E-Companion of "On the Measurement of Economic Tail Risk" by Steven Kou and Xianhua Peng

EC.1. Proof of Lemma 1

Proof. Without loss of generality, we only need to prove for the case $s=1$, as ρ satisfies Axioms A1-A5 if and only if $\frac{1}{s}\rho$ satisfies Axioms A1-A5 (with $s=1$ in Axiom A3).

The "only if" part. First, we show that (2) holds for any $X \in \mathcal{L}^\infty(\Omega, \mathcal{F}, P)$. Define the set function $\nu(E) := \rho(1_E), E \in \mathcal{F}$. Then, it follows from Axiom A2 and A3 that ν is monotonic, $\nu(\emptyset)=0$, and $\nu(\Omega)=1$. For $M \geq 1$, define $\mathcal{L}^M := \{X \mid |X| \leq M\}$. For any $X \in \mathcal{L}^\infty(\Omega, \mathcal{F}, P)$, let M_0 be the essential supremum of $|X|$ and denote $X^{M_0} := \min(M_0, \max(X, -M_0))$. Then $X^{M_0} \in \mathcal{L}^{M_0}$ and $X = X^{M_0}$ a.s., which implies that $\rho(X) = \rho(X^{M_0})$ (by Axiom A4) and $\nu(X > x) = \nu(X^{M_0} > x)$, $\forall x$. Since ρ satisfies Axioms A1-A3 on $\mathcal{L}^\infty(\Omega, \mathcal{F}, P)$, it follows that ρ satisfies the conditions (i)-(iii) of the Corollary in Section 3 of Schmeidler (1986) (with $B(K)$ in the corollary defined to be $\mathcal{L}^{1+M_0}$). Hence, it follows from the Corollary that

$$\begin{aligned}\rho(X) = \rho(X^{M_0}) &= \int_0^\infty \nu(X^{M_0} > x)dx + \int_{-\infty}^0 (\nu(X^{M_0} > x) - 1)dx \\ &= \int_0^\infty \nu(X > x)dx + \int_{-\infty}^0 (\nu(X > x) - 1)dx. \qquad \text{(EC.1)}\end{aligned}$$

Let U be a uniform $U(0,1)$ random variable. Define the function h such that $h(0)=0$, $h(1)=1$, and $h(p) := \rho(1_{\{U \leq p\}})$, $\forall p \in (0,1)$. By Axiom A4, $h(\cdot)$ satisfies $\nu(A) = h(P(A))$ for all A. Therefore, by (EC.1), (2) holds for X. In addition, for any $0 < q < p < 1$, $h(p) = \rho(1_{\{U \leq p\}}) \geq \rho(1_{\{U \leq q\}}) = h(q)$. Hence, h is an increasing function.

Second, we show that (2) holds for any (possibly unbounded) $X \in \mathcal{X}$. For $M > 0$, since X^M belongs to $\mathcal{L}^\infty(\Omega, \mathcal{F}, P)$, it follows that (2) holds for X^M, which implies

$$\begin{aligned}\rho(X^M) &= \int_0^\infty h(P(X^M > x))dx + \int_{-\infty}^0 (h(P(X^M > x)) - 1)dx \\ &= \int_0^M h(P(X > x))dx + \int_{-M}^0 (h(P(X > x)) - 1)dx.\end{aligned}$$

Letting $M \to \infty$ on both sides of the above equation and using Axiom A5, we conclude that (2) holds for X.

The "if" part. Suppose h is a distortion function and ρ is defined by (2). Define the set function $\nu(A) := h(P(A)), \forall A \in \mathcal{F}$. Then $\rho(X)$ is the Choquet integral of X with respect to ν. By definition of ρ and simple verification, ρ satisfies Axioms A2-A5. It follows from Denneberg (1994, Proposition 5.1) that ρ satisfies positive homogeneity and comonotonic additivity, which implies that ρ satisfies Axiom A1. $\square$

EC.2. Proof of Theorem 1 and Theorem 2

First, we give the following definition; a similar definition for a set-valued (not single-valued) statistical functional is given in Osband (1985) and Gneiting (2011).

Definition EC.1. A single-valued statistical functional ρ is said to have convex level sets with respect to $\mathcal{P}$, if for any two distributions $F_1 \in \mathcal{P}$ and $F_2 \in \mathcal{P}$ and any $\lambda \in (0,1)$, $\rho(F_1) = \rho(F_2)$ and $\lambda F_1 + (1-\lambda)F_2 \in \mathcal{P}$ imply that $\rho(\lambda F_1 + (1-\lambda)F_2) = \rho(F_1)$.

The following Lemma EC.1 gives a necessary condition for a single-valued statistical functional to be general elicitable. The lemma is a variant of Proposition 2.5 of Osband (1985), Lemma 1 of Lambert, Pennock, and Shoham (2008), and Theorem 6 of Gneiting (2011), which concern set-valued statistical functionals.

Lemma EC.1. *If a single-valued statistical functional ρ is general elicitable with respect to $\mathcal{P}$, then ρ has convex level sets with respect to $\mathcal{P}$.*

Proof. Suppose ρ is general elicitable. Then there exists a forecasting objective function $S(x,y)$ such that (10) holds. For any two distribution F_1 and F_2 and any $\lambda \in (0,1)$, denote $F_\lambda := \lambda F_1 + (1-\lambda)F_2$. If $t = \rho(F_1) = \rho(F_2)$ and $F_\lambda \in \mathcal{P}$, then $t = \min\{x \mid x \in \arg\min_x \int S(x,y)dF_i(y)\}$, $i = 1,2$. Furthermore, since $\int S(x,y)dF_\lambda(y) = \lambda \int S(x,y)dF_1(y) + (1-\lambda)\int S(x,y)dF_2(y)$, it follows that $t \in \arg\min_x \int S(x,y)dF_\lambda(y)$. For any $t' \in \arg\min_x \int S(x,y)dF_\lambda(y)$, it holds that $\int S(t',y)dF_\lambda(y) \le \int S(t,y)dF_\lambda(y)$, which implies that $\lambda \int S(t',y)dF_1(y) + (1-\lambda)\int S(t',y)dF_2(y) \le \lambda \int S(t,y)dF_1(y) + (1-\lambda)\int S(t,y)dF_2(y)$. However, by definition of t, $\int S(t,y)dF_i(y) \le \int S(t',y)dF_i(y), i = 1,2$. Therefore, $\int S(t,y)dF_i(y) = \int S(t',y)dF_i(y), i = 1,2$, which implies that $t' \in \arg\min_x \int S(x,y)dF_i(y)$, $i = 1,2$. Since $t = \min\{x \mid x \in \arg\min_x \int S(x,y)dF_i(y)\}$, it follows that $t' \ge t$. Therefore, $t = \min\{x \mid x \in \arg\min_x \int S(x,y)dF_\lambda(y)\} = \rho(F_\lambda)$. $\square$

Lemma EC.2. *Let $c \in [0,1]$ be a constant. If ρ is defined in (2) with $s = 1$ and $h(u) = 1-c, \forall u \in (0,1)$, $h(0) = 0$, and $h(1) = 1$, then $\rho = c\,VaR_0 + (1-c)\,VaR_1$, where $VaR_0(F) := \inf\{x \mid F(x) > 0\}$ and $VaR_1(F) := \inf\{x \mid F(x) = 1\}$. In addition, ρ has convex level sets with respect to $\mathcal{P}^\rho = \{F \mid \rho(F)$ is well defined and finite$\}$.*

Proof. If $\mathrm{VaR}_0(F) \ge 0$, then

$$\begin{aligned}
\rho(F) &= \int_{(0,\mathrm{VaR}_0(F))} h(1-F(x))\,dx + \int_{(\mathrm{VaR}_0(F),\mathrm{VaR}_1(F))} h(1-F(x))\,dx \\
&\quad + \int_{(\mathrm{VaR}_1(F),\infty)} h(1-F(x))\,dx \\
&= \mathrm{VaR}_0(F) + (1-c)(\mathrm{VaR}_1(F) - \mathrm{VaR}_0(F)) = c\mathrm{VaR}_0(F) + (1-c)\mathrm{VaR}_1(F).
\end{aligned}$$

If $\mathrm{VaR}_0(F) < 0$, similar calculation also leads to $\rho(F) = c\mathrm{VaR}_0(F) + (1-c)\mathrm{VaR}_1(F)$.

Suppose $t=\rho(F_1)=\rho(F_2)$. Denote $F_\lambda := \lambda F_1 + (1-\lambda)F_2$, $\lambda\in(0,1)$. There are three cases:

(i) $c=0$. Then, $t=\mathrm{VaR}_1(F_1)=\mathrm{VaR}_1(F_2)$. By definition of VaR_1, $F_i(x)<1$ for $x<t$ and $F_i(x)=1$ for $x\ge t$. Hence, for any $\lambda\in(0,1)$, it holds that $F_\lambda(x)<1$ for $x<t$ and $F_\lambda(x)=1$ for $x\ge t$. Hence, $\rho(F_\lambda)=\mathrm{VaR}_1(F_\lambda)=t$.

(ii) $c\in(0,1)$. Without loss of generality, suppose $\mathrm{VaR}_0(F_1)\le \mathrm{VaR}_0(F_2)$. Since $t=c\mathrm{VaR}_0(F_1)+(1-c)\mathrm{VaR}_1(F_1)=c\mathrm{VaR}_0(F_2)+(1-c)\mathrm{VaR}_1(F_2)$, $\mathrm{VaR}_1(F_1)\ge\mathrm{VaR}_1(F_2)$. Hence, for any $\lambda\in(0,1)$, $\mathrm{VaR}_0(F_\lambda)=\mathrm{VaR}_0(F_1)$ and $\mathrm{VaR}_1(F_\lambda)=\mathrm{VaR}_1(F_1)$. Hence, $\rho(F_\lambda)=t$.

(iii) $c=1$. Then, $t=\mathrm{VaR}_0(F_1)=\mathrm{VaR}_0(F_2)$. By definition of VaR_0, $F_i(x)=0$ for $x<t$ and $F_i(x)>0$ for $x>t$. Hence, for any $\lambda\in(0,1)$, it holds that $F_\lambda(x)=0$ for $x<t$ and $F_\lambda(x)>0$ for $x>t$. Hence, $\rho(F_\lambda)=\mathrm{VaR}_0(F_\lambda)=t$. $\square$

LEMMA EC.3. *Let $\alpha\in(0,1)$ and $c\in[0,1]$. Let ρ be defined in (2) with $s=1$ and h being defined as $h(x):=(1-c)\cdot 1_{\{x=1-\alpha\}}+1_{\{x>1-\alpha\}}$. Then*

$$\rho(F)=cq_\alpha^-(F)+(1-c)q_\alpha^+(F),\ \forall F\in\mathcal{P}, \tag{EC.2}$$

where $q_\alpha^-(F):=\inf\{x\mid F(x)\ge\alpha\}$ and $q_\alpha^+(F):=\inf\{x\mid F(x)>\alpha\}$. Furthermore, ρ has convex level sets with respect to $\mathcal{P}=\{F\mid \rho(F)$ is well-defined and finite$\}$.

Proof. Define $g(x):=1-h(1-x)$, $x\in[0,1]$. Then, $g(x)=c\cdot 1_{\{x=\alpha\}}+1_{\{x>\alpha\}}$, and ρ can be represented as

$$\rho(F)=-\int_{-\infty}^0 g(F(x))dx+\int_0^\infty(1-g(F(x)))dx.$$

Note that $F(x)=\alpha$ for $x\in[q_\alpha^-(F),q_\alpha^+(F))$. Consider three cases:

(i) $q_\alpha^-(F)\ge 0$. In this case,

$$\begin{aligned}\rho(F)&=\int_0^\infty(1-g(F(x)))dx\\&=\int_{[0,q_\alpha^-(F))}(1-g(F(x)))dx+\int_{[q_\alpha^-(F),q_\alpha^+(F))}(1-g(F(x)))dx+\int_{(q_\alpha^+(F),\infty)}(1-g(F(x)))dx\\&=q_\alpha^-(F)+(1-c)(q_\alpha^+(F)-q_\alpha^-(F))=cq_\alpha^-(F)+(1-c)q_\alpha^+(F).\end{aligned}$$

(ii) $q_\alpha^-(F)<0<q_\alpha^+(F)$. In this case,

$$\rho(F)=-\int_{(q_\alpha^-(F),0)}g(F(x))dx+\int_{(0,q_\alpha^+(F))}(1-g(F(x)))dx=cq_\alpha^-(F)+(1-c)q_\alpha^+(F).$$

(iii) $q_\alpha^+(F)\le 0$. In this case,

$$\begin{aligned}\rho(F)&=-\int_{(-\infty,q_\alpha^-(F))}g(F(x))dx-\int_{(q_\alpha^-(F),q_\alpha^+(F))}g(F(x))dx-\int_{(q_\alpha^+(F),0)}g(F(x))dx\\&=-c(q_\alpha^+(F)-q_\alpha^-(F))+q_\alpha^+(F)=cq_\alpha^-(F)+(1-c)q_\alpha^+(F),\end{aligned}$$

which completes the proof of (EC.2).

