, the operator $\mathcal{A}$ can be defined as follows:

Definition B.3 *An operator $\mathcal{A}$ acting on an OIDE (33) w.r.t. ρ_M produces another OIDE and is defined as:*

$$\begin{aligned} & \mathcal{A}_{\rho_M}(N, M, a_0, a_1, \cdots, a_N, r_1, \cdots, r_M, \rho_1, \cdots, \rho_M) \\ = \ & (N+1, M-1, \rho_M a_0 + \sum_{i=1}^{M} r_i \rho_i, \rho_M a_1 + a_0, \cdots, \rho_M a_N + a_{N-1}, a_N, \\ & r_1(\rho_M - \rho_1), \cdots, r_{M-1}(\rho_M - \rho_{M-1}), \rho_1, \cdots, \rho_{M-1}), \end{aligned}$$

where the original OIDE and the new OIDE are simply represented by their respective characteristic vectors.

On the other hand, the operation of the operator $\mathcal{B}$ on the equation (34) w.r.t. ρ_M can be done by multiplying both sides of (34) by $\rho_M + x$ and the final equation is given by:

$$a_N x^{N+1} + \sum_{k=1}^{N}(\rho_M a_k + a_{k-1})x^k + \left(\rho_M a_0 + \sum_{i=1}^{M} r_i\rho_i\right) + \sum_{i=1}^{M-1}\frac{r_i(\rho_M - \rho_i)\rho_i}{\rho_i + x} = 0. \qquad (37)$$

Thus, by performing the operator $\mathcal{B}$ on the equation (34), we decrease the number of fractions of (34) by one, and increase the order of the polynomial by one as well. Moreover, the final equation (37) has the same form as before, and its characteristic vector is given by: $\Big(N+1,$ $M-1,$ $\rho_M a_0 + \sum_{i=1}^{M} r_i\rho_i,$ $\rho_M a_1 + a_0, \cdots, \rho_M a_N + a_{N-1},$ $a_N,$ $r_1(\rho_M - \rho_1), \cdots, r_{M-1}(\rho_M - \rho_{M-1}),$ $\rho_1, \cdots, \rho_{M-1}\Big)$, which is the same as that of the OIDE (36).

To summarize, the operator $\mathcal{B}$ can be defined as follows:

Definition B.4 *An operator $\mathcal{B}$ acting on an equation (34) w.r.t. ρ_M produces a new equation and is defined as:*

$$\begin{aligned} & \mathcal{B}_{\rho_M}(N, M, a_0, a_1, \cdots, a_N, r_1, \cdots, r_M, \rho_1, \cdots, \rho_M) \\ = \; & \Big(N+1, M-1, \rho_M a_0 + \sum_{i=1}^{M} r_i\rho_i, \rho_M a_1 + a_0, \cdots, \rho_M a_N + a_{N-1}, a_N, \\ & r_1(\rho_M - \rho_1), \cdots, r_{M-1}(\rho_M - \rho_{M-1}), \rho_1, \cdots, \rho_{M-1}\Big), \end{aligned}$$

where the original equation and the new equation are simply represented by their respective characteristic vectors.

The description above implies that operators $\mathcal{A}$ and $\mathcal{B}$, acting on an OIDE (33) and an equation (34) w.r.t. ρ_j for any $j = 1, 2, \cdots, M$ respectively, do not change the equivalence of their characteristic vectors. We state this result as the following lemma.

