Numerical pricing of discrete barrier and lookback options via Laplace transforms

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Most contracts of barrier and lookback options specify discrete monitoring policies. However, unlike their continuous counterparts, discrete barrier and lookback options essentially have no analytical solution. For a broad class of models, including the classical Brownian model and jump-diffusion models, we show that the Laplace transforms of discrete barrier and lookback options can be obtained via a recursion involving only analytical formulae of standard European call and put options, thanks to Spitzer's formula. The Laplace transforms can be numerically inverted to get option prices fast and accurately. Furthermore, the same method can be used to compute the hedging parameters (the greeks) of these products.

1 Introduction

Among the most popular path-dependent options are lookback and barrier options, the payoff of which depend on the extrema of the underlying stochastic process. One important feature of these options is that the values of the options are quite sensitive to whether the extrema are monitored discretely or continuously; see, for example, Broadie, Glasserman, and Kou (1997, 1999).

In the continuously monitored case, the analytical solutions for lookback and barrier options are available under the classical Brownian model; see, for example, Gatto, Goldman and Sosin (1979), Goldman, Sosin and Shepp (1979), and Conze and Viswanathan (1991) for lookback options; and see, for example, Merton (1973), Heynen and Kat (1994a, 1994b), Rubinstein and Reiner (1991), Chance (1998) for barrier options. Recently, Boyle and Tian (1999) and Davydov and Linetsky (2001) have priced continuously monitored barrier and lookback options under the CEV model using lattice and Laplace transform methods, respectively.

In practice most of the lookback and barrier options are discretely monitored; for some (regulatory and practical) reasons of this, see Broadie, Glasserman, and Kou (1997). However, unlike the continuous monitoring case, there is essentially no analytical solution for discrete barrier and lookback options, except by using the m-dimensional multivariate normal distribution (m being the number of monitoring points), which is hardly computable if $m > 5$; see, for example, Heynan and Kat (1995).

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Because of this, various numerical methods have been proposed for discrete barrier and lookback options under the classical Brownian model, including, for example, lattice methods (Babbs, 1992; Boyle and Lau, 1994; Cheuk and Vorst, 1997; Hull and White, 1993; Kat, 1995; and Ritchken, 1995) and numerical integration (AitSahlia and Lai, 1997; Sullivan, 2000; Tse *et al*, 2001).

Broadie, Glasserman and Kou (1997, 1999) propose an enhanced trinomial tree method and develop analytical approximations to relate the prices of continuous and discrete lookback and barrier options under the classical Brownian model. (Chuang, 1996, also independently suggested the approximation for barrier options in a heuristic way.) The derivation in Broadie *et al* (1997) for discrete barrier options is further simplified and extended in Hörfelt (2003) and Kou (2003). The approximations are very simple to use and give very good results when the number of monitoring points is large; however, they may not be sufficiently accurate when there is a limited number of monitoring points or the option is close to maturity. The enhanced trinomial trees may still be time-consuming. Furthermore, it is not clear how to generalize the results outside the classical Brownian setting.

In an interesting paper Ohgren (2001) shows how to compute the characteristic function of the discretely monitored maximum stock price, by using the celebrated Spitzer's (1956) formula, and then uses the result to price discrete lookbacks at the inception of the contract and at monitoring points (but not at any generic point in time), if the previous achieved maximum (minimum) stock price can be ignored.

In this paper, building on the result in Ohgren (2001) and the Laplace transform (with respect to strike prices) introduced in Carr and Madan (1999), we develop a method based on Laplace transform that easily allows us to compute the price and hedeging parameters (the Greeks) of discretely monitored lookback and barrier options at *any* point in time, even if the previous achieved maximum (minimum) cannot be ignored. The proposed method has several distinctive features:

- ❑ It allows us to compute, via a simple recursion only involving the standard European call and put options, the Laplace transforms of the discrete barrier and lookback options; see Sections 3 and 4.
- ❑ The Laplace transforms can then be numerically inverted easily via a two-sided Euler algorithm, and the inversion is fast and accurate; see Section 5 and Appendix B.
- ❑ The method can compute the prices of barrier and lookback options at any time point (not just at the monitoring points and at the inception of the contract). Because of this flexibility, we are also able to compute, at almost no additional computational cost, the main hedging parameters (the Greeks); see Sections 3 and 4.
- ❑ It can be implemented not only under the classical Brownian model, but also under more *general* models (eg, jump-diffusion models) with stationary independent increaments; see Section 5.

After the paper had been accepted by the journal, we found out that a similar method using Fourier transforms in the case of pricing discrete lookback options at the monitoring points (but not at any time points, hence with no discussion of the hedging parameters) was independently suggested on pp. 893–4 in Borovkov and Novikov (2002). The method proposed here is more general, as it is applicable to the pricing of both discrete lookback and barrier options at any time points (and hence to the computing of the hedging parameters).

The rest of the paper is organized as follows. Section 2 introduces some notations. Laplace transforms are derived in Section 3. The main algorithm is summarized in Section 4. We then show in Section 5 how to implement the algorithm and provide some numerical results. Appendix A discusses possible extensions of our methodology to various products. Details on the Laplace inversion algorithms are deferred to Appendix B.

The reader who is mainly interested in practical aspects of the method may want to go directly to the algorithm in Section 4 and then to the numerical examples in Section 5.

2 Notation

2.1 Lookback options

A standard (also called floating) lookback call (put) gives the option holder the right to buy (sell) an asset at its lowest (highest) price during the life of the option. In a discrete time setting the minimum (maximum) of the asset price will be determined at discrete monitoring instants. We assume that the monitoring instants are equally spaced in time. More precisely, consider the asset value $S(t)$, monitored in the interval $[0, T]$ at a sequence of equally spaced monitoring points, $0 \equiv t(0) < t(1) < \cdots < t(m) \equiv T$. Let $X_i := \log\{S(t(i))/S(t(i-1))\}$, where X_i is the return between $t(i-1)$ and $t(i)$, and

$$S_k := S(t(k)) = S_{k-1}\mathrm{e}^{X_k} = S_0\mathrm{e}^{(X_1+\cdots+X_k)}, \quad k = 0,1,\ldots,m \tag{1}$$

Introduce the maxima and minima of the asset price only at the monitoring points,

$$M_{t(l),t(k)} := \max_{l\le j\le k} S_j,\ 0 \le l \le k \le m;\ \ M_{0,T} := \max_{0\le k\le m} S_k,\ \ m_{0,T} := \min_{0\le k\le m} S_k \tag{2}$$