We then show that ρ has convex level sets with respect to $\mathcal{P}$. Suppose that $\rho(F_1)=\rho(F_2)$. Then

$$cq_\alpha^-(F_1)+(1-c)q_\alpha^+(F_1)=cq_\alpha^-(F_2)+(1-c)q_\alpha^+(F_2). \tag{EC.3}$$

For $\lambda\in(0,1)$, define $F_\lambda:=\lambda F_1+(1-\lambda)F_2$. There are three cases:

(i) $c=0$. Then, $\rho=q_\alpha^+$. Denote $t=q_\alpha^+(F_1)=q_\alpha^+(F_2)$, then $F_i(x)>\alpha$ for $x>t$ and $F_i(x)\le\alpha$ for $x<t$, $i=1,2$. Hence, $F_\lambda(x)>\alpha$ for $x>t$ and $F_\lambda(x)\le\alpha$ for $x<t$, which implies $t=q_\alpha^+(F_\lambda)$, i.e., q_α^+ has convex level sets with respect to $\mathcal{P}$.

(ii) $c\in(0,1)$. Without loss of generality, assume $q_\alpha^-(F_1)\ge q_\alpha^-(F_2)$. Then it follows from (EC.3) that $q_\alpha^+(F_1)\le q_\alpha^+(F_2)$. Therefore, $[q_\alpha^-(F_1),q_\alpha^+(F_1)]\subset[q_\alpha^-(F_2),q_\alpha^+(F_2)]$. There are two subcases: (ii.i) $q_\alpha^-(F_1)<q_\alpha^+(F_1)$. In this case, $F_\lambda(x)<\alpha$ for $x<q_\alpha^-(F_1)$; $F_\lambda(x)=\alpha$ for $x\in[q_\alpha^-(F_1),q_\alpha^+(F_1))$; and $F_\lambda(x)>\alpha$ for $x>q_\alpha^+(F_1)$. Therefore, $q_\alpha^-(F_\lambda)=q_\alpha^-(F_1)$ and $q_\alpha^+(F_\lambda)=q_\alpha^+(F_1)$, which implies that $\rho(F_\lambda)=\rho(F_1)$. (ii.ii) $q_\alpha^-(F_1)=q_\alpha^+(F_1)$. In this case, $F_\lambda(x)<\alpha$ for $x<q_\alpha^-(F_1)$ and $F_\lambda(x)>\alpha$ for $x>q_\alpha^+(F_1)$. Therefore, $q_\alpha^-(F_\lambda)=q_\alpha^-(F_1)$ and $q_\alpha^+(F_\lambda)=q_\alpha^+(F_1)$, which implies that $\rho(F_\lambda)=\rho(F_1)$. Therefore, ρ has convex level sets.

(iii) $c=1$. Then, $\rho=q_\alpha^-=\mathrm{VaR}_\alpha$. Denote $t=q_\alpha^-(F_1)=q_\alpha^-(F_2)$, then $F_i(x)<\alpha$ for $x<t$ and $F_i(x)\ge\alpha$ for $x\ge t$, $i=1,2$. Hence, $F_\lambda(x)<\alpha$ for $x<t$ and $F_\lambda(x)\ge\alpha$ for $x\ge t$, which implies that $q_\alpha^-(F_\lambda)=t$, i.e., q_α^- has convex level sets with respect to $\mathcal{P}$. □

Next, we prove Theorem 2, which shows that among the class of risk measures that satisfy Axioms A1-A5, only four kinds of risk measures satisfy the necessary condition of being general elicitable with respect to $\mathcal{D}_{disc}$, the class of discrete distributions that have positive probabilities only on a finite number of values.

Proof of Theorem 2. Define $g(u):=1-h(1-u)$, $u\in[0,1]$. Then $g(0)=0$, $g(1)=1$, and g is increasing on $[0,1]$. And then, ρ can be represented as

$$\rho(F)=-\int_{-\infty}^0 g(F(x))dx+\int_0^\infty(1-g(F(x)))dx.$$

For a discrete distribution $F=\sum_{i=1}^n p_i\delta_{x_i}$, where $0\le x_1<x_2<\cdots<x_n$, $p_i>0$, $i=1,\dots,n$, and $\sum_{i=1}^n p_i=1$, it can be shown by simple calculation that $\rho(F)=g(p_1)x_1+\sum_{i=2}^n(g(\sum_{j=1}^i p_j)-g(\sum_{j=1}^{i-1}p_j))x_i$.

Suppose ρ has convex level sets with respect to $\mathcal{D}_{disc}$. There are three cases for g:

Case (i): for any $q\in(0,1)$, $g(q)=0$. Then $g(u)=1_{\{u=1\}}$. By Lemma EC.2 (with $c=0$), $\rho=\mathrm{VaR}_1$ and ρ has convex level sets with respect to $\mathcal{P}^\rho$.

Case (ii): there exists $q_0\in(0,1)$ such that $g(q_0)=1$ and $g(q)\in\{0,1\}$ for all $q\in(0,1)$. Let $\alpha=\inf\{q\mid g(q)=1\}$. There are three subcases: (ii.i) $\alpha=0$. Then, $g(u)=1_{\{u>0\}}$. By Lemma EC.2

(with $c=1$), $\rho=\mathrm{VaR}_0$ and ρ has convex level sets with respect to $\mathcal{P}^\rho$. (ii.ii) $\alpha\in(0,1)$ and $g(\alpha)=1$. Then, $g(u)=1_{\{u\geq\alpha\}}$. By Lemma EC.3 (with $c=1$), $\rho=q_\alpha^-=\mathrm{VaR}_\alpha$ and ρ has convex level sets with respect to $\mathcal{P}^\rho$. (ii.iii) $\alpha\in(0,1)$ and $g(\alpha)=0$. Then, $g(u)=1_{\{u>\alpha\}}$. By Lemma EC.3 (with $c=0$), $\rho=q_\alpha^+$ and ρ has convex level sets with respect to $\mathcal{P}^\rho$.

Case (iii): there exists $q\in(0,1)$ such that $g(q)\in(0,1)$. For any $0<x_1<x_2$ and any $q\in(0,1)$ that satisfy

$$1=\rho(\delta_1)=\rho(q\delta_{x_1}+(1-q)\delta_{x_2})=x_1g(q)+x_2(1-g(q)), \tag{EC.4}$$

since ρ has convex level sets, it follows that

$$1=\rho(v(q\delta_{x_1}+(1-q)\delta_{x_2})+(1-v)\delta_1),\ \forall v\in(0,1). \tag{EC.5}$$

For any $q\in(0,1)$ such that $g(q)\in(0,1)$, (EC.4) holds for any $(x_1,x_2)=(1-c,-\frac{g(q)}{1-g(q)}(1-c)+\frac{1}{1-g(q)})$, $\forall c\in(0,1)$. Noting that $x_1<1<x_2$, (EC.5) implies

$$\begin{aligned}
1&=\rho(v(q\delta_{x_1}+(1-q)\delta_{x_2})+(1-v)\delta_1)\\
&=x_1g(vq)+g(vq+1-v)-g(vq)+x_2(1-g(vq+1-v))\\
&=(1-c)g(vq)+g(vq+1-v)-g(vq)\\
&\quad+\left[-\frac{g(q)}{1-g(q)}(1-c)+\frac{1}{1-g(q)}\right](1-g(vq+1-v))\\
&=1+c\left[-g(vq)+\frac{g(q)}{1-g(q)}(1-g(vq+1-v))\right],\ \forall v\in(0,1),\forall c\in(0,1).
\end{aligned}$$

Therefore,

$$-g(vq)+\frac{g(q)}{1-g(q)}(1-g(vq+1-v))=0,\ \forall v\in(0,1),\forall q \text{ such that } g(q)\in(0,1). \tag{EC.6}$$

Let $\alpha=\sup\{q\mid g(q)=0,q\in[0,1]\}$ and $\beta=\inf\{q\mid g(q)=1,q\in[0,1]\}$. Since there exists $q_0\in(0,1)$ such that $g(q_0)\in(0,1)$, it follows that $\alpha\leq q_0<1$, $g(\alpha)\leq g(q_0)<1$, $\beta\geq q_0>0$, and $g(\beta)\geq g(q_0)>0$.

There are four subcases:

Case (iii.i) $\alpha=\beta$ and $g(\alpha)=c\in(0,1)$. In this case, $\alpha=\beta\in(0,1)$. By the definition of α and β, $g(x)=0$ for $x<\alpha$ and $g(x)=1$ for $x>\alpha$. By Lemma EC.3, $\rho=cq_\alpha^-+(1-c)q_\alpha^+$ and ρ has convex level sets with respect to $\mathcal{P}^\rho$.

Case (iii.ii) $\alpha<\beta$ and $g(\alpha)\in(0,1)$. In this case, $\alpha\in(0,1)$. It follows from the definition of β that $g((\alpha+\beta)/2)<1$. Let $\epsilon_0=\beta-\alpha$. By the definition of β, $g(\alpha+\epsilon)<1$ for all $\epsilon\in(0,\epsilon_0)$. In addition, $g(\alpha+\epsilon)\geq g(\alpha)>0$ for all $\epsilon\in(0,\epsilon_0)$. Hence, $g(\alpha+\epsilon)\in(0,1)$ for all $\epsilon\in(0,\epsilon_0)$. For any $\eta\in(0,\alpha)$ and $\epsilon\in(0,\epsilon_0)$, let $q=\alpha+\epsilon$ and $v=\frac{\alpha-\eta}{\alpha+\epsilon}$. Then it follows from the definition of α that $g(vq)=g(\alpha-\eta)=0$, which implies from (EC.6) that $1=g(vq+1-v)=g(\alpha-\eta+\frac{\epsilon+\eta}{\alpha+\epsilon})$, for any $\epsilon\in(0,\epsilon_0),\eta\in(0,\alpha)$.

Then, $g(\alpha+) = \lim_{\epsilon\downarrow 0,\eta\downarrow 0} g(\alpha-\eta+\frac{\epsilon+\eta}{\alpha+\epsilon}) = 1$, which contradicts $g(\alpha+) \le g((\alpha+\beta)/2) < 1$. Therefore, this case does not hold.

Case (iii.iii) $\alpha < \beta$, $g(\alpha) = 0$, and $g(\beta) \in (0,1)$. Since $g(\beta) \in (0,1)$, it follows that $\beta \in (0,1)$. By the definition of β, for any $\eta \in (0, 1-\beta)$, $g(\beta+\eta) = 1$. By the definition of α, $g((\beta+\alpha)/2) > 0$. Hence, $g(\beta-) \ge g((\beta+\alpha)/2) > 0$. Hence, there exists $\epsilon_0 > 0$ such that $g(\beta-\epsilon) > 0$ for any $\epsilon \in (0,\epsilon_0)$. On the other hand, $g(\beta-\epsilon) \le g(\beta) < 1$ for any $\epsilon \in (0,\epsilon_0)$. Hence, $g(\beta-\epsilon) \in (0,1)$ for any $\epsilon \in (0,\epsilon_0)$. Then, for any $\eta \in (0,1-\beta)$ and $\epsilon \in (0,\epsilon_0)$, let $q = \beta-\epsilon$ and $v = \frac{1-\beta-\eta}{1-\beta+\epsilon}$. Then, we have $g(vq+1-v) = g(\beta+\eta) = 1$. Since $g(\beta-\epsilon) \in (0,1)$ for $\epsilon \in (0,\epsilon_0)$, it follows from (EC.6) that $0 = g(vq) = g(\frac{1-\beta-\eta}{1-\beta+\epsilon}(\beta-\epsilon))$, which implies that $g(\beta-) = \lim_{\eta\downarrow 0,\epsilon\downarrow 0} g(\frac{1-\beta-\eta}{1-\beta+\epsilon}(\beta-\epsilon)) = 0$. This contradicts $g(\beta-) > 0$. Therefore, this case does not hold.

Case (iii.iv) $\alpha < \beta$, $g(\alpha) = 0$, $g(\beta) = 1$. By the specification of case (iii), let $q_0 \in (0,1)$ such that $g(q_0) \in (0,1)$. Then, $\alpha < q_0 < \beta$. We will show that either there exists a constant $c \in (0,1)$ such that $g(u) = c$, $\forall u \in (0,1)$, or $g(u) = u$, $\forall u \in (0,1)$.