Lemma B.5 *Given an OIDE (33) and an equation (34), which have the same characteristic vectors, performing the operators $\mathcal{A}$ and $\mathcal{B}$ on (33) and (34) w.r.t. ρ_j for any $j = 1, 2, \cdots, M$ respectively, yields a new OIDE with the number of differential terms increased by one and the number of integral terms decreased by one; and a new equation with the number of the polynomial terms increased by one and the number of fraction terms decreased by one. Moreover the new OIDE and the new equation still have the same characteristic vectors.*

Using the above lemma, performing the operator $\mathcal{A}$ and $\mathcal{B}$ on (31) and (32) repeatedly for M times changes the OIDE (31) to an $M + 2$ order homogeneous linear ODE with constant coefficients, and changes the equation (32) to an $M + 2$ order polynomial equation, which is exactly $(G(x)-\alpha)(\rho_1+x)\cdots(\rho_M+x)=0$. Lemma B.5 tells us that the final OIDE, actually an ODE, should have the same characteristic vectors as the final equation. Therefore, we conclude that the equation $(G(x)-\alpha)(\rho_1+x)\cdots(\rho_M+x)=0$ is actually the characteristic function of the $M+2$ order homogeneous linear ODE with constant coefficients, and Theorem 3.2 is proved. □

C Proof of Theorem 3.3

Proof. For notation simplicity, we use $\beta_1, \cdots, \beta_{m+1}, \gamma_1, \cdots, \gamma_{n+1}$ to represent $\beta_{1,\alpha}, \cdots, \beta_{m+1,\alpha}$, $\gamma_{1,\alpha}, \cdots, \gamma_{n+1,\alpha}$, which are the $(m+n+2)$ roots of the equation $G(x)=\alpha$ and satisfy (7).

If $x \geq b$, $E^x[e^{-\alpha\tau_b+\theta X_{\tau_b}}]=e^{\theta x}$ because $\tau_b=0$ and $X_0=x$. When $x<b$, $E^x[e^{-\alpha\tau_b+\theta X_{\tau_b}}]$, as a function of x, is closely related to the solution of the OIDE $(Lu)(x)=\alpha u(x)$. If we can solve this OIDE explicitly, then we can prove this theorem by using a similar technique as in [35]. Therefore, we intend to prove Theorem 3.3 in the following three steps.

(I). Prove that the OIDE $(Lu)(x)=\alpha u(x)$ when $x<b$ and $u(x)=e^{\theta x}$ when $x \geq b$ has a unique solution given that $u(x)$ satisfies the following two conditions: (i) the smooth pasting condition, i.e., $u(b-)=u(b+)=e^{\theta b}$ and (ii) the bounded boundary condition, i.e., $u(x)$ is bounded near $-\infty$. Furthermore, the solution can be expressed explicitly as $u(x)=\sum_{i=1}^{m+1} w_i e^{\beta_i x}$ when $x<b$, where w is the solution of the linear system $ABw=J$. Here A, B and J are the same as in Theorem 3.3;

(II). Prove that the following function

$$u(x) := \begin{cases} e^{\theta x} & \text{if } x \geq b \\ \sum_{l=1}^{m+1} w_l e^{\beta_l x} & \text{if } x < b \end{cases}$$

is exactly $E^x[e^{-\alpha\tau_b+\theta X_{\tau_b}}]$.

(III). The matrix A is nonsingular.

Proof of (I): Applying Theorem 3.2 yields that when $x<b$, $u(x)$ is of the following form

$$u(x)=\sum_{i=1}^{m+1} w_i e^{\beta_i x}+\sum_{j=1}^{n+1} v_j e^{\gamma_j x},$$

where $w_1, w_2, \cdots, w_{m+1}, v_1, v_2, \cdots, v_{n+1}$ are undetermined constants. Since $u(x)$ is bounded near $-\infty$, we conclude that

$$v_1=v_2=\cdots=v_{n+1}=0.$$

On the other hand, the smooth pasting condition implies that

$$w_1 e^{\beta_1 b}+w_2 e^{\beta_2 b}+\cdots+w_{m+1} e^{\beta_{m+1} b}=e^{\theta b}. \tag{38}$$