At any time t, lying between the $(l-1)$th and lth monitoring points, ie,

$$(t(l-1)) \le t < t(l)$$

standard finance theory gives the values of the standard (floating) lookback call and put option at any time $t \in [0,T]$ as

$$LC(t,T) = \mathrm{e}^{-r(T-t)}\,\mathsf{E}^*\left[S(T) - m_{0,T} \mid \mathcal{F}_t\right],$$
$$LP(t,T) = \mathrm{e}^{-r(T-t)}\,\mathsf{E}^*\left[M_{0,T} - S(T) \mid \mathcal{F}_t\right]$$

respectively, where r is the risk-free interest rate and E^* represents the expectation under the risk-neutral measure (the measure could be specified by arbitrage arguments for the Brownian model or by equilibrium arguments for general models). In the same way, at any time $t \in [0,T]$, for the fixed strike put and call we have $FP(t, T) = e^{-r(T-t)}\mathsf{E}^*[(K - m_{0,T})^+|\mathcal{F}_t]$ and $FC(t, T) = e^{-r(T-t)}\mathsf{E}^*[(M_{0,T} - K)^+ | \mathcal{F}_t]$. Other types of lookback options include percentage lookbacks in which the extreme values are multiplied by a constant, and partial-lookback options in which the monitoring interval for the extremum is a subinterval of $[0, T]$. We will not attempt to price such derivatives and refer the interested reader to Andreasen (1998) for a detailed description.

2.2 Barrier options

Barrier options can be classified according to whether the asset price needs to pass or to avoid a certain level to receive a payoff. In the first case they are called knock-in options, in the second knock-out. For example, the up-and-out call and put options (UOC and UOP from now on) with strike K, barrier H and maturity T, have payoffs $(S(T) - K)^+\mathbf{1}_{\{M_{0,T} < H\}}$ and $(K - S(T))^+\mathbf{1}_{\{M_{0,T} < H\}}$, where $\mathbf{1}_{\{\cdot\}}$ is the indicator function of the event $\{\cdot\}$. Similarly, up-and-in call and put options (UIC and UIP from now on) with the same parameters have payoffs $(S(T) - K)^+\mathbf{1}_{\{M_{0,T} \geq H\}}$ and $(K - S(T))^+\mathbf{1}_{\{M_{0,T} \geq H\}}$. Down-and-in and down-and-out options have a similar structure with $M_{0,T}$ substituted by $m_{0,T}$. As seen for lookbacks, we can value barrier options by taking the discounted expected value of the payoff at maturity under the risk-neutral measure; for example, the price of an up-and-out put option is

$$UOP(t,T) = e^{-r(T-t)}\,\mathsf{E}^*\left[(K - S(T))^+\mathbf{1}_{\{M_{0,T<H}\}}\,|\,\mathcal{F}_t\right] \tag{3}$$

All other barrier options can be priced in the same way.

2.3 Some mathematical notation

Define the maxima of the return process between the monitoring points to be

$$\tilde{M}_{l,k} := \max\left(0, X_{l+1}, X_{l+1} + X_{l+2}, \ldots, X_{l+1} + \cdots + X_k\right) = \max_{l \leq j \leq k} \sum_{i=l+1}^{j} X_i, \quad l = 0, \ldots, k$$

where we have used the convention that the sum is zero if the index set is empty. Throughout the paper, we shall assume that $X_1, X_2, \ldots$, are independent identically distributed (iid) random variables. With $X_{s,t} := \log\{S(t)/S(s)\}$ being the return between time s and time t, $t \geq s$, define

$$A(u;t) := \mathsf{E}^*\left[e^{uY^t_{l,m}}\right] = x_{l,m}\mathsf{E}^*\left[e^{uX_{t,t(l)}}\right], Y^t_{l,m} := X_{t,t(l)} + \tilde{M}_{l,m} \tag{4}$$

$$C(u,v;t) := \mathsf{E}^*\left[e^{uX_{t,t(l)}}\right]\mathsf{E}^*\left[e^{u\tilde{M}_{l,m} + vX_{t,T}}\right] = \hat{x}_{l,m}\mathsf{E}^*\left[e^{(u+v)X_{t,t(l)}}\right] \tag{5}$$

where

$$x_{l,k} := \mathsf{E}^*\left[e^{u\tilde{M}_{l,k}}\right], \quad \hat{x}_{l,k} := \mathsf{E}^*\left[e^{u\tilde{M}_{l,k} + vB_{l,k}}\right], \quad l \leq k; \quad B_{l,k} := \sum_{i=l+1}^{k} X_i \tag{6}$$

3 Laplace transforms for discrete lookback and barrier options

3.1 Characteristic function computation

The results in this subsection generalize the results in Ohgren (2001) by showing how to compute $x_{l,k}$ and $\hat{x}_{l,k}$(6) recursively via Spitzer's formula for the sum of iid random variables.

LEMMA 1 *Define for* $0 \le l \le k$,

$$a_{l,k} := \mathsf{E}^*\left[\mathrm{e}^{uB^+_{l,k}}\right],\ \ \hat{a}_{l,k} := \mathsf{E}^*\left[\mathrm{e}^{(u+v)B^+_{l,k}}\right] + \mathsf{E}^*\left[\mathrm{e}^{-vB^-_{l,k}}\right] - 1,\ \ u, v \in \mathbb{C} \tag{7}$$

where $B^+_{l,k}$ *and* $B^-_{l,k}$ *denote the positive and negative part of the* $B_{l,k}$, *respectively. Then for any given l, we have*

$$x_{l,k+1} = \frac{1}{k-l+1}\sum_{j=0}^{k-l} a_{l,k+1-j}\, x_{l,l+j} \tag{8}$$

$$\hat{x}_{l,k+1} = \frac{1}{k-l+1}\sum_{j=0}^{k-l} \hat{a}_{l,k+1-j}\, \hat{x}_{l,l+j} \tag{9}$$

PROOF Equation (8) is a slight generalization of the recursion given in Ohgren (2001), in which the case $l = 0$ is discussed. To show (9), first note that Spitzer (1956) also proves that, for $s < 1$ and $u, v \in C$, with Im $(u) \ge 0$ and Im $(v) \ge 0$:

$$\sum_{k=0}^{\infty} s^k \mathsf{E}^*\left[\mathrm{e}^{-u\tilde{M}_{l,k} + vB_{l,k}}\right] = \exp\left(\sum_{k=1}^{\infty} \frac{s^k}{k}\left(\mathsf{E}^*\left[\mathrm{e}^{(u+v)B^+_{l,k}}\right] + \mathsf{E}^*\left[\mathrm{e}^{-vB^-_{l,k}}\right] - 1\right)\right) \tag{10}$$