First, we will show that $\alpha = 0$ and $\beta = 1$. Suppose for the sake of contradiction that $\alpha > 0$. Since $\alpha < q_0$, it follows that $g(\alpha+\epsilon) < 1$ for all $\epsilon \in (0,\epsilon_0)$, where $\epsilon_0 = q_0 - \alpha$. Furthermore, by the definition of α, $g(\alpha+\epsilon) > 0$ for all $\epsilon \in (0,\epsilon_0)$. Hence, $g(\alpha+\epsilon) \in (0,1)$ for all $\epsilon \in (0,\epsilon_0)$. For any $\eta \in (0,\alpha)$ and $\epsilon \in (0,\epsilon_0)$, let $q = \alpha+\epsilon$ and $v = \frac{\alpha-\eta}{\alpha+\epsilon}$. Then it follows from the definition of α that $g(vq) = g(\alpha-\eta) = 0$, which implies from (EC.6) that $1 = g(vq+1-v) = g(\alpha-\eta+\frac{\epsilon+\eta}{\alpha+\epsilon})$, for any $\epsilon \in (0,\epsilon_0), \eta \in (0,\alpha)$. Then, $g(\alpha+) = \lim_{\epsilon\downarrow 0,\eta\downarrow 0} g(\alpha-\eta+\frac{\epsilon+\eta}{\alpha+\epsilon}) = 1$, which contradicts $g(\alpha+) \le g(q_0) < 1$. Therefore, $\alpha = 0$.

In addition, suppose for the sake of contradiction that $\beta < 1$. Then, by the definition of β, for any $\eta \in (0,1-\beta)$, $g(\beta+\eta) = 1$. Let $\epsilon_0 = \beta - q_0$. Since $\beta > q_0$, $g(\beta-\epsilon) \ge g(q_0) > 0$ for any $\epsilon \in (0,\epsilon_0)$. By the definition of β, $g(\beta-\epsilon) < 1$ for any $\epsilon \in (0,\epsilon_0)$. Hence, $g(\beta-\epsilon) \in (0,1)$ for any $\epsilon \in (0,\epsilon_0)$. Then, for any $\eta \in (0,1-\beta)$ and $\epsilon \in (0,\epsilon_0)$, let $q = \beta-\epsilon$ and $v = \frac{1-\beta-\eta}{1-\beta+\epsilon}$. Then, we have $g(vq+1-v) = g(\beta+\eta) = 1$. Since $g(\beta-\epsilon) \in (0,1)$ for any $\epsilon \in (0,\epsilon_0)$, it follows from (EC.6) that $0 = g(vq) = g(\frac{1-\beta-\eta}{1-\beta+\epsilon}(\beta-\epsilon))$, which implies that $g(\beta-) = \lim_{\eta\downarrow 0,\epsilon\downarrow 0} g(\frac{1-\beta-\eta}{1-\beta+\epsilon}(\beta-\epsilon)) = 0$. This contradicts $g(\beta-) \ge g(q_0) > 0$. Therefore, $\beta = 1$.

Then, it follows from $\alpha = 0$ and $\beta = 1$ that

$$g(q) \in (0,1), \ \forall q \in (0,1). \tag{EC.7}$$

Therefore, it follows from (EC.6) and (EC.7) that

$$-g(vq) + \frac{g(q)}{1-g(q)}(1 - g(vq+1-v)) = 0, \ \forall v \in (0,1), \forall q \in (0,1). \tag{EC.8}$$

For any $q \in (0,1)$ and $v \in (0,1)$, $vq+1-v > q$ and $\lim_{v\uparrow 1}(vq+1-v)=q$. It then follows from (EC.8) that

$$g(q-)=\lim_{v\uparrow 1} g(vq)=\lim_{v\uparrow 1}\frac{g(q)}{1-g(q)}(1-g(vq+1-v))=\frac{g(q)}{1-g(q)}(1-g(q+)),\ \forall q\in(0,1). \quad \text{(EC.9)}$$

Second, we consider two further cases for g:

Case (iii.iv.i) There exist $0<u_1<u_2<1$ such that $g(u_1)=g(u_2)$. Let $w_1=\inf\{u \mid g(u)=g(u_1)\}$ and $w_2=\sup\{u \mid g(u)=g(u_2)\}$. Consider three further cases: (a) $w_1>0$. Since $\lim_{q\downarrow w_1}\frac{1-u_2}{1-q}=\frac{1-u_2}{1-w_1}<1=\lim_{q\downarrow w_1}\frac{w_1}{q}$, there exists $q_0\in(w_1,u_2)$ such that $\frac{1-u_2}{1-q_0}<\frac{w_1}{q_0}$. Choose $v_0\in(0,1)$ such that $\frac{1-u_2}{1-q_0}<v_0<\frac{w_1}{q_0}$. Since $v_0q_0<w_1$, $g(v_0q_0)<g(u_1)$. And, since $w_1<q_0<v_0q_0+1-v_0<u_2$, $g(q_0)=g(v_0q_0+1-v_0)=g(u_1)$. Therefore, $-g(v_0q_0)+\frac{g(q_0)}{1-g(q_0)}(1-g(v_0q_0+1-v_0))>0$, which contradicts (EC.8). Hence, this case cannot hold. (b) $w_2<1$. Since $\lim_{q\uparrow w_2}\frac{1-w_2}{1-q}=1>\frac{u_1}{w_2}=\lim_{q\uparrow w_2}\frac{u_1}{q}$, there exists $q_0\in(u_1,w_2)$ such that $\frac{1-w_2}{1-q_0}>\frac{u_1}{q_0}$. Choose $v_0\in(0,1)$ such that $\frac{1-w_2}{1-q_0}>v_0>\frac{u_1}{q_0}$. Since $w_2>q_0>v_0q_0>u_1$, $g(q_0)=g(v_0q_0)=g(u_1)$. And, since $v_0q_0+1-v_0>w_2$, $g(v_0q_0+1-v_0)>g(u_1)$. Therefore, $-g(v_0q_0)+\frac{g(q_0)}{1-g(q_0)}(1-g(v_0q_0+1-v_0))<0$, which contradicts (EC.8). Hence, this case cannot hold. (c) $w_1=0$ and $w_2=1$. In this case, $g(u)=c, \forall u\in(0,1)$, for some constant $c\in(0,1)$. By Lemma EC.2, $\rho=c\mathrm{VaR}_0+(1-c)\mathrm{VaR}_1$, and ρ has convex level sets with respect to $\mathcal{P}^\rho$.

Case (iii.iv.ii) g is strictly increasing on $(0,1)$. Then, $g(p_1)-g(p_2)\neq 0$ for any $p_1\neq p_2$. We will show that $g(1-)=1$ and $g(0+)=0$. Consider $0<x_1<x_2<x_3$ and $p_1,p_2\in(0,1)$ such that

$$\rho(p_1\delta_{x_1}+(1-p_1)\delta_{x_2})=\rho(p_2\delta_{x_1}+(1-p_2)\delta_{x_3}),$$

which is equivalent to

$$x_1g(p_1)+x_2(1-g(p_1))=x_1g(p_2)+(1-g(p_2))x_3. \quad \text{(EC.10)}$$

Let $\frac{x_1}{x_2}=c_1$ and $\frac{x_3}{x_2}=c_3$. Then, $c_1\in(0,1)$, $c_3>1$, and (EC.10) is equivalent to

$$c_1=\frac{1-g(p_2)}{g(p_1)-g(p_2)}c_3-\frac{1-g(p_1)}{g(p_1)-g(p_2)}. \quad \text{(EC.11)}$$

For any fixed $0<p_1<p_2<1$ and $1<c_3<\frac{1-g(p_1)}{1-g(p_2)}$, define c_1 as in (EC.11). Then, $c_1\in(0,1)$. For any such p_1,p_2,c_3, and c_1, it follows from the convexity of the level sets of ρ that

$$\begin{aligned}
&x_1g(p_1)+x_2(1-g(p_1))=\rho(p_1\delta_{x_1}+(1-p_1)\delta_{x_2})\\
&=\rho(v(p_1\delta_{x_1}+(1-p_1)\delta_{x_2})+(1-v)(p_2\delta_{x_1}+(1-p_2)\delta_{x_3}))\\
&=\rho((vp_1+(1-v)p_2)\delta_{x_1}+v(1-p_1)\delta_{x_2}+(1-v)(1-p_2)\delta_{x_3})\\
&=x_1g(vp_1+(1-v)p_2)+x_2(g(v+(1-v)p_2)-g(vp_1+(1-v)p_2))\\
&\quad +x_3(1-g(v+(1-v)p_2)),\ \forall v\in(0,1),
\end{aligned}$$

which is equivalent to

$$
\begin{aligned}
&c_1[g(p_1)-g(vp_1+(1-v)p_2)]+1-g(p_1)-g(v+(1-v)p_2)+g(vp_1+(1-v)p_2)\\
&=c_3[1-g(v+(1-v)p_2)],\ \forall v\in(0,1).
\end{aligned}
$$

Plugging (EC.11) into the above equation, we obtain that for any $0<p_1<p_2<1$, any $1<c_3<\frac{1-g(p_1)}{1-g(p_2)}$, and any $v\in(0,1)$, it holds that

$$
\begin{aligned}
0=c_3&\left[\frac{1-g(p_2)}{g(p_1)-g(p_2)}(g(p_1)-g(vp_1+(1-v)p_2))-1+g(v+(1-v)p_2)\right]\\
&-\frac{1-g(p_1)}{g(p_1)-g(p_2)}[g(p_1)-g(vp_1+(1-v)p_2)]+1-g(p_1)\\
&-g(v+(1-v)p_2)+g(vp_1+(1-v)p_2).
\end{aligned}\tag{EC.12}
$$

Therefore,

$$
\begin{aligned}
0=&-\frac{1-g(p_1)}{g(p_1)-g(p_2)}[g(p_1)-g(vp_1+(1-v)p_2)]+1-g(p_1)\\
&-g(v+(1-v)p_2)+g(vp_1+(1-v)p_2),\ \forall v\in(0,1),\forall p_1<p_2,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
0=&\,g(vp_1+(1-v)p_2)(1-g(p_2))+g(v+(1-v)p_2)(g(p_2)-g(p_1))\\
&+g(p_1)g(p_2)-g(p_2),\ \forall v\in(0,1),\forall p_1<p_2.
\end{aligned}\tag{EC.13}
$$

Letting $v\uparrow 1$ in (EC.13), we obtain

$$
0=g(p_1+)(1-g(p_2))+g(1-)(g(p_2)-g(p_1))+g(p_1)g(p_2)-g(p_2),\forall p_1<p_2.\tag{EC.14}
$$

Since g is increasing on $(0,1)$, there exists $p_1^*\in(0,1)$, such that g is continuous at p_1^*. Choose any $p_2^*>p_1^*$. Letting $p_1=p_1^*$ and $p_2=p_2^*$ in (EC.14) leads to $(g(p_1^*)-g(p_2^*))(1-g(1-))=0$. Since g is strictly increasing, it follows that

$$
g(1-)=1.\tag{EC.15}
$$

Letting $q=\frac{1}{2}$ in (EC.8) leads to

$$
\frac{g(\frac{v}{2})}{1-g(1-\frac{v}{2})}=\frac{g(\frac{1}{2})}{1-g(\frac{1}{2})},\forall v\in(0,1).\tag{EC.16}
$$

It follows from (EC.16) and (EC.15) that

$$
g(0+)=\lim_{v\downarrow 0}g(\frac{v}{2})=\lim_{v\downarrow 0}\frac{g(\frac{1}{2})}{1-g(\frac{1}{2})}(1-g(1-\frac{v}{2}))=\frac{g(\frac{1}{2})}{1-g(\frac{1}{2})}(1-g(1-))=0.\tag{EC.17}
$$

We will then show that g is continuous on $(0,1)$. By (EC.8), we have

$$\begin{aligned}
g(v-) &= \lim_{q\uparrow 1} g(vq) = \lim_{q\uparrow 1} \frac{g(q)}{1-g(q)}(1-g(vq+1-v)) \\
&= \lim_{q\uparrow 1} g(q) \lim_{q\uparrow 1} \frac{1-g(vq+1-v)}{1-g(q)} \\
&= g(1-) \lim_{q\uparrow 1} \frac{1-g(vq+1-v)}{g((1-q)v)} \frac{g((1-q)v)}{g(1-q)} \frac{g(1-q)}{1-g(q)} \\
&= \lim_{q\uparrow 1} \frac{1-g(\frac{1}{2})}{g(\frac{1}{2})} \frac{g((1-q)v)}{g(1-q)} \frac{g(\frac{1}{2})}{1-g(\frac{1}{2})} \text{ (by (EC.15) and (EC.16))} \\
&= \lim_{q\uparrow 1} \frac{g((1-q)v)}{g(1-q)} = \lim_{q\downarrow 0} \frac{g(qv)}{g(q)}, \ \forall v \in (0,1). \qquad \text{(EC.18)}
\end{aligned}$$