It is easy to see that to determine $w_1, w_2, \cdots, w_{m+1}$, m more equations are required. To achieve this, we substitute $u(x) = \sum_{l=1}^{m+1} w_l e^{\beta_l x}$ back into the original OIDE. Note that for any $x < b$ and $\theta < \eta_1$,

$$\begin{aligned}
&\int_{-\infty}^{+\infty} u(x+y) f_Y(y) dy \\
= \;& \int_{-\infty}^{0} u(x+y) \left(q_d \sum_{j=1}^{n} q_j \theta_j e^{\theta_j y} \right) dy + \int_{0}^{b-x} u(x+y) \left(p_u \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i y} \right) dy \\
& + \int_{b-x}^{+\infty} e^{\theta(x+y)} \left(p_u \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i y} \right) dy \\
= \;& q_d \sum_{j=1}^{n} \left(q_j \theta_j e^{-\theta_j x} \int_{-\infty}^{x} u(z) e^{\theta_j z} dz \right) + p_u \sum_{i=1}^{m} \left(p_i \eta_i e^{\eta_i x} \int_{x}^{b} u(z) e^{-\eta_i z} dz \right) \\
& + \int_{b-x}^{+\infty} e^{\theta x} \left(p_u \sum_{i=1}^{m} p_i \eta_i e^{-(\eta_i - \theta) y} \right) dy \\
= \;& q_d \sum_{j=1}^{n} \sum_{l=1}^{m+1} \left(w_l \frac{q_j \theta_j}{\theta_j + \beta_l} e^{\beta_l x} \right) + p_u \sum_{i=1}^{m} \sum_{l=1}^{m+1} \left(w_l \frac{p_i \eta_i}{\eta_i - \beta_l} e^{\beta_l x} \right) \\
& - p_u \sum_{i=1}^{m} \sum_{l=1}^{m+1} \left(w_l \frac{p_i \eta_i}{\eta_i - \beta_l} e^{\beta_l b} e^{\eta_i (x-b)} \right) + p_u \sum_{i=1}^{m} \left(\frac{p_i \eta_i}{\eta_i - \theta} e^{\theta b} e^{\eta_i (x-b)} \right).
\end{aligned}$$

So for any $x < b$ and $\theta < \eta_1$, we have:

$$\begin{aligned}
& \frac{\sigma^2}{2} u''(x) + \mu u'(x) - (\lambda + \alpha) u(x) + \lambda \int_{-\infty}^{+\infty} u(x+y) f_Y(y) dy \\
= \;& \sum_{l=1}^{m+1} \left[w_l e^{\beta_l x} (G(\beta_l) - \alpha) \right] - \lambda p_u \sum_{i=1}^{m} \left\{ p_i e^{\eta_i (x-b)} \left[\sum_{l=1}^{m+1} \left(w_l \frac{\eta_i e^{\beta_l b}}{\eta_i - \beta_l} \right) - \frac{\eta_i}{\eta_i - \theta} e^{\theta b} \right] \right\} \\
= \;& -\lambda p_u \sum_{i=1}^{m} \left\{ p_i e^{\eta_i (x-b)} \left[\sum_{l=1}^{m+1} \left(w_l \frac{\eta_i e^{\beta_l b}}{\eta_i - \beta_l} \right) - \frac{\eta_i}{\eta_i - \theta} e^{\theta b} \right] \right\} \\
= \;& 0.
\end{aligned}$$

Because $\eta_1, \eta_2, \cdots, \eta_m$ are distinct, we conclude that:

$$\sum_{l=1}^{m+1} \left(\frac{\eta_i e^{\beta_l b}}{\eta_i - \beta_l} w_l \right) = \frac{\eta_i}{\eta_i - \theta} e^{\theta b} \qquad \text{for } i = 1, 2, \cdots, m \tag{39}$$

Combining (38) and (39) results in $ABw = J$ immediately.

Proof of (II): Consider the following function

$$u(x) = \begin{cases} e^{\theta x} & \text{if } x \geq b \\ \sum_{l=1}^{m+1} w_l e^{\beta_l x} & \text{if } x < b. \end{cases}$$

Now we begin to prove this function $u(x)$ is exactly $E^x[e^{-\alpha\tau_b+\theta X_{\tau_b}}]$.