We can again extend (10) to any $u, v \in C$, by limiting $s \le s'_0$ for some s'_0 small enough. In fact, the result will still hold for $s \le s'_0 = 1/c'$, with $c' = \max\left(\mathsf{E}^*[\mathrm{e}^{L'|X|}], 2c_X\right)$, where $L' = 2\max(|u|, |v|)$ and

$$c_X = \max\left(\mathsf{E}^*\left[\mathrm{e}^{L'B^+_{l,k}}\right], \mathsf{E}^*\left[\mathrm{e}^{L'B^-_{l,k}}\right]\right)$$

Now (9) follows by using Leibniz's formula at $s = 0$, as in Ohgren (2001). □

LEMMA 2 *When u and v are real numbers, we have*

$$\mathsf{E}^*\left[\mathrm{e}^{uB^+_{l,k}}\right] = \begin{cases} 1 + \mathsf{E}^*\left[(\mathrm{e}^{uB_{l,k}} - 1)\mathbf{1}_{\{uB_{l,k} > 0\}}\right] = 1 + C_1(u,k), & \text{if } u \ge 0 \\ 1 - \mathsf{E}^*\left[(1 - \mathrm{e}^{uB_{l,k}})\mathbf{1}_{\{uB_{l,k} < 0\}}\right] = 1 - P_1(u,k), & \text{if } u < 0 \end{cases}$$

$$\mathsf{E}^*\left[e^{-vB_{l,k}^-}\right]=\begin{cases}1+\mathsf{E}^*\left[(e^{-vB_{l,k}}-1)\mathbf{1}_{\{vB_{l,k}<0\}}\right]=1+C_1(-v,k), & \text{if } v\geq 0\\ 1-\mathsf{E}^*\left[(1-e^{-vB_{l,k}})\mathbf{1}_{\{vB_{l,k}>0\}}\right]=1-P_1(-v,k), & \text{if } v<0\end{cases}$$

where $C_1(u, k)$ *is the value of a European call option with strike* $K = 1$ *on the asset* $\bar{S}_t$ *with* $\bar{S}_0 = 1$ *and return* $u \cdot X_{t(l),\, t(k)}$ *(ignoring the discount factor), and* $P_1(u, k)$ *is the value of a European put option with strike* $K = 1$ *on the asset* $\bar{S}_t$ *with* $\bar{S}_0 = 1$ *and return* $u \cdot X_{t(l),\, t(k)}$.

PROOF Note that

$$\mathsf{E}^*\left[e^{uB_{l,k}^+}\right]=\mathsf{P}^*\left(B_{l,k}<0\right)+\mathsf{E}^*\left[e^{uB_{l,k}}\mathbf{1}_{\{B_{l,k}>0\}}\right]=1+\mathsf{E}^*\left[(e^{uB_{l,k}}-1)\mathbf{1}_{\{B_{l,k}>0\}}\right]$$

When $u \geq 0$, we have

$$\begin{aligned}\mathsf{E}^*\left[e^{uB_{l,k}^+}\right]&=1+\mathsf{E}^*\left[(e^{uB_{l,k}}-1)\mathbf{1}_{\{B_{l,k}>0\}}\right]=1+\mathsf{E}^*\left[(e^{uB_{l,k}}-1)\mathbf{1}_{\{uB_{l,k}>0\}}\right]\\&=1+C_1(u,k)\end{aligned}$$

When $u < 0$, we have

$$\begin{aligned}\mathsf{E}^*\left[e^{uB_{l,k}^+}\right]&=1+\mathsf{E}^*\left[(e^{uB_{l,k}}-1)\mathbf{1}_{\{uB_{l,k}<0\}}\right]=1-\mathsf{E}^*\left[(1-e^{uB_{l,k}})\mathbf{1}_{\{uB_{l,k}<0\}}\right]\\&=1-P_1(u,k)\end{aligned}$$

In addition,

$$\begin{aligned}\mathsf{E}^*\left[e^{-vB_{l,k}^-}\right]&=P(B_{l,k}>0)+\mathsf{E}^*\left[e^{-vB_{l,k}}\mathbf{1}_{\{B_{l,k}<0\}}\right]\\&=1+\mathsf{E}^*\left[(e^{-vB_{l,k}}-1)\mathbf{1}_{\{B_{l,k}<0\}}\right]\end{aligned}$$

As before, if $v \geq 0$ then

$$\begin{aligned}\mathsf{E}^*\left[e^{-vB_{l,k}^-}\right]&=1+\mathsf{E}^*\left[(e^{-vB_{l,k}}-1)\mathbf{1}_{\{B_{l,k}<0\}}\right]=1+\mathsf{E}^*\left[(e^{-vB_{l,k}}-1)\mathbf{1}_{\{-vB_{l,k}>0\}}\right]\\&=1+C_1(-v,k)\end{aligned}$$

while if $v < 0$ then

$$\begin{aligned}\mathsf{E}^*\left[e^{-vB_{l,k}^-}\right]&=1+\mathsf{E}^*\left[(e^{-vB_{l,k}}-1)\mathbf{1}_{\{B_{l,k}<0\}}\right]=1-\mathsf{E}^*\left[(1-e^{-vB_{l,k}})\mathbf{1}_{\{-vB_{l,k}<0\}}\right]\\&=1-P_1(-v,k)\end{aligned}$$

and the proof is terminated. □

Lemma 2 indicates that whenever u and v are real numbers, we can easily compute $a_{l,\,k}$ and $\hat{a}_{l,\,k}$ via analytical solutions of the standard call/put options. Often, the formulae for call/put options are analytic functions, which can then be extended to the complex plane via analytical extensions even when u and v are complex parameters. This is useful when we numerically invert Laplace transforms, as the inversion will be performed in the complex plane.

3.2 Laplace transform for discrete lookback options

Under a given risk-neutral measure, the price of a lookback option is

$$LP(t,T) = \mathrm{e}^{-r(T-t)}\,\mathsf{E}^*\big[M_{0,T} - S(T)\,|\,\mathcal{F}_t\big] = \mathrm{e}^{-r(T-t)}\,\mathsf{E}^*\big[M_{0,T}\,|\,\mathcal{F}_t\big] - S(t) \tag{11}$$

Therefore, we need to compute the value of $\mathsf{E}^*[M_{0,T}\,|\,\mathcal{F}_t]$. Consider any time $t \in [t(l-1), t(l))$, with $m \geq l \geq 1$. Since $\max(a, b) = a + \max(b-a, 0)$, we can write:

$$\begin{aligned}\mathsf{E}^*\big[M_{0,T}\,|\,\mathcal{F}_t\big] &= \mathsf{E}^*\big[M_{t(l),T}\,|\,\mathcal{F}_t\big] \\ &+ \mathsf{E}^*\Big[(M_{0,t(l-1)} - M_{t(l),T})\,\mathbf{1}_{\{M_{0,t(l-1)} \geq M_{t(l),T}\}}\,|\,\mathcal{F}_t\Big]\end{aligned} \tag{12}$$

Noting that $M_{t(l),T} = \max_{l \leq j \leq m} S_j = S(t)\mathrm{e}^{X_{t,t(l)} + \tilde{M}_{l,m}} = S(t)\mathrm{e}^{Y^t_{l,m}}$, we have

$$\mathsf{E}^*\big[M_{t(l),T}\,|\,\mathcal{F}_t\big] = S(t)\mathsf{E}^*\big[\mathrm{e}^{Y^t_{l,m}}\big] = S(t)A(1;t) \tag{13}$$

using the notation in (4). Since $A(1, t)$ can be computed via $x_{l,m}$ and (4), we only need to compute the second term in (12).