Now consider $0 = x_1 < x_2 < x_3 < x_4$ and $p_1, p_2 \in (0,1)$ such that

$$\rho(p_1\delta_{x_1} + (1-p_1)\delta_{x_3}) = \rho(p_2\delta_{x_2} + (1-p_2)\delta_{x_4}),$$

which is equivalent to

$$x_1 g(p_1) + x_3(1-g(p_1)) = x_2 g(p_2) + x_4(1-g(p_2)). \qquad \text{(EC.19)}$$

Since ρ has convex level sets, it follows that for any $v \in (0,1)$, it holds that

$$\begin{aligned}
x_3(1-g(p_1)) &= x_1 g(p_1) + x_3(1-g(p_1)) = \rho(p_1\delta_{x_1} + (1-p_1)\delta_{x_3}) \\
&= \rho(v(p_1\delta_{x_1} + (1-p_1)\delta_{x_3}) + (1-v)(p_2\delta_{x_2} + (1-p_2)\delta_{x_4})) \\
&= \rho(vp_1\delta_{x_1} + (1-v)p_2\delta_{x_2} + v(1-p_1)\delta_{x_3} + (1-v)(1-p_2)\delta_{x_4}) \\
&= x_2(g(vp_1 + (1-v)p_2) - g(vp_1)) \\
&\quad + x_3(g(v + (1-v)p_2) - g(vp_1 + (1-v)p_2)) + x_4(1 - g(v + (1-v)p_2)). \qquad \text{(EC.20)}
\end{aligned}$$

Let $\frac{x_3}{x_2} = 1 + c_3$ and $\frac{x_4}{x_2} = 1 + c_3 + c_4$. Then, $c_3 > 0$, $c_4 > 0$, and (EC.19) becomes

$$c_3 = \frac{1-g(p_2)}{g(p_2)-g(p_1)} c_4 + \frac{g(p_1)}{g(p_2)-g(p_1)}. \qquad \text{(EC.21)}$$

Furthermore, (EC.20) is equivalent to

$$\begin{aligned}
0 &= g(vp_1 + (1-v)p_2) - g(vp_1) + (1+c_3+c_4)(1-g(v+(1-v)p_2)) \\
&\quad + (1+c_3)(g(v+(1-v)p_2) - g(vp_1 + (1-v)p_2) - 1 + g(p_1)), \forall v \in (0,1). \qquad \text{(EC.22)}
\end{aligned}$$

For any $0 < p_1 < p_2 < 1$ and $c_4 > 0$, let c_3 be defined in (EC.21). Then, $c_3 > 0$. Hence, (EC.22) holds for any such p_1, p_2, c_3, and c_4. Plugging (EC.21) into (EC.22), we obtain that for any $0 < p_1 < p_2 < 1$ and any $c_4 > 0$, it holds that

$$0 = g(vp_1 + (1-v)p_2) - g(vp_1) + \frac{g(p_2)}{g(p_2)-g(p_1)}[g(p_1) - g(vp_1 + (1-v)p_2)]$$

$$
\begin{aligned}
&+c_4\frac{1-g(p_2)}{g(p_2)-g(p_1)}[g(v+(1-v)p_2)-g(vp_1+(1-v)p_2)-1+g(p_1)]\\
&+c_4\frac{1-g(p_1)}{g(p_2)-g(p_1)}[1-g(v+(1-v)p_2)],\forall v\in(0,1),
\end{aligned}\tag{EC.23}
$$

which implies that

$$
\begin{aligned}
0=\,&g(vp_1+(1-v)p_2)-g(vp_1)\\
&+\frac{g(p_2)}{g(p_2)-g(p_1)}[g(p_1)-g(vp_1+(1-v)p_2)],\ \forall 0<p_1<p_2<1,\forall v\in(0,1),
\end{aligned}
$$

which can be simplified to be

$$
-g(vp_1+(1-v)p_2)-(g(p_2)-g(p_1))\frac{g(vp_1)}{g(p_1)}+g(p_2)=0,\ \forall p_1<p_2,\forall v\in(0,1).\tag{EC.24}
$$

Letting $p_2\uparrow 1$ in (EC.24) and applying (EC.15), we obtain

$$
-g((vp_1+1-v)-)-(1-g(p_1))\frac{g(vp_1)}{g(p_1)}+1=0,\ \forall 0<p_1<1,\forall v\in(0,1).\tag{EC.25}
$$

Then, it follows from (EC.8) and (EC.25) that

$$
g((vp_1+1-v)-)=g(vp_1+1-v),\forall 0<p_1<1,\forall v\in(0,1),
$$

which implies that

$$
g(v-)=g(v),\forall v\in(0,1).\tag{EC.26}
$$

It follows from (EC.9) and (EC.26) that g is continuous on $(0,1)$, i.e.,

$$
g(v-)=g(v)=g(v+),\forall v\in(0,1).\tag{EC.27}
$$

Lastly, we will show that $g(u)=u$ for any $u\in(0,1)$. Letting $p_1\downarrow 0$ in (EC.24), we obtain

$$
-g(((1-v)p_2)+)-(g(p_2)-g(0+))\lim_{p_1\downarrow 0}\frac{g(vp_1)}{g(p_1)}+g(p_2)=0,\ \forall 0<p_2<1,\forall v\in(0,1).\tag{EC.28}
$$

Applying (EC.17), (EC.18), and (EC.27) to (EC.28), we obtain

$$
g((1-v)p_2)=g(p_2)(1-g(v)),\ \forall 0<p_2<1,\forall v\in(0,1).\tag{EC.29}
$$

Letting $p_2\uparrow 1$ in (EC.29) and using (EC.15) and (EC.27), we obtain

$$
g(1-v)=g(1-)(1-g(v))=1-g(v),\ \forall v\in(0,1),\tag{EC.30}
$$

which in combination with (EC.29) implies

$$
g(vp_2)=g(v)g(p_2),\ \forall 0<p_2<1,\forall v\in(0,1).\tag{EC.31}
$$

In the following, we will show by induction that

$$g(\frac{k}{2^n}) = \frac{k}{2^n},\ k = 1, 2, \dots, 2^n - 1, \forall n \in \mathbb{N}. \tag{EC.32}$$

Letting $v = \frac{1}{2}$ in (EC.30), we obtain $g(\frac{1}{2}) = \frac{1}{2}$. Hence, (EC.32) holds for $n = 1$. Suppose (EC.32) holds for n. We will show that it also holds for $n+1$. In fact, for any $0 \le k \le 2^{n-1} - 1$, since $1 \le 2k+1 \le 2^n - 1$, it follows from (EC.31) that

$$g(\frac{2k+1}{2^{n+1}}) = g(\frac{1}{2}) g(\frac{2k+1}{2^n}) = \frac{2k+1}{2^{n+1}},\ 0 \le k \le 2^{n-1} - 1. \tag{EC.33}$$

For any $2^{n-1} \le k \le 2^n - 1$, it holds that $1 \le 2^{n+1} - (2k+1) \le 2^n - 1$. Hence, it follows from (EC.30) that

$$\begin{aligned} & g(\frac{2k+1}{2^{n+1}}) = 1 - g(\frac{2^{n+1} - (2k+1)}{2^{n+1}}) = 1 - \frac{2^{n+1} - (2k+1)}{2^{n+1}} \text{ (by (EC.33))} \\ = & \frac{2k+1}{2^{n+1}},\ 2^{n-1} \le k \le 2^n - 1. \end{aligned} \tag{EC.34}$$

In addition, for any $1 \le k \le 2^n - 1$, $g(\frac{2k}{2^{n+1}}) = g(\frac{k}{2^n}) = \frac{k}{2^n}$, which in combination with (EC.33) and (EC.34) implies that (EC.32) holds for $n+1$, and hence holds for any n. Since $\{k/2^n, k = 1, \dots, 2^n - 1, n \in \mathbb{N}\}$ is dense on $(0,1)$ and g is continuous on $(0,1)$, it follows from (EC.32) that $g(u) = u$ for all $u \in (0,1)$, which completes the proof. □

Finally, the proof of Theorem 1 is as follows.

Proof of Theorem 1. By Lemma EC.1 and Theorem 2, only those risk measures listed in cases (i)-(iv) of Theorem 2 satisfy the necessary condition for a risk measure to be general elicitable with respect to $\mathcal{D}_{disc}$. Therefore, we only need to check if those risk measures are general elicitable with respect to $\mathcal{D}_{disc}$.

First, we will show that for $c \in (0,1]$, $\rho = c\mathrm{VaR}_0 + (1-c)\mathrm{VaR}_1$ is not general elicitable with respect to $\mathcal{D}_{disc}$. Suppose for the sake of contradiction that ρ is general elicitable with respect to $\mathcal{D}_{disc}$, then there exists a function S such that (10) holds. For any u, letting $F = \delta_u$ in (10) and noting $\rho(\delta_u) = u$ yields

$$S(u,u) \le S(x,u), \forall x, \forall u, \text{ and the equality holds only if } u \le x. \tag{EC.35}$$

For any $u < v$ and $p \in (0,1)$, letting $F = p\delta_u + (1-p)\delta_v$ in (10) yields $pS(cu + (1-c)v, u) + (1-p)S(cu + (1-c)v, v) \le pS(x,u) + (1-p)S(x,v)$, $\forall x$. Letting $p \to 0$ leads to

$$S(cu + (1-c)v, v) \le S(x,v), \forall u < v, \forall x. \tag{EC.36}$$

Letting $x = v$ in (EC.36), we obtain

$$S(cu + (1-c)v, v) \le S(v,v), \forall u < v. \tag{EC.37}$$

By (EC.35), $S(v,v) \le S(cu+(1-c)v,v)$, $\forall u<v$, which in combination with (EC.37) implies $S(v,v) = S(cu+(1-c)v,v)$, $\forall u<v$; however, by (EC.35), $S(v,v) = S(cu+(1-c)v,v)$ implies $v \le cu+(1-c)v$, which contradicts $u<v$. Hence, ρ is not general elicitable with respect to $\mathcal{D}_{disc}$.

Second, we will show that for $c=0$, $\rho = c\mathrm{VaR}_0 + (1-c)\mathrm{VaR}_1 = \mathrm{VaR}_1$ is general elicitable with respect to $\mathcal{D}^\infty$. Let $a>0$ be a constant and define the forecasting objective function

$$S(x,y) = \begin{cases} 0, & \text{if } x \ge y, \\ a, & \text{else.} \end{cases}$$

Then for any $F \in \mathcal{D}^\infty$ and any $x \ge \rho(F)$,

$$\int_{\mathbb{R}} S(x,y)dF(y) = \int_{y \le \rho(F)} S(x,y)dF(y) = 0.$$

On the other hand, for any $F \in \mathcal{D}^\infty$ and any $x<\rho(F)$,

$$\int_{\mathbb{R}} S(x,y)dF(y) = \int_{x<y\le\rho(F)} S(x,y)dF(y) = a\int_{x<y\le\rho(F)} dF(y) = a(1-F(x)) > 0.$$

Therefore, for any $F \in \mathcal{D}^\infty$, $\rho(F) = \min\{x \mid x \in \arg\min_x \int S(x,y)dF(y)\}$.

Third, we will show that for any $\alpha \in (0,1)$, VaR_α is general elicitable with respect to $\mathcal{D}^1$. Define

$$S(x,y) = (1_{\{x\ge y\}} - \alpha)(x-y). \tag{EC.38}$$

It follows from part (c) of Theorem 9 in Gneiting (2011) that

$$[q_\alpha^-(F), q_\alpha^+(F)] = \arg\min_x \int S(x,y)dF(y),\ \forall F \in \mathcal{D}^1,$$

where $q_\alpha^-(F) := \inf\{y \mid F(y) \ge \alpha\}$ and $q_\alpha^+(F) := \inf\{y \mid F(y) > \alpha\}$. Therefore, $\mathrm{VaR}_\alpha = q_\alpha^-$ satisfies (10) for any $F \in \mathcal{D}^1$ with S defined in (EC.38).

Fourth, we will show that for any $\alpha \in (0,1)$, any $c \in [0,1)$, $\rho = cq_\alpha^- + (1-c)q_\alpha^+$ is not general elicitable with respect to $\mathcal{D}_{disc}$. Suppose for the purpose of contradiction that ρ is general elicitable with respect to $\mathcal{D}_{disc}$. Then, for any $u<v$, letting $F = \alpha\delta_u + (1-\alpha)\delta_v$ in (10) leads to

$$cu + (1-c)v = \min\left\{x \mid x \in \arg\min_x [\alpha S(x,u) + (1-\alpha)S(x,v)]\right\}, \forall u<v. \tag{EC.39}$$

For any $u<v$ and any $p>\alpha$, letting $F = p\delta_u + (1-p)\delta_v$ in (10) and noting $q_\alpha^-(F) = q_\alpha^+(F) = u$ yields

$$u = \min\left\{x \mid x \in \arg\min_x [pS(x,u) + (1-p)S(x,v)]\right\}, \forall u<v, \forall p>\alpha,$$

which implies that

$$pS(u,u) + (1-p)S(u,v) \le pS(x,u) + (1-p)S(x,v), \forall u<v, \forall p>\alpha, \forall x. \tag{EC.40}$$

Letting $p \downarrow \alpha$ in (EC.40), we have

$$\alpha S(u,u) + (1-\alpha)S(u,v) \leq \alpha S(x,u) + (1-\alpha)S(x,v), \forall u < v, \forall x, \quad \text{(EC.41)}$$

which implies

$$u \in \arg\min_x [\alpha S(x,u) + (1-\alpha)S(x,v)], \forall u < v,$$

which contradicts (EC.39) because $u < cu + (1-c)v$ for $c \in [0,1)$ and $u < v$.