First, from Part (I), we know

$$-\alpha u(x) + (Lu)(x) = 0, \quad \text{for any } x < b. \tag{40}$$

Since $u(x)$ may not be continuously differentiable (e.g., when $m = n = 1$, i.e., for the DEP, Kou and Wang [35] pointed out that if $\theta = 0$, then $u'(b-) > 0 = u'(b+)$), we cannot apply Itô's formula to the process $\{e^{-\alpha t}u(X_t) : t \geq 0\}$ directly. However, we can construct a series of functions $\{u_n(x) : n = 1, 2, \cdots\}$ smooth enough to approximate $u(x)$. More precisely, $\{u_n(x) : n = 1, 2, \cdots\}$ can be selected such that: (1) $u_n(x)$ converges to $u(x)$ as n goes to $+\infty$ for any x; (2) $u_n(x)$ is twice continuously differentiable for any $n = 1, 2, \cdots$; (3) $u_n(x) \equiv u(x)$ for any $x \leq b$ or $x \geq b + \frac{1}{n}$; (4) for all $x \in (b, b + \frac{1}{n})$ and n, $0 \leq u_n(x) \leq M_1$, where M_1 is a positive constant. In addition, for all $x \leq b$, we have $|u_n(x)| \equiv |u(x)| \leq M_2$ for any $n = 1, 2, \cdots$, where $M_2 := \sum_{l=1}^{m+1} |w_l| e^{\beta_l b}$.

Similar algebra as on p. 510 of Kou and Wang [35] yields that for any $x < b$,

$$(Lu_n)(x) = \alpha u(x) + \lambda \int_{b-x}^{b-x+1/n} [u_n(x+y) - u(x+y)] f_Y(y) dy,$$

thanks to (40). Note that by construction,

$$\begin{aligned} |u_n(x) - u(x)| &\leq \max_{x\in(b,b+1/n)} u_n(x) + \max_{x\in(b,b+1/n)} u(x) \\ &\leq M_1 + \max(e^{\theta b}, e^{\theta(b+1)}) =: M \quad \text{for any } x \text{ and } n. \end{aligned}$$

We obtain that

$$\begin{aligned} & |-\alpha u_n(x) + (Lu_n)(x)| \\ \leq\ & \lambda p_u \sum_{i=1}^{m} |p_i| \eta_i \int_{b-x}^{b-x+1/n} |u_n(x+y) - u(x+y)| dy \\ \leq\ & \frac{\lambda p_u M \sum_{i=1}^{m} |p_i| \eta_i}{n} \to 0, \quad \text{uniformly for all } x < b, \text{ as } n \to +\infty. \end{aligned} \tag{41}$$

Applying Itô's formula to the jump processes $\{e^{-\alpha t}u_n(X_t) : t \geq 0\}$ for any $n = 1, 2, \cdots$, we obtain a series of local martingales $\{M_t^{(n)} : t \geq 0\}$ for $n = 1, 2, \cdots$, as follows:

$$M_t^{(n)} := e^{-\alpha(t\wedge\tau_b)} u_n(X_{t\wedge\tau_b}) - \int_0^{t\wedge\tau_b} e^{-\alpha s}[-\alpha u_n(X_s) + (Lu_n)(X_s)] ds.$$

Note that for any $t \geq 0$ and $n = 1, 2 \cdots$,

$$\begin{aligned} & |u_n(X_{t\wedge\tau_b})| \\ =\ & |u_n(X_{t\wedge\tau_b}) I_{\{t<\tau_b\}} + u_n(X_{t\wedge\tau_b}) I_{\{t\geq\tau_b, X_{\tau_b}<b+1/n\}} + u_n(X_{t\wedge\tau_b}) I_{\{t\geq\tau_b, X_{\tau_b}>b+1/n\}}| \\ \leq\ & \max(M_2, M_1, e^{\theta X_{\tau_b}} I_{\{t\geq\tau_b\}}), \end{aligned} \tag{42}$$