If t is a monitoring point $t(l)$ and $S_{t(l)} \geq M_{0,t(l-1)}$, that is, whenever the previous maximum of the asset price is less than the value at the lth monitoring point and can, therefore, be ignored, then the second term in (12) is zero. This is exactly the case studied in Ohgren (2001). However, in general, when either t is not a monitoring point or t is a monitoring point $t(l)$ but $S_{t(l)} < M_{0,t(l-1)}$, it is necessary to compute the second term in (12). For this purpose, following a Laplace transform approach first introduced by Carr and Madan (1999), we now derive the Laplace transform of the second term in (12).

Theorem 1 *Let* $\xi > 1$ *and assume that* $A(1-\xi; t) < \infty$. *At any time* $t \in [t(l-1), t(l))$, $m \geq l \geq 1$, *the Laplace transform of*

$$f(x; S(t)) := \mathsf{E}^*\Big[\big(\mathrm{e}^x - M_{t(l),T}\big)\,\mathbf{1}_{\{\mathrm{e}^x \geq M_{t(l),T}\}}\,|\,\mathcal{F}_t\Big] \tag{14}$$

with respect to x is given by

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} \mathrm{e}^{-\xi x} f(x; S(t))\,\mathrm{d}x = \frac{(S(t))^{-(\xi-1)}}{\xi(\xi-1)} A(1-\xi; t) \tag{15}$$

using the notation in (4).

PROOF Letting the risk-neutral density of $Y^t_{l,m}$ be $\varphi(Y^t_{l,m}; y)$, we can rewrite (14) as

$$f(x;S(t)) = \int_{-\infty}^{x-\log(S(t))} \left(e^{x} - S(t)e^{y}\right)\varphi\left(Y_{l,m}^{t};y\right)dy$$

The Laplace transform is then given by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\xi x}\left[\int_{-\infty}^{x-\log(S(t))} \left(e^{x} - S(t)e^{y}\right)\varphi\left(Y_{l,m}^{t};y\right)dy\right]dx$$

Applying Fubini's Theorem, we can interchange the order of integration and obtain:

$$\begin{aligned}\hat{f}(\xi) &= \int_{-\infty}^{\infty}\left[\int_{y+\log S(t)}^{\infty} e^{-(\xi-1)x}dx\right]\varphi\left(Y_{l,m}^{t};y\right)dy \\ &\quad - \int_{-\infty}^{\infty}\left[\int_{y+\log S(t)}^{\infty} e^{-\xi x}dx\right]S(t)e^{y}\varphi\left(Y_{l,m}^{t};y\right)dy \\ &= \frac{(S(t))^{-(\xi-1)}}{\xi-1}\int_{-\infty}^{\infty} e^{-(\xi-1)y}\varphi\left(Y_{l,m}^{t};y\right)dy \\ &\quad - \frac{(S(t))^{-(\xi-1)}}{\xi}\int_{-\infty}^{\infty} e^{-(\xi-1)y}\varphi\left(Y_{l,m}^{t};y\right)dy \\ &= \frac{(S(t))^{-(\xi-1)}}{\xi(\xi-1)}\mathsf{E}^{*}\left[e^{-(\xi-1)Y_{l,m}^{t}}\right]\end{aligned}$$

from which the conclusion follows. □

Corollary 1 *At any time* $t \in [t(l-1), t(l))$, with $1 \le l \le m$, we have

$$LP(t,T) = e^{-r(T-t)}\left[S(t)A(1,t) + \mathcal{L}_{\xi}^{-1}\left(\frac{(S(t))^{-(\xi-1)}}{\xi(\xi-1)}A(1-\xi;t)\right)\Bigg|_{\log M_{0,t(l-1)}}\right] - S(t) \quad (16)$$

$$\begin{aligned}\Delta(LP(t,T)) &= \frac{\partial}{\partial S(t)}LP(t,T) \\ &= e^{-r(T-t)}\left[A(1,t) - \mathcal{L}_{\xi}^{-1}\left(\frac{(S(t))^{-\xi}}{\xi}A(1-\xi;t)\right)\Bigg|_{\log M_{0,t(l-1)}}\right] - 1\end{aligned}$$

$$\Gamma(LP(t,T)) = \frac{\partial^2}{\partial^2 S(t)} LP(t,T)$$
$$= e^{-r(T-t)} \cdot \mathcal{L}_\xi^{-1}\left((S(t))^{-(\xi+1)} A(1-\xi;t)\right)\Big|_{\log M_{0,t(l-1)}}$$
$$VG(LP(t,T)) = \frac{\partial}{\partial\sigma} LP(t,T)$$
$$= e^{-r(T-t)}\left[S(t)\frac{\partial A(1,t)}{\partial\sigma} - \mathcal{L}_\xi^{-1}\left(\frac{(S(t))^{-(\xi-1)}}{\xi(\xi-1)}\frac{\partial}{\partial\sigma}A(1-\xi;t)\right)\Bigg|_{\log M_{0,t(l-1)}}\right]$$

where $\mathcal{L}_\xi^{-1}$ means the Laplace inversion with respect to ξ, and σ is the volatility parameter.