Fifth, it follows from Theorem 7 in Gneiting (2011) that the mean functional $\rho(F) := \int x dF(x)$ is elicitable and hence general elicitable with respect to $\mathcal{D}^1$. The proof is thus completed. □

EC.3. Proof of Theorem 3 and Theorem 4

The proof of Theorem 3 is almost identical to that of Theorem 1. By Theorem 6 of Gneiting (2011), a necessary condition for ρ to be elicitable is that ρ has convex level sets. First, we will identify risk measures having convex level sets by proving Theorem 4, which is a stronger version of Theorem 2.

Proof of Theorem 4. Define $g(u) := 1 - h(1-u)$, $u \in [0,1]$. Then $g(0) = 0$, $g(1) = 1$, and g is increasing on $[0,1]$. And then, ρ can be represented as

$$\rho(F) = -\int_{-\infty}^{0} g(F(x))dx + \int_{0}^{\infty} (1 - g(F(x)))dx.$$

For a discrete distribution $F = \sum_{i=1}^n p_i \delta_{x_i}$, where $0 \leq x_1 < x_2 < \cdots < x_n$, $p_i > 0$, $i = 1, \ldots, n$, and $\sum_{i=1}^n p_i = 1$, it can be shown by simple calculation that $\rho(F) = g(p_1)x_1 + \sum_{i=2}^n (g(\sum_{j=1}^i p_j) - g(\sum_{j=1}^{i-1} p_j))x_i$. In addition, $F = \sum_{i=1}^n p_i \delta_{x_i} \in \mathcal{D}_{disc} \cap \{F \mid q_{\alpha_0}^-(F) = q_{\alpha_0}^+(F)\}$ if and only if $\sum_{j=1}^i p_j \neq \alpha_0, \forall i$.

Suppose ρ has convex level sets with respect to $\mathcal{D}_{disc} \cap \{F \mid q_{\alpha_0}^-(F) = q_{\alpha_0}^+(F)\}$. There are three cases for g.

Case (i): for any $q \in (0,1)$, $g(q) = 0$. Then $g(u) = 1_{\{u=1\}}$. By Lemma EC.2 (with $c = 0$), $\rho = \mathrm{VaR}_1$ and ρ has convex level sets with respect to $\mathcal{P}^\rho$.

Case (ii): there exists $q_0 \in (0,1)$ such that $g(q_0) = 1$ and $g(q) \in \{0,1\}$ for all $q \in (0,1)$. Let $\alpha = \inf\{q \mid g(q) = 1\}$. There are three subcases: (ii.i) $\alpha = 0$. Then, $g(u) = 1_{\{u>0\}}$. By Lemma EC.2 (with $c = 1$), $\rho = \mathrm{VaR}_0$ and ρ has convex level sets with respect to $\mathcal{P}^\rho$. (ii.ii) $\alpha \in (0,1)$ and $g(\alpha) = 1$. Then, $g(u) = 1_{\{u \geq \alpha\}}$. By Lemma EC.3 (with $c = 1$), $\rho = q_\alpha^- = \mathrm{VaR}_\alpha$ and ρ has convex level sets with respect to $\mathcal{P}^\rho$. (ii.iii) $\alpha \in (0,1)$ and $g(\alpha) = 0$. Then, $g(u) = 1_{\{u>\alpha\}}$. By Lemma EC.3 (with $c = 0$), $\rho = q_\alpha^+$ and ρ has convex level sets with respect to $\mathcal{P}^\rho$.

Case (iii): there exists $q \in (0,1)$ such that $g(q) \in (0,1)$. For any $0 < x_1 < x_2$ and any $q \in (0,1)$ that satisfy $q \neq \alpha_0$, $g(q) \in (0,1)$, and

$$1 = \rho(\delta_1) = \rho(q\delta_{x_1} + (1-q)\delta_{x_2}) = x_1 g(q) + x_2(1 - g(q)), \quad \text{(EC.42)}$$

since ρ has convex level sets, it follows that

$$1=\rho(v(q\delta_{x_1}+(1-q)\delta_{x_2})+(1-v)\delta_1),\ \forall v\in(0,1). \tag{EC.43}$$

For any $q\in(0,1)$ such that $q\neq\alpha_0$ and $g(q)\in(0,1)$, (EC.42) holds for any $(x_1,x_2)=(1-c,-\frac{g(q)}{1-g(q)}(1-c)+\frac{1}{1-g(q)})$, $\forall c\in(0,1)$. Noting that $x_1<1<x_2$, (EC.43) implies

$$\begin{aligned}1&=\rho(v(q\delta_{x_1}+(1-q)\delta_{x_2})+(1-v)\delta_1)\\&=x_1g(vq)+g(vq+1-v)-g(vq)+x_2(1-g(vq+1-v))\\&=(1-c)g(vq)+g(vq+1-v)-g(vq)\\&\quad+\left[-\frac{g(q)}{1-g(q)}(1-c)+\frac{1}{1-g(q)}\right](1-g(vq+1-v))\\&=1+c\left[-g(vq)+\frac{g(q)}{1-g(q)}(1-g(vq+1-v))\right],\ \forall v\in(0,1),\forall c\in(0,1).\end{aligned}$$

Therefore,

$$-g(vq)+\frac{g(q)}{1-g(q)}(1-g(vq+1-v))=0,\ \forall v\in(0,1),\forall q \text{ such that } q\neq\alpha_0 \text{ and } g(q)\in(0,1). \tag{EC.44}$$

Let $\alpha=\sup\{q\mid g(q)=0,q\in[0,1]\}$ and $\beta=\inf\{q\mid g(q)=1,q\in[0,1]\}$. Since there exists $q_0\in(0,1)$ such that $g(q_0)\in(0,1)$, it follows that $\alpha\le q_0<1$, $g(\alpha)\le g(q_0)<1$, $\beta\ge q_0>0$, and $g(\beta)\ge g(q_0)>0$.

There are four subcases:

Case (iii.i) $\alpha=\beta$ and $g(\alpha)=c\in(0,1)$. In this case, $\alpha=\beta\in(0,1)$. By the definition of α and β, $g(x)=0$ for $x<\alpha$ and $g(x)=1$ for $x>\alpha$. By Lemma EC.3, $\rho=cq_\alpha^-+(1-c)q_\alpha^+$ and ρ has convex level sets with respect to $\mathcal{P}^\rho$.

Case (iii.ii) $\alpha<\beta$ and $g(\alpha)\in(0,1)$. In this case, $\alpha\in(0,1)$. There are two subcases. (iii.ii.i) $\alpha\ge\alpha_0$. It follows from the definition of β that $g((\alpha+\beta)/2)<1$. Let $\epsilon_0=\beta-\alpha$. By the definition of β, $g(\alpha+\epsilon)<1$ for all $\epsilon\in(0,\epsilon_0)$. In addition, $g(\alpha+\epsilon)\ge g(\alpha)>0$ for all $\epsilon\in(0,\epsilon_0)$. Hence, $g(\alpha+\epsilon)\in(0,1)$ for all $\epsilon\in(0,\epsilon_0)$. For any $\eta\in(0,\alpha)$ and $\epsilon\in(0,\epsilon_0)$, let $q=\alpha+\epsilon$ and $v=\frac{\alpha-\eta}{\alpha+\epsilon}$. Then, $q>\alpha_0$ for all $\epsilon\in(0,\epsilon_0)$. Furthermore, it follows from the definition of α that $g(vq)=g(\alpha-\eta)=0$, which implies from (EC.44) that $1=g(vq+1-v)=g(\alpha-\eta+\frac{\epsilon+\eta}{\alpha+\epsilon})$, for any $\epsilon\in(0,\epsilon_0),\eta\in(0,\alpha)$. Then, $g(\alpha+)=\lim_{\epsilon\downarrow0,\eta\downarrow0}g(\alpha-\eta+\frac{\epsilon+\eta}{\alpha+\epsilon})=1$, which contradicts $g(\alpha+)\le g((\alpha+\beta)/2)<1$. Therefore, the subcase (iii.ii.i) does not hold. (iii.ii.ii) $\alpha<\alpha_0$. It follows from the definition of β that $g((\alpha+\min(\beta,\alpha_0))/2)<1$. Let $\epsilon_0=\min(\beta,\alpha_0)-\alpha$. By the definition of β, $g(\alpha+\epsilon)<1$ for all $\epsilon\in(0,\epsilon_0)$. In addition, $g(\alpha+\epsilon)\ge g(\alpha)>0$ for all $\epsilon\in(0,\epsilon_0)$. Hence, $g(\alpha+\epsilon)\in(0,1)$ for all $\epsilon\in(0,\epsilon_0)$. For any $\eta\in(0,\alpha)$ and $\epsilon\in(0,\epsilon_0)$, let $q=\alpha+\epsilon$ and $v=\frac{\alpha-\eta}{\alpha+\epsilon}$. Then, $q=\alpha+\epsilon<\alpha_0$ for all $\epsilon\in(0,\epsilon_0)$. Furthermore, it follows from the definition of α that $g(vq)=g(\alpha-\eta)=0$, which implies from (EC.44) that $1=g(vq+1-v)=g(\alpha-\eta+\frac{\epsilon+\eta}{\alpha+\epsilon})$, for any $\epsilon\in(0,\epsilon_0),\eta\in(0,\alpha)$. Then, $g(\alpha+)=$

$\lim_{\epsilon\downarrow 0,\eta\downarrow 0} g(\alpha-\eta+\frac{\epsilon+\eta}{\alpha+\epsilon})=1$, which contradicts $g(\alpha+)\le g((\alpha+\min(\beta,\alpha_0))/2)<1$. Therefore, the subcase (iii.ii.ii) does not hold. Hence, the case (iii.ii) does not hold.

Case (iii.iii) $\alpha<\beta$, $g(\alpha)=0$, and $g(\beta)\in(0,1)$. Since $g(\beta)\in(0,1)$, it follows that $\beta\in(0,1)$. There are two subcases. (iii.iii.i) $\beta\le\alpha_0$. By the definition of β, for any $\eta\in(0,1-\beta)$, $g(\beta+\eta)=1$. By the definition of α, $g((\beta+\alpha)/2)>0$. Then, $g(\beta-)\ge g((\beta+\alpha)/2)>0$, and $g(\beta-\epsilon)\ge g((\beta+\alpha)/2)>0$ for any $\epsilon\in(0,\epsilon_0)$, where $\epsilon_0=(\beta-\alpha)/2$. On the other hand, $g(\beta-\epsilon)\le g(\beta)<1$ for any $\epsilon\in(0,\epsilon_0)$. Hence, $g(\beta-\epsilon)\in(0,1)$ for any $\epsilon\in(0,\epsilon_0)$. Then, for any $\eta\in(0,1-\beta)$ and $\epsilon\in(0,\epsilon_0)$, let $q=\beta-\epsilon$ and $v=\frac{1-\beta-\eta}{1-\beta+\epsilon}$. Then, $q=\beta-\epsilon<\alpha_0$ for all $\epsilon\in(0,\epsilon_0)$. Furthermore, we have $g(vq+1-v)=g(\beta+\eta)=1$. Since $g(q)=g(\beta-\epsilon)\in(0,1)$ and $q<\alpha_0$ for all $\epsilon\in(0,\epsilon_0)$, it follows from (EC.44) that $0=g(vq)=g(\frac{1-\beta-\eta}{1-\beta+\epsilon}(\beta-\epsilon))$, which implies that $g(\beta-)=\lim_{\eta\downarrow 0,\epsilon\downarrow 0} g(\frac{1-\beta-\eta}{1-\beta+\epsilon}(\beta-\epsilon))=0$. This contradicts $g(\beta-)>0$. Therefore, the subcase (iii.iii.i) does not hold. (iii.iii.ii) $\beta>\alpha_0$. By the definition of β, for any $\eta\in(0,1-\beta)$, $g(\beta+\eta)=1$. By the definition of α, $g((\beta+\alpha)/2)>0$. Then, $g(\beta-)\ge g((\beta+\alpha)/2)>0$, and $g(\beta-\epsilon)\ge g((\beta+\alpha)/2)>0$ for any $\epsilon\in(0,\epsilon_0)$, where $\epsilon_0=\min((\beta-\alpha)/2,\beta-\alpha_0)$. On the other hand, $g(\beta-\epsilon)\le g(\beta)<1$ for any $\epsilon\in(0,\epsilon_0)$. Hence, $g(\beta-\epsilon)\in(0,1)$ for any $\epsilon\in(0,\epsilon_0)$. Then, for any $\eta\in(0,1-\beta)$ and $\epsilon\in(0,\epsilon_0)$, let $q=\beta-\epsilon$ and $v=\frac{1-\beta-\eta}{1-\beta+\epsilon}$. Then, $q>\alpha_0$ for all $\epsilon\in(0,\epsilon_0)$. Furthermore, we have $g(vq+1-v)=g(\beta+\eta)=1$. Since $g(q)=g(\beta-\epsilon)\in(0,1)$ and $q=\beta-\epsilon>\alpha_0$ for all $\epsilon\in(0,\epsilon_0)$, it follows from (EC.44) that $0=g(vq)=g(\frac{1-\beta-\eta}{1-\beta+\epsilon}(\beta-\epsilon))$, which implies that $g(\beta-)=\lim_{\eta\downarrow 0,\epsilon\downarrow 0} g(\frac{1-\beta-\eta}{1-\beta+\epsilon}(\beta-\epsilon))=0$. This contradicts $g(\beta-)>0$. Therefore, the subcase (iii.iii.ii) does not hold. Hence, the case (iii.iii) does not hold.