where the inequality holds because $|u_n(x)|$ is bounded by M_2 when $x \le b$, bounded by M_1 when $x \in (b, b+1/n)$, and equal to $u(x) \equiv e^{\theta x}$ when $x \ge b+1/n$. Hence, for any $t \ge 0$ and $n = 1, 2\cdots$, we have

$$\begin{aligned} |M_t^{(n)}| &\le |u_n(X_{t\wedge\tau_b})| + \int_0^{t\wedge\tau_b -} |-\alpha u_n(X_s) + (Lu_n)(X_s)| ds \\ &\le \max(M_1, M_2, e^{\theta X_{\tau_b}} I_{\{t\ge\tau_b\}}) + \frac{\lambda p_u M \sum_{i=1}^m |p_i|\eta_i}{n} t. \end{aligned} \tag{43}$$

If $\theta \le 0$, then

$$\sup_{0\le s\le t} |M_s^{(n)}| \le \max(M_1, M_2, e^{\theta b}) + \frac{\lambda p_u M \sum_{i=1}^m |p_i|\eta_i}{n} t,$$

which implies that $\{M_t^{(n)} : t \ge 0\}$ is a true martingale for any $n = 1, 2, \cdots$.

If $\theta \in (0, \eta_1)$, we have

$$\sup_{0\le s\le t} |M_s^{(n)}| \le \max(M_1, M_2, e^{\theta \sup_{0\le s\le t} X_s}) + \frac{\lambda p_u M \sum_{i=1}^m |p_i|\eta_i}{n} t,$$

In order to show that in this case, $\{M_t^{(n)} : t \ge 0\}$ is also a true martingale for any $n = 1, 2, \cdots$, it suffices to prove that

$$E^x \left[e^{\theta \sup_{0\le s\le t} X_s} \right] < +\infty, \qquad \text{for any } t \ge 0.$$

Note that

$$\sup_{0\le s\le t} X_s \le X_0 + |\mu| t + \sigma \max_{0\le s\le t} W_s + \sum_{i=1}^{N_t} Y_i^+,$$

where $Y_i^+ := \max\{Y_i, 0\}$, and $\max_{0\le s\le t} W_s$ has the same distribution as $|W_t|$. It follows that for any $t \ge 0$ and $\theta \in (0, \eta_1)$,

$$E^x \left[e^{\theta \sup_{0\le s\le t} X_s} \right] \le e^{\theta(|x| + |\mu| t)} \cdot E\left[e^{\theta\sigma |W_t|} \right] \cdot E\left[e^{\theta \sum_{i=1}^{N_t} Y_i^+} \right] < +\infty \tag{44}$$

because on the one hand, the fact that $\theta < \eta_1$ leads to

$$E\left[e^{\theta \sum_{i=1}^{N_t} Y_i^+} \right] = e^{\lambda t (E e^{\theta Y_1^+} - 1)} = e^{\lambda t (q_d + p_u \sum_{i=1}^m \frac{p_i \eta_i}{\eta_i - \theta} - 1)} < +\infty,$$

and on the other hand, $E\left[e^{\theta\sigma |W_t|} \right] = 2 e^{\theta^2 \sigma^2 t/2} \Phi(\theta\sigma\sqrt{t}) < +\infty$, where $\Phi(\cdot)$ is the cdf of the standard normal random variable.