PROOF (16) is a direct consequence of (15), (13), (12), and (11). All other results follow easily by interchanging derivatives and integrals, which is legitimate by using Theorem A. 12 on pp. 203–4 in Schiff (1999). □

COROLLARY 2 *At any time $t \in [t(l-1), t(l))$, $l \geq 1$, the price of a fixed strike lookback call option, $FC(t, T)$, is given by*

$$FC(t,T) = \begin{cases} LP(t,T) + S(t) - Ke^{-r(T-t)} & \text{if } M_{0,t(l-1)} \geq K \\ e^{-r(T-t)}\left\{S(t)A(1;t) + f(\log(K), S(t)) - K\right\} & \text{if } M_{0,t(l-1)} < K \end{cases}$$

PROOF If $M_{0,\,t(l-1)} \geq K$, then clearly $M_{0,\,T} \geq K$. Thus,

$$FC(t,T) = e^{-r(T-t)}\mathsf{E}^*[M_{0,\,T} \mid \mathcal{F}_t] - Ke^{-r(T-t)}.$$

If $M_{0,\,t(l-1)} < K$, then

$$FC(t,T) = e^{-r(T-t)}\left\{\mathsf{E}^*\left[\max(M_{t(l),T}, K) \mid \mathcal{F}_t\right] - K\right\}$$
$$= e^{-r(T-t)}\left\{\mathsf{E}^*\left[M_{t(l),T} \mid \mathcal{F}_t\right] + \mathsf{E}^*\left[(K - M_{t(l),T})^+ \mid \mathcal{F}_t\right] - K\right\}$$

and the conclusion follows via (13) and (14). □

The derivation of the greeks for fixed strike lookback options is similar to that in Corollary 1 and hence is omitted. Analogous results for lookback calls with both fixed and floating strikes are deferred to Appendix A.

3.3 Laplace transform for barrier options

In this subsection we derive a Laplace Transform for the up-and-out put option. In Appendix B we will also show how the same approach can be extended to price other barrier options via symmetry.

THEOREM 2 *Let $\xi > 1$ and $\zeta > 0$ and assume that $C(-\zeta, 1-\xi; t) < \infty$. At any time $t \in [t(l-1), t(l))$, $m \geq l \geq 1$, the double Laplace transform of*

$$f(\kappa,h;S(t)) = \mathsf{E}^*\left[\left(e^{\kappa} - S(T)\right)^+ \mathbf{1}_{\{M_{0,T}<e^h\}} \middle| \mathcal{F}_t\right]$$

is given by

$$\hat{f}(\xi,\zeta) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\xi\kappa-\zeta h} f(\kappa,h;S(t))\,d\kappa\,dh = (S(t))^{-(\xi+\zeta-1)} \cdot \frac{C(-\zeta,1-\xi;t)}{\xi(\xi-1)\zeta} \quad (17)$$

with the function C defined in (5).

PROOF Letting the risk-neutral density of $(X_{t,T}, Y_{l,m}^t)$ be $\phi(x, y)$, we can write

$$f(\kappa,h;S(t)) = \int_{-\infty}^{\kappa-\log(S(t))}\int_{-\infty}^{h-\log(S(t))} \left(e^{\kappa} - S(t)e^{x}\right)\phi(x,y)\,dy\,dx$$

The Laplace transform is then given by

$$\hat{f}(\xi,\zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\xi\kappa-\zeta h}$$
$$\times\left(\int_{-\infty}^{\kappa-\log(S(t))}\int_{-\infty}^{h-\log(S(t))} \left(e^{\kappa} - S(t)e^{x}\right)\phi(x,y)\,dy\,dx\right)d\kappa\,dh$$

Consider the integral with respect to κ first. Applying Fubini's Theorem, we can interchange the order of integration and obtain

$$\int_{-\infty}^{\infty} e^{-\xi\kappa-\zeta h}\cdot\left(\int_{-\infty}^{\kappa-\log(S(t))}\int_{-\infty}^{h-\log(S(t))} \left(e^{\kappa} - S(t)e^{x}\right)\phi(x,y)\,dy\,dx\right)d\kappa$$
$$= \int_{-\infty}^{\infty} e^{-\zeta h}\cdot\left(\int_{-\infty}^{h-\log(S(t))}\left(\int_{x+\log(S(t))}^{\infty} e^{-\xi\kappa}e^{\kappa}\,d\kappa\right)\phi(x,y)\,dy\right)dx$$
$$- \int_{-\infty}^{\infty} e^{-\zeta h}\cdot\left(\int_{-\infty}^{h-\log(S(t))}\left(\int_{x+\log(S(t))}^{\infty} e^{-\xi\kappa}\,d\kappa\right)S(t)e^{x}\phi(x,y)\,dy\right)dx$$
$$= \left\{\frac{(S(t))^{-(\xi-1)}}{\xi-1} - \frac{(S(t))^{-(\xi-1)}}{\xi}\right\}\int_{-\infty}^{\infty} e^{-\zeta h}\cdot\left(\int_{-\infty}^{h-\log(S(t))}\phi(x,y)e^{-(\xi-1)x}\,dy\right)dx$$
$$= \frac{(S(t))^{-(\xi-1)}}{\xi(\xi-1)}\int_{-\infty}^{\infty} e^{-\zeta h}\cdot\left(\int_{-\infty}^{h-\log(S(t))}\phi(x,y)e^{-(\xi-1)x}\,dy\right)dx$$

Now consider the integral with respect to h and apply once again Fubini's Theorem:

$$\hat{f}(\xi,\zeta)=\frac{(S(t))^{-(\xi-1)}}{\xi(\xi-1)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\zeta h}\cdot\left(\int_{-\infty}^{h-\log(S(t))}\phi(x,y)\mathrm{e}^{-(\xi-1)x}\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}h$$
$$=\frac{(S(t))^{-(\xi-1)}}{\xi(\xi-1)}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_{y+\log(S(t))}^{\infty}\mathrm{e}^{-\zeta h}\mathrm{d}h\right)\phi(x,y)\mathrm{d}y\right)\mathrm{e}^{-(\xi-1)x}\mathrm{d}x$$
$$=\frac{(S(t))^{-(\xi-1)}}{\xi(\xi-1)}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{(S(t))^{-\zeta}}{\zeta}\phi(x,y)\mathrm{e}^{-\zeta y}\mathrm{d}y\right)\mathrm{e}^{-(\xi-1)x}\mathrm{d}x$$

from which the conclusion follows. □

By inverting (17) and its derivatives with respect to $S(t)$ and the volatility σ, we can explicitly find the price of the up-and-out option and its Delta, Gamma and Vega values.