Case (iii.iv) $\alpha<\beta$, $g(\alpha)=0$, $g(\beta)=1$. By the specification of case (iii), there exsits $q_0\in(0,1)$ such that $g(q_0)\in(0,1)$. Then, $\alpha<q_0<\beta$. We will show that either there exists a constant $c\in(0,1)$ such that $g(u)=c$, $\forall u\in(0,1)$, or $g(u)=u$, $\forall u\in(0,1)$.

First, we will show that $\alpha=0$ and $\beta=1$. Suppose for the sake of contradiction that $\alpha>0$. There are two subcases. (a) The first subcase is that $\alpha\ge\alpha_0$. Since $\alpha<q_0$, it follows that $g(\alpha+\epsilon)\le g(q_0)<1$ for all $\epsilon\in(0,\epsilon_0)$, where $\epsilon_0=q_0-\alpha$. Furthermore, by the definition of α, $g(\alpha+\epsilon)>0$ for all $\epsilon\in(0,\epsilon_0)$. Hence, $g(\alpha+\epsilon)\in(0,1)$ for all $\epsilon\in(0,\epsilon_0)$. For any $\eta\in(0,\alpha)$ and $\epsilon\in(0,\epsilon_0)$, let $q=\alpha+\epsilon$ and $v=\frac{\alpha-\eta}{\alpha+\epsilon}$. Then, $q=\alpha+\epsilon>\alpha_0$ for all $\epsilon\in(0,\epsilon_0)$. Furthermore, it follows from the definition of α that $g(vq)=g(\alpha-\eta)=0$, which implies from (EC.44) that $1=g(vq+1-v)=g(\alpha-\eta+\frac{\epsilon+\eta}{\alpha+\epsilon})$, for any $\epsilon\in(0,\epsilon_0),\eta\in(0,\alpha)$. Then, $g(\alpha+)=\lim_{\epsilon\downarrow 0,\eta\downarrow 0} g(\alpha-\eta+\frac{\epsilon+\eta}{\alpha+\epsilon})=1$, which contradicts $g(\alpha+)\le g(q_0)<1$. (b) The second subcase is that $0<\alpha<\alpha_0$. Since $\alpha<q_0$, it follows that $g(\alpha+\epsilon)\le g(q_0)<1$ for all $\epsilon\in(0,\epsilon_0)$, where $\epsilon_0=\min(q_0-\alpha,\alpha_0-\alpha)$. Furthermore, by the definition of α, $g(\alpha+\epsilon)>0$ for all $\epsilon\in(0,\epsilon_0)$. Hence, $g(\alpha+\epsilon)\in(0,1)$ for all $\epsilon\in(0,\epsilon_0)$. For any $\eta\in(0,\alpha)$ and $\epsilon\in(0,\epsilon_0)$, let $q=\alpha+\epsilon$ and $v=\frac{\alpha-\eta}{\alpha+\epsilon}$. Then, $q=\alpha+\epsilon<\alpha_0$ for all $\epsilon\in(0,\epsilon_0)$. Furthermore, it follows from the definition of α that $g(vq)=g(\alpha-\eta)=0$, which implies from (EC.44) that $1=g(vq+1-v)=g(\alpha-\eta+\frac{\epsilon+\eta}{\alpha+\epsilon})$, for any $\epsilon \in (0, \epsilon_0), \eta \in (0, \alpha)$. Then, $g(\alpha+) = \lim_{\epsilon \downarrow 0, \eta \downarrow 0} g(\alpha - \eta + \frac{\epsilon+\eta}{\alpha+\epsilon}) = 1$, which contradicts $g(\alpha+) \leq g(q_0) < 1$. Since both subcases lead to contradiction, it follows that $\alpha = 0$.

In addition, suppose for the sake of contradiction that $\beta < 1$. There are two subcases. (a) The first subcase is that $\beta \leq \alpha_0$. By the definition of β, for any $\eta \in (0, 1-\beta)$, $g(\beta+\eta) = 1$. Let $\epsilon_0 = \beta - q_0$. Since $\beta > q_0$, $g(\beta - \epsilon) \geq g(q_0) > 0$ for any $\epsilon \in (0, \epsilon_0)$. By the definition of β, $g(\beta - \epsilon) < 1$ for any $\epsilon \in (0, \epsilon_0)$. Hence, $g(\beta - \epsilon) \in (0, 1)$ for any $\epsilon \in (0, \epsilon_0)$. Then, for any $\eta \in (0, 1-\beta)$ and $\epsilon \in (0, \epsilon_0)$, let $q = \beta - \epsilon$ and $v = \frac{1-\beta-\eta}{1-\beta+\epsilon}$. Then, $q = \beta - \epsilon < \alpha_0$ for all $\epsilon \in (0, \epsilon_0)$, and $g(vq + 1 - v) = g(\beta + \eta) = 1$. Since $g(q) = g(\beta - \epsilon) \in (0, 1)$ and $q = \beta - \epsilon < \alpha_0$ for any $\epsilon \in (0, \epsilon_0)$, it follows from (EC.44) that $0 = g(vq) = g(\frac{1-\beta-\eta}{1-\beta+\epsilon}(\beta - \epsilon))$, which implies that $g(\beta-) = \lim_{\eta \downarrow 0, \epsilon \downarrow 0} g(\frac{1-\beta-\eta}{1-\beta+\epsilon}(\beta - \epsilon)) = 0$. This contradicts that $g(\beta-) \geq g(q_0) > 0$. (b) The second subcase is that $\beta > \alpha_0$. By the definition of β, for any $\eta \in (0, 1-\beta)$, $g(\beta + \eta) = 1$. Let $\epsilon_0 = \min(\beta - q_0, \beta - \alpha_0)$. Since $\beta > q_0$, $g(\beta - \epsilon) \geq g(q_0) > 0$ for any $\epsilon \in (0, \epsilon_0)$. By the definition of β, $g(\beta - \epsilon) < 1$ for any $\epsilon \in (0, \epsilon_0)$. Hence, $g(\beta - \epsilon) \in (0, 1)$ for any $\epsilon \in (0, \epsilon_0)$. Then, for any $\eta \in (0, 1-\beta)$ and $\epsilon \in (0, \epsilon_0)$, let $q = \beta - \epsilon$ and $v = \frac{1-\beta-\eta}{1-\beta+\epsilon}$. Then, $q = \beta - \epsilon > \alpha_0$ for all $\epsilon \in (0, \epsilon_0)$, and $g(vq + 1 - v) = g(\beta + \eta) = 1$. Since $g(q) = g(\beta - \epsilon) \in (0, 1)$ and $q = \beta - \epsilon > \alpha_0$ for any $\epsilon \in (0, \epsilon_0)$, it follows from (EC.44) that $0 = g(vq) = g(\frac{1-\beta-\eta}{1-\beta+\epsilon}(\beta - \epsilon))$, which implies that $g(\beta-) = \lim_{\eta \downarrow 0, \epsilon \downarrow 0} g(\frac{1-\beta-\eta}{1-\beta+\epsilon}(\beta - \epsilon)) = 0$. This contradicts that $g(\beta-) \geq g(q_0) > 0$. Since both subcases lead to contradiction, it follows that $\beta = 1$.

Then, it follows from $\alpha = 0$ and $\beta = 1$ that

$$g(q) \in (0, 1), \ \forall q \in (0, 1). \tag{EC.45}$$

Therefore, it follows from (EC.44) and (EC.45) that

$$-g(vq) + \frac{g(q)}{1 - g(q)}(1 - g(vq + 1 - v)) = 0, \ \forall v \in (0, 1), \forall q \in (0, 1), q \neq \alpha_0. \tag{EC.46}$$

For any $q \in (0, 1)$, $q \neq \alpha_0$, and $v \in (0, 1)$, $vq + 1 - v > q$ and $\lim_{v \uparrow 1}(vq + 1 - v) = q$. It then follows from (EC.46) that

$$g(q-) = \lim_{v \uparrow 1} g(vq) = \lim_{v \uparrow 1} \frac{g(q)}{1 - g(q)}(1 - g(vq + 1 - v)) = \frac{g(q)}{1 - g(q)}(1 - g(q+)), \ \forall q \in (0, 1), q \neq \alpha_0. \tag{EC.47}$$

Second, we consider two further cases for g:

Case (iii.iv.i) There exist $0 < u_1 < u_2 < 1$ such that $g(u_1) = g(u_2)$. Let $w_1 = \inf\{u \mid g(u) = g(u_1)\}$ and $w_2 = \sup\{u \mid g(u) = g(u_2)\}$. Consider three further cases: (a) $w_1 > 0$. Since $\lim_{q \downarrow w_1} \frac{1-u_2}{1-q} = \frac{1-u_2}{1-w_1} < 1 = \lim_{q \downarrow w_1} \frac{w_1}{q}$, there exists $q_0 \in (w_1, u_2)$ such that $q_0 \neq \alpha_0$ and $\frac{1-u_2}{1-q_0} < \frac{w_1}{q_0}$. Choose $v_0 \in (0, 1)$ such that $\frac{1-u_2}{1-q_0} < v_0 < \frac{w_1}{q_0}$. Since $v_0 q_0 < w_1$, $g(v_0 q_0) < g(u_1)$. And, since $w_1 < q_0 < v_0 q_0 + 1 - v_0 < u_2$, $g(q_0) = g(v_0 q_0 + 1 - v_0) = g(u_1)$. Therefore, $-g(v_0 q_0) + \frac{g(q_0)}{1-g(q_0)}(1 - g(v_0 q_0 + 1 - v_0)) > 0$, which contradicts (EC.46). Hence, this case cannot hold. (b) $w_2 < 1$. Since $\lim_{q\uparrow w_2} \frac{1-w_2}{1-q} = 1 > \frac{u_1}{w_2} = \lim_{q\uparrow w_2} \frac{u_1}{q}$, there exists $q_0 \in (u_1, w_2)$ such that $q_0 \neq \alpha_0$ and $\frac{1-w_2}{1-q_0} > \frac{u_1}{q_0}$. Choose $v_0 \in (0,1)$ such that $\frac{1-w_2}{1-q_0} > v_0 > \frac{u_1}{q_0}$. Since $w_2 > q_0 > v_0 q_0 > u_1$, $g(q_0) = g(v_0 q_0) = g(u_1)$. And, since $v_0 q_0 + 1 - v_0 > w_2$, $g(v_0 q_0 + 1 - v_0) > g(u_1)$. Therefore, $-g(v_0 q_0) + \frac{g(q_0)}{1-g(q_0)}(1 - g(v_0 q_0 + 1 - v_0)) < 0$, which contradicts (EC.46). Hence, this case cannot hold. (c) $w_1 = 0$ and $w_2 = 1$. In this case, $g(u) = c, \forall u \in (0,1)$, for some constant $c \in (0,1)$. By Lemma EC.2, $\rho = c\mathrm{VaR}_0 + (1-c)\mathrm{VaR}_1$, and ρ has convex level sets with respect to $\mathcal{P}^\rho$.