Therefore, we conclude that for any $\theta < \eta_1$, $\{M_t^{(n)} : t \ge 0\}$ is a true martingale for any $n = 1, 2, \cdots$. Thus, for any $t \ge 0$ and $x < b$, we have

$$\begin{aligned} E^x M_t^{(n)} &= E^x \left[e^{-\alpha(t\wedge\tau_b)} u_n(X_{t\wedge\tau_b}) \right] - E^x \Big[\int_0^{t\wedge\tau_b} e^{-\alpha s} [-\alpha u_n(X_s) + (Lu_n)(X_s)] ds \Big] \\ &= E^x M_0^{(n)} = u_n(X_0) = u_n(x) = u(x), \end{aligned}$$

where $x = X_0$ is the starting point of $\{X_t : t \geq 0\}$. Letting n go to $+\infty$ and applying the dominated convergence theorem (DCT) yields that

$$\lim_{n\to+\infty} E^x\left[e^{-\alpha(t\wedge\tau_b)}u_n(X_{t\wedge\tau_b})\right] = E^x\left[e^{-\alpha(t\wedge\tau_b)}u(X_{t\wedge\tau_b})\right], \tag{45}$$

and

$$\begin{aligned} &\lim_{n\to+\infty} E^x\left[\int_0^{t\wedge\tau_b} e^{-\alpha s}[-\alpha u_n(X_s) + (Lu_n)(X_s)]ds\right] \\ =& \lim_{n\to+\infty} E^x\left[\int_0^{t\wedge\tau_b -} e^{-\alpha s}[-\alpha u_n(X_s) + (Lu_n)(X_s)]ds\right] = 0, \end{aligned} \tag{46}$$

where the DCT applies for (45) because (42) and

$$E^x[e^{\theta X_{\tau_b}} I_{\{t\geq\tau_b\}}] \leq e^{\theta b} I_{\{\theta\leq 0\}} + I_{\{\theta>0\}} \cdot E^x\left[e^{\theta \sup_{0\leq s\leq t} X_s}\right] < +\infty, \quad \text{for any } x\in\mathbb{R}$$

thanks to (44); while the last equality in (46) holds thanks to the uniform convergence in (41). Consequently, we obtain that for any $t \geq 0$ and $x < b$,

$$\begin{aligned} u(x) =& E^x\left[e^{-\alpha(t\wedge\tau_b)}u(X_{t\wedge\tau_b})\right] \\ =& E^x\left[e^{-\alpha\tau_b}u(X_{\tau_b})I_{\{\tau_b\leq t\}}\right] + E^x\left[e^{-\alpha t}u(X_t)I_{\{\tau_b>t\}}\right]. \\ =& E^x\left[e^{-\alpha\tau_b+\theta X_{\tau_b}}I_{\{\tau_b\leq t\}}\right] + E^x\left[e^{-\alpha t}u(X_t)I_{\{\tau_b>t\}}\right], \end{aligned} \tag{47}$$

where the last equality holds because $u(X_{\tau_b}) = e^{\theta X_{\tau_b}}$ on the set $\{\tau_b < +\infty\}$

Note that on the set $\{\tau_b > t\}$, we have $X_t < b$ and hence $|u(X_t)| \leq M_2$. It follows from the DCT that the second term on the right-hand side of (47) converges to zero as t goes to infinity. Besides, applying the monotone convergence theorem yields that the first term on the right-hand side of (47) converges to $E^x\left[e^{-\alpha\tau_b+\theta X_{\tau_b}}I_{\{\tau_b<+\infty\}}\right]$ as t goes to infinity.

Consequently, letting t go to $+\infty$ in (47) yields

$$u(x) = E^x\left[e^{-\alpha\tau_b+\theta X_{\tau_b}}I_{\{\tau_b<+\infty\}}\right] = E^x\left[e^{-\alpha\tau_b+\theta X_{\tau_b}}\right], \quad \text{for any } x < b.$$

Proof of (III). We will prove it by contradiction. Assume that A is singular. Then the $(m+1)$ row vectors of A are linearly dependant. Thus, there exist $(m+1)$ constants $k_0, k_1, \cdots, k_m$, not all of which are zero, such that

$$k_0 + k_1\frac{\eta_1}{\eta_1-\beta_l} + k_2\frac{\eta_2}{\eta_2-\beta_l} + \cdots + k_m\frac{\eta_m}{\eta_m-\beta_l} = 0, \qquad \text{for } l = 1, 2, \cdots, m+1.$$