COROLLARY 3 At any time $t\in[t(l-1),t(l))$, with $1\le l\le m$, we have:

$$UOP(t,T)=\mathrm{e}^{-r(T-t)}\,\mathcal{L}^{-1}_{\xi,\zeta}\left(\frac{(S(t))^{-(\xi+\zeta-1)}}{\xi(\xi-1)\zeta}C(-\zeta,1-\xi;t)\right)\Bigg|_{\log(K),\log(H)},$$
$$\frac{\partial}{\partial S(t)}UOP(t,T)=\mathrm{e}^{-r(T-t)}\,\mathcal{L}^{-1}_{\xi,\zeta}\left(\frac{(\xi+\zeta-1)(S(t))^{-(\xi+\zeta)}}{\xi(\xi-1)\zeta}C(-\zeta,1-\xi;t)\right)\Bigg|_{\log(K),\log(H)},$$
$$\frac{\partial^2}{\partial^2 S(t)}UOP(t,T)=\mathrm{e}^{-r(T-t)}\,\mathcal{L}^{-1}_{\xi,\zeta}\left(\frac{(\xi+\zeta-1)(\xi+\zeta)(S(t))^{-(\xi+\zeta+1)}}{\xi(\xi-1)\zeta}C(-\zeta,1-\xi;t)\right)\Bigg|_{\log(K),\log(H)},$$
$$\frac{\partial}{\partial\sigma}UOP(t,T)=\mathrm{e}^{-r(T-t)}\,\mathcal{L}^{-1}_{\xi,\zeta}\left(\frac{(S(t))^{-(\xi+\zeta-1)}}{\xi(\xi-1)\zeta}\frac{\partial}{\partial\sigma}C(-\zeta,1-\xi;t)\right)\Bigg|_{\log(K),\log(H)}\tag{18}$$

4 The main algorithm

The results in the previous section lead to an algorithm for computation of the prices and hedging parameters (the Greeks) for discrete lookback and barrier options under a quite general class of asset pricing models, essentially only requiring that the return process is a Lévy process (with independent and stationary increments). To illustrate the algorithm, without loss of generality, we shall focus on computing the price and the Greeks for a floating lookback put (or an up-and-out put option).

The Algorithm

Input: Analytical formulae of standard European call and put options.

Step 1 Use the European call and put formulae to calculate $a_{i,k}$ ($\hat{a}_{i,k}$), via (7) and Lemma 2.

Step 2 Use the recursion in equation (8) (eq. (9)) to compute $x_{l,k}$ ($\hat{x}_{l,k}$).

Step 3 Compute $A(u; t)$ ($C(u, v; t)$) from equation (4) (eq. (5)).

Step 4 Numerically invert the Laplace transforms given in Corollary 1 (Corollary 3).

For other lookback and barrier options, the only change is in Step 4; more precisely, one simply uses the results in Appendix A instead of Corollaries 1 and 3. In Step 4 the Laplace transforms are inverted by using two-sided Euler inversion algorithms (See Appendix B and Petrella, 2004), which are extensions of one-sided Euler algorithms in Abate and Whitt (1992) and Choudhury, Lucantoni and Whitt (1994).

As will be demonstrated in our numerical examples, for a wide variety of parameters (including the cases where the barrier is very close to the initial asset price and there are many monitoring points), the algorithm is quite fast (typically only requires a few seconds), and is quite accurate (typically up to three decimal points). Furthermore, the algorithm essentially only requires to input the standard European call and put prices, thanks to Spitzer's formula; the implication of this in terms of hedging discrete lookback and barrier options using standard European call and put options remains an interesting open problem.

The workload in the algorithm is quadratic in the number of monitoring points. Indeed, the computation of $a_{0,j}$ for $j = 1, \ldots, m$ is equivalent to computing m European call/put prices ($2m$ for the computation of ($\hat{a}_{i,k}$)). Therefore, the recursion in Step 2 requires $m(m+1)/2$ operations to compute either $x_{m,k}$ or $\hat{x}_{m,k}$. Thus, the total workload for both barrier and lookback options is of the order $O(Nm^2)$, where N is the number terms needed for Laplace inversion.

5 Examples and numerical implementation

5.1 Lookback options under the Brownian model

In the setting of Black and Scholes (1973), the underlying asset follows a geometric Brownian motion

$$S(t) = S(s)\exp\left\{\left(\mu - \tfrac{1}{2}\sigma^2\right)(t-s) + \sigma\left(W(t) - W(s)\right)\right\} \tag{19}$$

where μ and σ are constant and $W(t)$ is a Wiener process with a zero mean and a variance of t. Under the risk-neutral measure $\mu^* = r - q$, where q is the continuous dividend rate.

At any generic point in time $t \in [t(l-1), t(l))$, we must compute $A(1-\xi; t)$ in (4) with $\xi \in \mathbb{C}$. Using (19), we can immediately derive

$$\mathsf{E}^*\left[e^{\xi' X_{t,t(l)}}\right] = e^{\xi'\mu(t(l)-t)+\frac{1}{2}\sigma^2\xi'^2(t(l)-t)} \tag{20}$$

where $\xi' = 1-\xi$. In order to implement the recursive equation (8) to compute $\mathsf{E}^*[e^{\xi'\tilde{M}_{l,m}}]$, we must first find the coefficients $a_{l,k}$ as defined in (7).

For notation simplicity, without loss of generality we will let $l = 0$, and consider $a_k \equiv a_{0,k}$ for $k = 1, 2, \ldots, (m-l)$. Since $B_{0,k} = X_1 + X_2 + \cdots + X_k$ is $N(\mu t(k), \sigma^2 t(k))$ with $t(k) = k\Delta T$, we can simply compute via Lemma 2

$$a_k = \mathsf{E}^*\left[e^{\xi' B^+_{0,k}}\right] =$$
$$\Phi\left(-\frac{\mu\sqrt{t(k)}}{\sigma}\right) + e^{\xi'\mu t(k)+\frac{1}{2}(\xi'\sigma)^2 t(k)} \int_{-\frac{\mu+\xi'\sigma^2}{\sigma}\sqrt{t(k)}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution. Since the function is analytic, it can be extended to the complex domain as

$$a_k = \Phi\left(-\frac{\mu\sqrt{t(k)}}{\sigma}\right) + \frac{1}{2} e^{\xi'\mu t(k)+\frac{1}{2}(\xi'\sigma)^2 t(k)} \operatorname{Erfc}\left(-\frac{\mu+\xi'\sigma^2}{\sigma\sqrt{2}}\sqrt{t(k)}\right) \tag{21}$$

where $\operatorname{Erfc}(\cdot)$ is the complex complementary error function.

Many algorithms and built-in software functions have been developed to accurately compute the complex complementary error function. In our implementation we compute it via the Faddeeva Function $W(z)$, defined by $W(z) = \exp(-z^2)\operatorname{Erfc}(-iz)$, using an algorithm by Poppe and Wijers (1990), which ensures an accuracy of 13 digits in almost the entire complex plane.