Case (iii.iv.ii) g is strictly increasing on $(0,1)$. Then, $g(p_1) - g(p_2) \neq 0$ for any $p_1 \neq p_2$. We will show that $g(1-) = 1$ and $g(0+) = 0$. Consider $0 < x_1 < x_2 < x_3$, $p_1, p_2 \in (0,1)$, and $p_i \neq \alpha_0$, $i = 1, 2$, such that

$$\rho(p_1 \delta_{x_1} + (1-p_1)\delta_{x_2}) = \rho(p_2 \delta_{x_1} + (1-p_2)\delta_{x_3}),$$

which is equivalent to

$$x_1 g(p_1) + x_2(1 - g(p_1)) = x_1 g(p_2) + (1 - g(p_2)) x_3. \tag{EC.48}$$

Let $\frac{x_1}{x_2} = c_1$ and $\frac{x_3}{x_2} = c_3$. Then, $c_1 \in (0,1)$, $c_3 > 1$, and (EC.48) is equivalent to

$$c_1 = \frac{1 - g(p_2)}{g(p_1) - g(p_2)} c_3 - \frac{1 - g(p_1)}{g(p_1) - g(p_2)}. \tag{EC.49}$$

For any fixed $0 < p_1 < p_2 < 1$, $p_i \neq \alpha_0$, $i = 1, 2$, and any $1 < c_3 < \frac{1-g(p_1)}{1-g(p_2)}$, define c_1 as in (EC.49). Then, $c_1 \in (0,1)$. For any such p_1, p_2, c_3, and c_1, since ρ has convex level sets with respect to $\mathcal{D}_{disc} \cap \{F \mid q^-_{\alpha_0}(F) = q^+_{\alpha_0}(F)\}$, it follows that

$$\begin{aligned}
&x_1 g(p_1) + x_2(1 - g(p_1)) = \rho(p_1 \delta_{x_1} + (1-p_1)\delta_{x_2}) \\
&= \rho(v(p_1 \delta_{x_1} + (1-p_1)\delta_{x_2}) + (1-v)(p_2 \delta_{x_1} + (1-p_2)\delta_{x_3})) \\
&= \rho((v p_1 + (1-v)p_2)\delta_{x_1} + v(1-p_1)\delta_{x_2} + (1-v)(1-p_2)\delta_{x_3}) \\
&= x_1 g(v p_1 + (1-v)p_2) + x_2(g(v + (1-v)p_2) - g(v p_1 + (1-v)p_2)) \\
&\quad + x_3(1 - g(v + (1-v)p_2)), \ \forall v \in (0,1),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&c_1[g(p_1) - g(v p_1 + (1-v)p_2)] + 1 - g(p_1) - g(v + (1-v)p_2) + g(v p_1 + (1-v)p_2) \\
&= c_3[1 - g(v + (1-v)p_2)], \ \forall v \in (0,1).
\end{aligned}$$

Plugging (EC.49) into the above equation, we obtain that for any $0 < p_1 < p_2 < 1$, $p_i \neq \alpha_0$, $i = 1, 2$, any $1 < c_3 < \frac{1-g(p_1)}{1-g(p_2)}$, and any $v \in (0,1)$, it holds that

$$\begin{aligned}
0 = &\, c_3 \left[\frac{1 - g(p_2)}{g(p_1) - g(p_2)} (g(p_1) - g(v p_1 + (1-v)p_2)) - 1 + g(v + (1-v)p_2) \right] \\
&- \frac{1 - g(p_1)}{g(p_1) - g(p_2)} [g(p_1) - g(v p_1 + (1-v)p_2)] + 1 - g(p_1) \\
&- g(v + (1-v)p_2) + g(v p_1 + (1-v)p_2).
\end{aligned} \tag{EC.50}$$

Therefore,

$$\begin{aligned}0=&-\frac{1-g(p_1)}{g(p_1)-g(p_2)}[g(p_1)-g(vp_1+(1-v)p_2)]+1-g(p_1)\\&-g(v+(1-v)p_2)+g(vp_1+(1-v)p_2),\ \forall v\in(0,1),\forall p_1<p_2,p_i\neq\alpha_0,i=1,2.\end{aligned}$$

which is equivalent to

$$\begin{aligned}0=&g(vp_1+(1-v)p_2)(1-g(p_2))+g(v+(1-v)p_2)(g(p_2)-g(p_1))\\&+g(p_1)g(p_2)-g(p_2),\ \forall v\in(0,1),\forall p_1<p_2,p_i\neq\alpha_0,i=1,2.\end{aligned}\tag{EC.51}$$

Letting $v\uparrow 1$ in (EC.51), we obtain

$$0=g(p_1+)(1-g(p_2))+g(1-)(g(p_2)-g(p_1))+g(p_1)g(p_2)-g(p_2),\forall p_1<p_2,p_i\neq\alpha_0,i=1,2.\tag{EC.52}$$

Since g is increasing on $(0,1)$, there exists $p_1^*\in(0,1)$, such that $p_1^*\neq\alpha_0$ and g is continuous at p_1^*. Choose any $p_2^*>p_1^*$ and $p_2^*\neq\alpha_0$. Letting $p_1=p_1^*$ and $p_2=p_2^*$ in (EC.52) leads to $(g(p_1^*)-g(p_2^*))(1-g(1-))=0$. Since g is strictly increasing, it follows that

$$g(1-)=1.\tag{EC.53}$$

Letting $q=q_0\neq\alpha_0$ in (EC.46) leads to

$$\frac{g(vq_0)}{g(q_0)}=\frac{1-g(vq_0+1-v)}{1-g(q_0)},\forall v\in(0,1).\tag{EC.54}$$

It follows from (EC.54) and (EC.53) that

$$g(0+)=\lim_{v\downarrow 0}g(vq_0)=\lim_{v\downarrow 0}\frac{g(q_0)}{1-g(q_0)}(1-g(vq_0+1-v))=\frac{g(q_0)}{1-g(q_0)}(1-g(1-))=0.\tag{EC.55}$$

We will then show that g is continuous on $(0,1)$. Consider $0=x_1<x_2<x_3<x_4$, $p_1,p_2\in(0,1)$, and $p_i\neq\alpha_0$, $i=1,2$, such that

$$\rho(p_1\delta_{x_1}+(1-p_1)\delta_{x_3})=\rho(p_2\delta_{x_2}+(1-p_2)\delta_{x_4}),$$

which is equivalent to

$$x_1g(p_1)+x_3(1-g(p_1))=x_2g(p_2)+x_4(1-g(p_2)).\tag{EC.56}$$

Since ρ has convex level sets with respect to $\mathcal{D}_{disc}\cap\{F\mid q_{\alpha_0}^-(F)=q_{\alpha_0}^+(F)\}$, it follows that for any $v\in(0,1)$, it holds that

$$\begin{aligned}&x_3(1-g(p_1))=x_1g(p_1)+x_3(1-g(p_1))=\rho(p_1\delta_{x_1}+(1-p_1)\delta_{x_3})\\=&\rho(v(p_1\delta_{x_1}+(1-p_1)\delta_{x_3})+(1-v)(p_2\delta_{x_2}+(1-p_2)\delta_{x_4}))\\=&\rho(vp_1\delta_{x_1}+(1-v)p_2\delta_{x_2}+v(1-p_1)\delta_{x_3}+(1-v)(1-p_2)\delta_{x_4})\\=&x_2(g(vp_1+(1-v)p_2)-g(vp_1))\\&+x_3(g(v+(1-v)p_2)-g(vp_1+(1-v)p_2))+x_4(1-g(v+(1-v)p_2)).\end{aligned}\tag{EC.57}$$

Let $\frac{x_3}{x_2} = 1 + c_3$ and $\frac{x_4}{x_2} = 1 + c_3 + c_4$. Then, $c_3 > 0$, $c_4 > 0$, and (EC.56) becomes

$$c_3 = \frac{1 - g(p_2)}{g(p_2) - g(p_1)} c_4 + \frac{g(p_1)}{g(p_2) - g(p_1)}. \tag{EC.58}$$

Furthermore, (EC.57) is equivalent to

$$\begin{aligned} 0 = {} & g(vp_1 + (1-v)p_2) - g(vp_1) + (1 + c_3 + c_4)(1 - g(v + (1-v)p_2)) \\ & + (1 + c_3)(g(v + (1-v)p_2) - g(vp_1 + (1-v)p_2) - 1 + g(p_1)), \forall v \in (0,1). \end{aligned} \tag{EC.59}$$

For any $0 < p_1 < p_2 < 1$, $p_i \neq \alpha_0$, $i = 1, 2$, and $c_4 > 0$, let c_3 be defined in (EC.58). Then, $c_3 > 0$. Hence, (EC.59) holds for any such p_1, p_2, c_3, and c_4. Plugging (EC.58) into (EC.59), we obtain that for any $0 < p_1 < p_2 < 1$, $p_i \neq \alpha_0$, $i = 1, 2$, and any $c_4 > 0$, it holds that

$$\begin{aligned} 0 = {} & g(vp_1 + (1-v)p_2) - g(vp_1) + \frac{g(p_2)}{g(p_2) - g(p_1)}[g(p_1) - g(vp_1 + (1-v)p_2)] \\ & + c_4 \frac{1 - g(p_2)}{g(p_2) - g(p_1)}[g(v + (1-v)p_2) - g(vp_1 + (1-v)p_2) - 1 + g(p_1)] \\ & + c_4 \frac{1 - g(p_1)}{g(p_2) - g(p_1)}[1 - g(v + (1-v)p_2)], \forall v \in (0,1), \end{aligned} \tag{EC.60}$$

which implies that

$$\begin{aligned} 0 = {} & g(vp_1 + (1-v)p_2) - g(vp_1) \\ & + \frac{g(p_2)}{g(p_2) - g(p_1)}[g(p_1) - g(vp_1 + (1-v)p_2)], \ \forall 0 < p_1 < p_2 < 1, p_i \neq \alpha_0, i = 1, 2, \forall v \in (0,1), \end{aligned}$$

which can be simplified to be

$$-g(vp_1 + (1-v)p_2) - (g(p_2) - g(p_1))\frac{g(vp_1)}{g(p_1)} + g(p_2) = 0, \ \forall 0 < p_1 < p_2 < 1, p_i \neq \alpha_0, i = 1, 2, \forall v \in (0,1). \tag{EC.61}$$

Letting $p_2 \uparrow 1$ in (EC.61) and applying (EC.53), we obtain

$$-g((vp_1 + 1 - v)-) - (1 - g(p_1))\frac{g(vp_1)}{g(p_1)} + 1 = 0, \ \forall 0 < p_1 < 1, p_1 \neq \alpha_0, \forall v \in (0,1). \tag{EC.62}$$

Then, it follows from (EC.46) and (EC.62) that

$$g((vp_1 + 1 - v)-) = g(vp_1 + 1 - v), \forall 0 < p_1 < 1, p_1 \neq \alpha_0, \forall v \in (0,1),$$

which implies that

$$g(v-) = g(v), \forall v \in (0,1). \tag{EC.63}$$

It follows from (EC.47) and (EC.63) that g is continuous on $(0,1) \setminus \{\alpha_0\}$, i.e.,

$$g(v-) = g(v) = g(v+), \forall v \in (0,1), v \neq \alpha_0. \tag{EC.64}$$

Letting $p_1 \downarrow 0$ in (EC.61), we obtain

$$-g(((1-v)p_2)+)-(g(p_2)-g(0+))\lim_{p_1\downarrow 0}\frac{g(vp_1)}{g(p_1)}+g(p_2)=0,\ \forall 0<p_2<1, p_2\neq\alpha_0, \forall v\in(0,1). \quad \text{(EC.65)}$$

By (EC.46), we have

$$\lim_{p_1\downarrow 0}\frac{g(vp_1)}{g(p_1)}=\lim_{p_1\downarrow 0}\frac{1-g(vp_1+1-v)}{1-g(p_1)}=\frac{1-g((1-v)+)}{1-g(0+)}=1-g((1-v)+), \forall v\in(0,1), \quad \text{(EC.66)}$$

where the last step follows from (EC.55). Plugging (EC.55) and (EC.66) into (EC.65), we obtain

$$g(((1-v)p_2)+)=g(p_2)g((1-v)+),\ \forall 0<p_2<1, p_2\neq\alpha_0, \forall v\in(0,1). \quad \text{(EC.67)}$$

For any $\epsilon\in(-\alpha_0,0)\cup(0,1-\alpha_0)$, letting $v=v_0\in(1-\alpha_0,1)$ and $p_2=\alpha_0+\epsilon$ in (EC.67), we obtain

$$g(((1-v_0)(\alpha_0+\epsilon))+)=g(\alpha_0+\epsilon)g((1-v_0)+),\ \forall\epsilon\in(-\alpha_0,0)\cup(0,1-\alpha_0). \quad \text{(EC.68)}$$

Noting that $(1-v_0)(\alpha_0+\epsilon)<1-v_0<\alpha_0$ for $\forall\epsilon\in(-\alpha_0,0)\cup(0,1-\alpha_0)$, it follows from (EC.64) and (EC.68) that

$$g((1-v_0)(\alpha_0+\epsilon))=g(\alpha_0+\epsilon)g(1-v_0),\ \forall\epsilon\in(-\alpha_0,0)\cup(0,1-\alpha_0). \quad \text{(EC.69)}$$