It implies that the function $f_A(\beta) := k_0 + k_1\frac{\eta_1}{\eta_1-\beta} + k_2\frac{\eta_2}{\eta_2-\beta} + \cdots + k_m\frac{\eta_m}{\eta_m-\beta}$ has at least $(m+1)$ roots: $\beta_1, \beta_2, \cdots, \beta_{m+1}$. However, $f_A(\beta)\prod_{i=1}^m\left[(\eta_i-\beta)I_{\{k_i\neq 0\}}\right]$ is a polynomial with an order at most m, and therefore has at most m roots. Thus, $f_A(\beta)$ also has at most m roots, which renders a contradiction. Therefore, the matrix A is non-singular. □

D Proof of Theorem 3.4

Proof. Note that for $\theta > 0$,

$$\begin{aligned}
\mathcal{L}(\alpha,\theta) &= \int_0^{+\infty}\int_{-\infty}^{+\infty} e^{-\theta\hat{a}-\alpha t} E^0\left[I_{\{X_t\geq -\hat{a},\tau_b\leq t\}}\right] d\hat{a}dt \\
&= E^0\left\{\int_{\tau_b}^{+\infty}\left[\int_{-X_t}^{+\infty} e^{-\theta\hat{a}-\alpha t}d\hat{a}\right]dt\right\} \\
&= \frac{1}{\theta}E^0\left\{\int_{\tau_b}^{+\infty} e^{\theta X_t-\alpha t}dt\right\} \\
&= \frac{1}{\theta}E^0\left[e^{-\alpha\tau_b}\int_0^{+\infty} e^{\theta X_{t+\tau_b}-\alpha t}dt\right].
\end{aligned}$$

On the other hand, the strong Markov property implies that for any $\alpha > \max(G(\theta), 0)$,

$$\begin{aligned}
& E^0\left[e^{-\alpha\tau_b}\int_0^{+\infty} e^{\theta X_{t+\tau_b}-\alpha t}dt\Big|\mathcal{F}_{\tau_b}\right] \\
=\ & e^{-\alpha\tau_b+\theta X_{\tau_b}}E^0\left[\int_0^{+\infty} e^{\theta X_t-\alpha t}dt\right] \\
=\ & e^{-\alpha\tau_b+\theta X_{\tau_b}}\int_0^{+\infty} e^{(G(\theta)-\alpha)t}dt = \frac{e^{-\alpha\tau_b+\theta X_{\tau_b}}}{\alpha-G(\theta)}.
\end{aligned}$$

Combining them together and applying (9) yields (16) immediately. □

E Distribution of the Running Maxima

This section gives the closed-form Laplace transform of the running maxima of the process X_t, i.e. $M_X(t) := \max_{\{0\leq s\leq t\}} X_s$, where $X_0 = 0$.

Theorem E.1 *Denote by $\mathcal{L}_M(s)$ the Laplace transforms of $E^0[e^{vM_X(t)}]$ w.r.t. t evaluated at sufficiently large $s > 0$. More precisely, $\mathcal{L}_M(s) = \int_0^\infty e^{-st}E^0[e^{vM_X(t)}]dt$, Then for any $v \in (-\infty, \beta_{1,s})$, we have*

$$\mathcal{L}_M(s) = \frac{1}{s} + \frac{v}{s}\sum_{l=1}^{m+1}\frac{d_l}{\beta_{l,s}-v}, \qquad s > 0, \tag{48}$$

where $\beta_{1,s}, \beta_{2,s}, \cdots, \beta_{m+1,s}$ are the $(m+1)$ positive roots of the equation $G(x) = s$ in (3) and $d = (d_1, \cdots, d_{m+1})'$ is uniquely determined by $Ad = \mathbf{1}$, where A associated with s is defined in Theorem 3.3 and $\mathbf{1} = (1, 1, \cdots, 1)'$.