We now proceed to compare the prices from the Laplace transform method (LT, from now on) with results obtained by Broadie *et al* (1999) (BGK, from now on) using enhanced trinomial trees, and to compare the Greeks with the estimates from Monte Carlo simulation (MC, from now on). In our implementation of the MC simulation, we follow the methodology suggested by Broadie and Glasserman (1996) and estimate the delta via their pathwise MC method, while the gamma is estimated via re-simulation. The numerical results for a floating lookback put, reported in Table 1, indicate that the accuracy of the LT method is high. We have found the algorithm to be extremely robust for various levels of volatility and maturity. Furthermore, our implementation of the Euler algorithm allows us to price lookback options with high accuracy even when the stock price at time t is close to the previous maximum $M_{0,t(1-1)}$.

5.2 Lookback options under Merton's and double exponential jump-diffusion models

Under jump diffusion models (JD from now on), the asset price is assumed to have

TABLE 1 Floating lookback put under the Brownian model.

Points (m)	Price LT	Price BGK	Price MC (Std err)	Δ: LT	Δ: MC (Std err)	Γ: LT	Γ: MC (Std err.)	Time (sec)
Previous max. M = 110								
5	13.300	13.300	13.294 (0.0039)	−0.3568	−0.3565 (0.0001)	0.0287	0.0285 (0.0002)	0.02
10	14.123	14.123	14.120 (0.0038)	−0.3034	−0.3031 (0.0001)	0.0309	0.0309 (0.0002)	0.03
20	14.806	14.806	14.802 (0.0037)	−0.2633	−0.2634 (0.0001)	0.0319	0.0321 (0.0002)	0.06
40	15.345	15.345	15.342 (0.0037)	−0.2333	−0.2332 (0.0001)	0.0324	0.0324 (0.0002)	0.19
80	15.754	15.755	15.760 (0.0037)	−0.2112	−0.2111 (0.0001)	0.0327	0.0325 (0.0002)	0.64
160	16.059	16.059	16.061 (0.0036)	−0.1952	−0.1952 (0.0001)	0.0329	0.0328 (0.0002)	2.34
Previous max. M = 120								
5	18.837	18.837	18.827 (0.0047)	−0.5924	−0.5921 (0.0001)	0.0244	0.0244 (0.0002)	0.01
10	19.323	19.323	19.316 (0.0046)	−0.5547	−0.5543 (0.0001)	0.0260	0.0264 (0.0002)	0.02
20	19.743	19.743	19.740 (0.0045)	−0.5238	−0.5240 (0.0002)	0.0273	0.0273 (0.0002)	0.07
40	20.083	20.083	20.081 (0.0045)	−0.4999	−0.5001 (0.0002)	0.0281	0.0282 (0.0002)	0.19
80	20.346	20.346	20.353 (0.0044)	−0.4819	−0.4822 (0.0002)	0.0287	0.0289 (0.0002)	0.64
160	20.544	20.544	20.548 (0.0044)	−0.4687	−0.4689 (0.0002)	0.0291	0.0293 (0.0002)	2.35

LT, BGK, and MC stand for the proposed Laplace transform method, the lattice method in Broadie, Glasserman, and Kou (1999), and the Monte Carlo method (based on 10 million simulation runs), respectively. The parameters are $S = 100$, $\sigma = 0.3$, $r = 0.1$, $T = 0.5$. The reported time is the CPU time on a Pentium 1.8 Ghz to compute both the price and the Greeks (delta and gamma) via the LT method. As a comparison, the MC method takes several minutes on the same machine, and the BGK method takes more than one hour CPU time on a Pentium 133 for $m \geq 40$.

the following dynamics:

$$S(t) = S(s)\exp\left\{\left(\mu - \tfrac{1}{2}\sigma^2\right)(t-s) + \sigma\left(W(t) - W(s)\right)\right\} \prod_{i=1}^{N(t)-N(s)} V_i, \quad t \geq s \tag{22}$$

where $W(t)$ is a standard Brownian motion, $N(t)$ a Poisson process with rate λ, and $\{V_i\}$ a sequence of iid non-negative random variables. The random variables V_i represents the price jump that occurs following the ith Poisson event. The probability of n jumps occurring in the time interval $[t_1, t_2]$ is given by

$$\pi_{t_1, t_2}(n) = e^{-\lambda(t_2 - t_1)} \frac{(\lambda(t_2 - t_1))^n}{n!} \tag{23}$$

In Merton's (1976) model (MJD, from now on), $J = \log(V)$ has a normal distribution with mean μ_J and variance σ_J^2. On the other hand, in the double exponential jump diffusion model (Kou, 2002), denoted by DEJD from now on, $Y = \log(V)$ has an asymmetric double exponential distribution with the density

$$f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y < 0\}}, \eta_1 > 1, \eta_2 > 0$$

where p, $q \geq 0$, $p + q = 1$, represent the probabilities of upward and downward jumps. Under the risk-neutral measure the drift becomes $\mu^* = r - q - \lambda\zeta$, with $\zeta = \mathsf{E}^*(V) - 1$. In MJD $\mathsf{E}^*(V) = \mathsf{E}^*(e^J) = e^{\mu_J + \sigma_J^2/2}$, while in DEJD

$$\mathsf{E}^*(V) = (q(\eta_2)/(\eta_2 + 1)) + (p(\eta_1)/(\eta_1 - 1)), \ \eta_1 > 1, \eta_2 > 0.$$

As for the previous case, at any point $t \in [t(1-1), t(l))$ we need to compute $\mathsf{E}^*[e^{\xi' Y_{l,m}^t}]$ in (4). Conditioning on the number of jumps occurring in $[t, t(l)]$, we can easily find

$$\mathsf{E}^*\left[e^{\xi' X_{t,t(l)}}\right] = \exp\left\{\left(\xi'\mu + \tfrac{1}{2}(\xi'\sigma)^2 + \lambda\left[\mathsf{E}^*[e^{\xi' J}] - 1\right]\right)(t(l) - t)\right\}$$

with $\mathsf{E}^*[e^{\xi' J}] = \exp\{\xi'\mu_J + (\xi'\sigma_j)^2/2\}$ in MJD and $\mathsf{E}^*[e^{\xi' J}] = (q\eta_2/(\eta_2 + \xi')) + (p\eta_1/(\eta_1 - \xi'))$ in DEJD. We can also find the values of the coefficients a_k, conditioning on the number of jumps occurring in $[t(l), t(k)]$. In MJD, defining $\mu_k^n = \mu^*(t(k) - t(l)) + n\mu_J$ and

$$\sigma_k^n = \sqrt{n\sigma_J^2 + \sigma^2(t(k) - t(l))}$$

we have, via Lemma 2, for $k = 1,2,\ldots\ m - l$,

$$a_k = \sum_{n=0}^{\infty} \left\{ \Phi\left(-\frac{\mu_k^n}{\sigma_k^n}\right) + \frac{1}{2} e^{\xi'\mu_k^n + \frac{1}{2}(\xi'\sigma_k^n)^2} \operatorname{Erfc}\left(-\frac{\mu_k^n + \xi'(\sigma_k^n)^2}{\sigma_k^n\sqrt{2}}\right)\right\} \pi_{t(l), t(k)}(n)$$

using the notation in (23). In practice, when computing the coefficients above, we will consider a truncated series up to the n^*th term, with $P(N(t(k)) - N(t(l)) > n^*) < 10^{-6}$. For DEJD, the computation of the coefficients a_k follows the same line of derivations and is omitted for brevity. We only point out that for the integrals $I_k(\lambda, \alpha, \beta, \delta)$ (using the notation in Kou 2002) the recursive relation $I_n = -(e^{\alpha\lambda}/\alpha)Hh_n(\beta\lambda - \delta) + (\beta/\alpha)I_{n-1}$ still holds for complex α substituting the normal cdf with the complex error function.