Letting $\epsilon\downarrow 0$ on both sides of (EC.69), we obtain

$$g(((1-v_0)\alpha_0)+)=g(\alpha_0+)g(1-v_0). \quad \text{(EC.70)}$$

Letting $\epsilon\uparrow 0$ on both sides of (EC.69), we obtain

$$g(((1-v_0)\alpha_0)-)=g(\alpha_0-)g(1-v_0). \quad \text{(EC.71)}$$

It follows from (EC.64) that $g(((1-v_0)\alpha_0)+)=g(((1-v_0)\alpha_0)-)=g((1-v_0)\alpha_0)$, which in combination with (EC.70), (EC.71), and $g(1-v_0)\in(0,1)$ implies that

$$g(\alpha_0+)=g(\alpha_0-)=g(\alpha_0). \quad \text{(EC.72)}$$

Combining (EC.64) and (EC.72), we obtain that g is continuous on $(0,1)$, i.e.,

$$g(v-)=g(v)=g(v+), \forall v\in(0,1). \quad \text{(EC.73)}$$

And then, (EC.67) becomes

$$g((1-v)p_2)=g(p_2)g(1-v),\ \forall 0<p_2<1, p_2\neq\alpha_0, \forall v\in(0,1). \quad \text{(EC.74)}$$

Letting $p_2\to\alpha_0$ and applying (EC.73), we obtain that

$$g((1-v)p_2)=g(p_2)g(1-v),\ \forall 0<p_2<1, \forall v\in(0,1). \quad \text{(EC.75)}$$

Lastly, we will show that $g(u)=u$ for any $u\in(0,1)$. By (EC.75) and (EC.66), we obtain

$$g(v)=1-g(1-v),\forall v\in(0,1). \tag{EC.76}$$

In the following, we will show by induction that

$$g(\frac{k}{2^n})=\frac{k}{2^n},\ k=1,2,\dots,2^n-1,\forall n\in\mathbb{N}. \tag{EC.77}$$

Letting $v=\frac{1}{2}$ in (EC.76), we obtain $g(\frac{1}{2})=\frac{1}{2}$. Hence, (EC.77) holds for $n=1$. Suppose (EC.77) holds for n. We will show that it also holds for $n+1$. In fact, for any $0\le k\le 2^{n-1}-1$, since $1\le 2k+1\le 2^n-1$, it follows from (EC.75) that

$$g(\frac{2k+1}{2^{n+1}})=g(\frac{1}{2})g(\frac{2k+1}{2^n})=\frac{2k+1}{2^{n+1}},\ 0\le k\le 2^{n-1}-1. \tag{EC.78}$$

For any $2^{n-1}\le k\le 2^n-1$, it holds that $1\le 2^{n+1}-(2k+1)\le 2^n-1$. Hence, it follows from (EC.76) that

$$\begin{aligned}g(\frac{2k+1}{2^{n+1}})&=1-g(\frac{2^{n+1}-(2k+1)}{2^{n+1}})=1-\frac{2^{n+1}-(2k+1)}{2^{n+1}}\ \text{(by (EC.78))}\\ &=\frac{2k+1}{2^{n+1}},\ 2^{n-1}\le k\le 2^n-1.\end{aligned} \tag{EC.79}$$

In addition, for any $1\le k\le 2^n-1$, $g(\frac{2k}{2^{n+1}})=g(\frac{k}{2^n})=\frac{k}{2^n}$, which in combination with (EC.78) and (EC.79) implies that (EC.77) holds for $n+1$, and hence holds for any n. Since $\{k/2^n,k=1,\dots,2^n-1,n\in\mathbb{N}\}$ is dense on $(0,1)$ and g is continuous on $(0,1)$, it follows from (EC.77) that $g(u)=u$ for all $u\in(0,1)$, which completes the proof. $\square$

Then, the proof of Theorem 3 is as follows.

Proof of Theorem 3. By Theorem 6 of Gneiting (2011), a necessary condition for ρ to be elicitable is that ρ has convex level sets. By Theorem 4, only those risk measures listed in cases (i)-(iv) of Theorem 4 satisfy the necessary condition for a risk measure to be elicitable with respect to $\mathcal{D}_{disc}\cap\{F\mid q^-_{\alpha_0}(F)=q^+_{\alpha_0}(F)\}$. Therefore, we only need to check if those risk measures are elicitable with respect to $\mathcal{D}_{disc}\cap\{F\mid q^-_{\alpha_0}(F)=q^+_{\alpha_0}(F)\}$.

First, we will show that for $c\in(0,1]$, $\rho=c\mathrm{VaR}_0+(1-c)\mathrm{VaR}_1$ is not elicitable with respect to $\mathcal{D}_{disc}\cap\{F\mid q^-_{\alpha_0}(F)=q^+_{\alpha_0}(F)\}$. Suppose for the sake of contradiction that ρ is elicitable with respect to $\mathcal{D}_{disc}\cap\{F\mid q^-_{\alpha_0}(F)=q^+_{\alpha_0}(F)\}$, then there exists a function S such that (7) holds. For any u, letting $F=\delta_u$ in (7) yields

$$u=\arg\min_x S(x,u),\forall u. \tag{EC.80}$$

For any $u<v$ and any $p\in(0,1)$, $p\ne\alpha_0$, letting $F=p\delta_u+(1-p)\delta_v$ in (7) yields

$$cu+(1-c)v=\arg\min_x[pS(x,u)+(1-p)S(x,v)],\forall u<v,\forall p\ne\alpha_0,$$

which implies that

$$pS(cu+(1-c)v,u)+(1-p)S(cu+(1-c)v,v)\leq pS(x,u)+(1-p)S(x,v),\forall u<v,\forall x,\forall p\neq\alpha_0.$$

Letting $p\downarrow 0$ leads to

$$S(cu+(1-c)v,v)\leq S(x,v),\forall u<v,\forall x, \tag{EC.81}$$

which implies that

$$cu+(1-c)v\in\arg\min_x S(x,v),\forall u<v,$$

which contradicts (EC.80).

Second, we will show that for $c=0$, $\rho=c\mathrm{VaR}_0+(1-c)\mathrm{VaR}_1=\mathrm{VaR}_1$ is not elicitable with respect to $\mathcal{D}_{disc}\cap\{F\mid q^-_{\alpha_0}(F)=q^+_{\alpha_0}(F)\}$. Suppose for the sake of contradiction that ρ is elicitable with respect to $\mathcal{D}_{disc}\cap\{F\mid q^-_{\alpha_0}(F)=q^+_{\alpha_0}(F)\}$, then there exists a function S such that (7) holds. For any u, letting $F=\delta_u$ in (7) yields

$$u=\arg\min_x S(x,u),\forall u. \tag{EC.82}$$

For any $u<v$ and any $p\in(0,1)$, $p\neq\alpha_0$, letting $F=p\delta_u+(1-p)\delta_v$ in (7) yields

$$v=\arg\min_x[pS(x,u)+(1-p)S(x,v)],\forall u<v,\forall p\in(0,1),p\neq\alpha_0,$$

which implies that

$$pS(v,u)+(1-p)S(v,v)\leq pS(x,u)+(1-p)S(x,v),\forall u<v,\forall p\in(0,1),p\neq\alpha_0,\forall x.$$

Letting $p\uparrow 1$, we obtain

$$S(v,u)\leq S(x,u),\forall u<v,\forall x,$$

which contradicts (EC.82).

Third, we will show that for any $\alpha\in(0,1)$, $\alpha\neq\alpha_0$, VaR_α is not elicitable with respect to $\mathcal{D}_{disc}\cap\{F\mid q^-_{\alpha_0}(F)=q^+_{\alpha_0}(F)\}$. Suppose for the sake of contradiction that VaR_α is elicitable with respect to $\mathcal{D}_{disc}\cap\{F\mid q^-_{\alpha_0}(F)=q^+_{\alpha_0}(F)\}$, then there exists a function S such that (7) holds. For any $u<v$, letting $F=\alpha\delta_u+(1-\alpha)\delta_v$ in (7) leads to

$$u=\arg\min_x[\alpha S(x,u)+(1-\alpha)S(x,v)],\forall u<v. \tag{EC.83}$$

For any $u<v$ and any $p<\alpha$, $p\neq\alpha_0$, letting $F=p\delta_u+(1-p)\delta_v$ in (7) yields

$$v=\arg\min_x[pS(x,u)+(1-p)S(x,v)],\forall u<v,\forall p<\alpha,p\neq\alpha_0,$$

which implies that

$$pS(v,u)+(1-p)S(v,v)\leq pS(x,u)+(1-p)S(x,v),\forall u<v,\forall p<\alpha,p\neq\alpha_0,\forall x. \tag{EC.84}$$

Letting $p \uparrow \alpha$ in (EC.84), we have

$$\alpha S(v,u) + (1-\alpha)S(v,v) \leq \alpha S(x,u) + (1-\alpha)S(x,v), \forall u < v, \forall x, \tag{EC.85}$$

which implies

$$v \in \arg\min_x [\alpha S(x,u) + (1-\alpha)S(x,v)], \forall u < v,$$

which contradicts (EC.83).

Fourth, we will show that VaR_{α_0} is elicitable with respect to $\mathcal{D}^1 \cap \{F \mid q^-_{\alpha_0}(F) = q^+_{\alpha_0}(F)\}$. Define

$$S(x,y) = (1_{\{x \geq y\}} - \alpha_0)(x-y). \tag{EC.86}$$

It follows from part (c) of Theorem 9 in Gneiting (2011) that

$$[q^-_{\alpha_0}(F), q^+_{\alpha_0}(F)] = \arg\min_x \int S(x,y)dF(y),\ \forall F \in \mathcal{D}^1,$$

which implies that

$$\mathrm{VaR}_{\alpha_0}(F) = \arg\min_x \int S(x,y)dF(y),\ \forall F \in \mathcal{D}^1 \cap \{F \mid q^-_{\alpha_0}(F) = q^+_{\alpha_0}(F)\}.$$

Fifth, we will show that for any $\alpha \in (0,1)$, $\alpha \neq \alpha_0$, and for any $c \in [0,1)$, $\rho = cq^-_\alpha + (1-c)q^+_\alpha$ is not elicitable with respect to $\mathcal{D}_{disc} \cap \{F \mid q^-_{\alpha_0}(F) = q^+_{\alpha_0}(F)\}$. Suppose for the sake of contradiction that ρ is elicitable with respect to $\mathcal{D}_{disc} \cap \{F \mid q^-_{\alpha_0}(F) = q^+_{\alpha_0}(F)\}$. Then, there exists a forecasting objective function $S(x,y)$ such that

$$cq^-_\alpha(F) + (1-c)q^+_\alpha(F) = \arg\min_x \int S(x,y)dF(y), \forall F \in \mathcal{D}_{disc} \cap \{F \mid q^-_{\alpha_0}(F) = q^+_{\alpha_0}(F)\}. \tag{EC.87}$$

Then, for any $u < v$, letting $F = \alpha\delta_u + (1-\alpha)\delta_v$ in (EC.87) leads to

$$cu + (1-c)v = \arg\min_x [\alpha S(x,u) + (1-\alpha)S(x,v)], \forall u < v. \tag{EC.88}$$

For any $u < v$ and any $p > \alpha$, $p \neq \alpha_0$, letting $F = p\delta_u + (1-p)\delta_v$ in (EC.87) yields

$$u = \arg\min_x [pS(x,u) + (1-p)S(x,v)], \forall u < v, \forall p > \alpha, p \neq \alpha_0,$$

which implies that

$$pS(u,u) + (1-p)S(u,v) \leq pS(x,u) + (1-p)S(x,v), \forall u < v, \forall p > \alpha, p \neq \alpha_0, \forall x. \tag{EC.89}$$

Letting $p \downarrow \alpha$ in (EC.89), we have

$$\alpha S(u,u) + (1-\alpha)S(u,v) \leq \alpha S(x,u) + (1-\alpha)S(x,v), \forall u < v, \forall x, \tag{EC.90}$$

which implies

$$u \in \arg\min_{x} [\alpha S(x,u) + (1-\alpha) S(x,v)], \forall u < v,$$

which contradicts (EC.88) because $u < cu + (1-c)v$ for any $c \in [0,1)$ and $u < v$.

Sixth, for any $c \in [0,1)$, $\rho = cq^-_{\alpha_0} + (1-c)q^+_{\alpha_0} = \mathrm{VaR}_{\alpha_0}$ on $\mathcal{D}^1 \cap \{F \mid q^-_{\alpha_0}(F) = q^+_{\alpha_0}(F)\}$ and hence has been shown to be elicitable with respect to $\mathcal{D}^1 \cap \{F \mid q^-_{\alpha_0}(F) = q^+_{\alpha_0}(F)\}$ in the fourth step.

Seventh, it follows from Theorem 7 in Gneiting (2011) that the mean functional $\rho(F) := \int x dF(x)$ is elicitable with respect to $\mathcal{D}^1$. The proof is thus completed. $\square$