Before proving Theorem E.1, we present the following Lemma E.2.

Lemma E.2 *Assume that $v \in (-\infty, \beta_{1,s})$, where $\beta_{1,s}$ is the smallest positive root of the equation $G(x) = s$ for sufficiently large $s > 0$. Then for any $t > 0$, we have that*

$$\lim_{y \to +\infty} e^{vy} P[M_X(t) \geq y] = 0 \tag{49}$$

Proof. When $v \leq 0$, the conclusion is trivial. When $v \in (0, \beta_{1,s})$, note that the process $\{e^{\theta X_t - G(\theta)t} : t \geq 0\}$ is a martingale for any $\theta \in (-\theta_1, \eta_1)$ since $G(\theta)$ is the exponent of the Lévy process $\{X_t : t \geq 0\}$. Fix $\theta \in (v, \beta_{1,s})$ such that $G(\theta) > 0$. This θ must exist because $G(\beta_{1,s}) = s > 0$ and $G(\theta)$ is continuous in the interval $(v, \beta_{1,s})$. Note that

$$e^{\theta y} P(\tau_y \leq t) \leq E[e^{\theta X_{t \wedge \tau_y}}] \leq e^{G(\theta)t} E[e^{\theta X_{t \wedge \tau_y} - G(\theta)(t \wedge \tau_y)}] \leq e^{G(\theta)t},$$

where the last equality holds owing to optional sampling theorem. So for any $y > 0$, we have

$$e^{vy} P[M_X(t) \geq y] = e^{(v-\theta)y} e^{\theta y} P[M_X(t) \geq y] = e^{(v-\theta)y} e^{\theta y} P(\tau_y \leq t) \leq e^{(v-\theta)y} e^{G(\theta)t}.$$

Note that $\theta > v$, so letting y go to infinity completes the proof of (49). □

Now we are ready to prove Theorem E.1.

Proof. First, using integration by parts and applying (49) leads to

$$\begin{aligned} E e^{v M_X(t)} &= \int_0^\infty e^{vy} f_M(y) dy = -\int_0^\infty e^{vy} dP(M_X(t) \geq y) \\ &= 1 + v \int_0^\infty P(M_X(t) \geq y) e^{vy} dy, \end{aligned}$$

where $f_M(y)$ represents the pdf of $M_X(t)$. Accordingly, $\mathcal{L}_M(s)$ can be expressed as

$$\begin{aligned} \mathcal{L}_M(s) &= \int_0^\infty e^{-st} E[e^{v M_X(t)}] dt \\ &= \frac{1}{s} + v \int_0^\infty e^{-st} \left[\int_0^\infty e^{vy} P(M_X(t) \geq y) dy \right] dt \\ &= \frac{1}{s} + v \int_0^\infty e^{vy} \left[\int_0^\infty e^{-st} P(M_X(t) \geq y) dt \right] dy. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\int_0^\infty e^{-st} P(M_X(t) \geq y) dt \\ =& -\frac{1}{s} \int_0^\infty P(M_X(t) \geq y) de^{-st} \\ =& \frac{1}{s} \int_0^\infty e^{-st} dP(M_X(t) \geq y) = \frac{1}{s} \int_0^\infty e^{-st} dP(\tau_y \leq t) = \frac{1}{s} E e^{-s\tau_y}. \end{aligned}$$

Thus, by (13) with $x = 0$,

$$\begin{aligned}\mathcal{L}_M(s) &= \frac{1}{s} + v\int_0^\infty e^{vy}\frac{1}{s}Ee^{-s\tau_y}dy \\ &= \frac{1}{s} + \frac{v}{s}\sum_{l=1}^{m+1} d_l \int_0^\infty e^{vy}e^{-\beta_{l,s}y}dy = \frac{1}{s} + \frac{v}{s}\sum_{l=1}^{m+1}\frac{d_l}{\beta_{l,s} - v},\end{aligned}$$

from which the conclusion follows. □