We now compare the LT method with MC, to verify the accuracy of the results. In Tables 2 and 3 we consider a floating lookback put option under the two jump-diffusion models, assuming two different values of the previous recorded maximum for various monitoring frequencies. In both models we set the total

TABLE 2 Floating lookback put under Merton's model: comparison of the LT and MC methods.

Points (m)	Price LT	Price MC (Std err)	Δ: LT	Δ: MC (Std err)	Γ: LT	Γ: MC (Std err.)	Time (sec)
M = 110							
5	12.683	12.682 (0.0039)	−0.3919	−0.3920 (0.0001)	0.0312	0.0313 (0.0002)	0.08
10	13.311	13.308 (0.0039)	−0.3488	−0.3489 (0.0001)	0.0331	0.0331 (0.0002)	0.18
20	13.812	13.813 (0.0038)	−0.3163	−0.3166 (0.0001)	0.0341	0.0345 (0.0002)	0.35
40	14.193	14.195 (0.0038)	−0.2924	−0.2926 (0.0001)	0.0348	0.0350 (0.0002)	0.81
80	14.476	14.478 (0.0038)	−0.2752	−0.2753 (0.0001)	0.0353	0.0353 (0.0002)	2.00
160	14.681	14.681 (0.0038)	−0.2629	−0.2627 (0.0001)	0.0356	0.0357 (0.0002)	5.55
M = 120							
5	18.528	18.528 (0.0047)	−0.6320	−0.6321 (0.0001)	0.0233	0.0233 (0.0002)	0.09
10	18.886	18.882 (0.0046)	−0.6031	−0.6033 (0.0001)	0.0249	0.0252 (0.0002)	0.17
20	19.180	19.181 (0.0046)	−0.5805	−0.5805 (0.0001)	0.0261	0.0262 (0.0002)	0.35
40	19.408	19.410 (0.0045)	−0.5634	−0.5633 (0.0002)	0.0269	0.0269 (0.0002)	0.82
80	19.580	19.583 (0.0045)	−0.5508	−0.5508 (0.0002)	0.0275	0.0277 (0.0002)	2.07
160	19.706	19.704 (0.0045)	−0.5417	−0.5415 (0.0002)	0.0279	0.0278 (0.0002)	5.69

MC is based on 10 million simulation runs. The parameters are $S = 100$, $r = 0.1$, $\sigma = 0.212\%$, $\lambda = 2.24$, $\mu_J = -0.01$, $\sigma_J = 0.141$, $T = 0.5$. The reported time is the time to compute both the price and sensitivity parameters using the LT method.

volatility to be 0.3. For MJD we set $\mu_J = -0.01$ and find the value of σ_J such that $\mathsf{E}(V) = 1$. Then, we choose the value of the jump rate λ and the diffusion volatility σ_D, by fixing the fraction of the total variance explained by jumps to be 0.5. For DEJD we follow the same approach, by setting the values of $\eta_1 = 10$ and then finding η_2 and p, so that $\mathsf{E}(V) \approx 1$ and the jump volatility is approximately equal to σ_J of MJD. The jump rate and diffusion volatility are once again determined by assigning half of the total variance to the jump component. We follow this approach in choosing the parameters throughout the paper for all the experiments regarding jump-diffusion models. For a detailed analysis of the calibration of jump-diffusion models, we refer the interest reader to Cont and Tankov (2004).

TABLE 3 Floating lookback put under the DE model: comparison of the LT and MC methods.

Points (m)	Price LT	Price MC (Std err)	Δ: LT	Δ: MC (Std err)	Γ: LT	Γ: MC (Std err.)	Time (sec)
M = 110							
5	13.634	13.626	−0.3722	−0.3724	0.0307	0.0311	0.11
		(0.0051)		(0.0002)		(0.0002)	
10	14.285	14.279	−0.3293	−0.3296	0.0326	0.0326	0.19
		(0.0052)		(0.0002)		(0.0002)	
20	14.802	14.791	−0.2972	−0.2975	0.0335	0.0334	0.35
		(0.0053)		(0.0002)		(0.0002)	
40	15.194	15.188	−0.2735	−0.2738	0.0342	0.0344	0.70
		(0.0054)		(0.0002)		(0.0002)	
80	15.482	15.476	−0.2565	−0.2567	0.0347	0.0351	1.41
		(0.0054)		(0.0002)		(0.0002)	
160	15.693	15.690	−0.2443	−0.2445	0.0350	0.0355	2.81
		(0.0055)		(0.0002)		(0.0002)	
M = 120							
5	19.370	19.364	−0.6092	−0.6094	0.0231	0.0231	0.11
		(0.0042)		(0.0002)		(0.0002)	
10	19.755	19.751	−0.5804	−0.5806	0.0247	0.0246	0.19
		(0.0044)		(0.0002)		(0.0002)	
20	20.067	20.060	−0.5578	−0.5580	0.0258	0.0261	0.36
		(0.0045)		(0.0002)		(0.0002)	
40	20.309	20.305	−0.5407	−0.5409	0.0266	0.0265	0.71
		(0.0046)		(0.0002)		(0.0002)	
80	20.488	20.484	−0.5282	−0.5285	0.0272	0.0270	1.41
		(0.0046)		(0.0002)		(0.0002)	
160	20.621	20.620	−0.5192	−0.5193	0.0276	0.0275	2.98
		(0.0047)		(0.0002)		(0.0002)	

MC is based on 10 million simulation runs. The parameters are $S = 100$, $r = 0.1$, $\sigma = 0.212$, $\lambda = 2.29$, $\eta_1 = 10.0$, $\eta_2 = 5.71$, $p = 0.6$, $T = 0.5$. The reported time is the CPU time to compute both the price and sensitivity parameters using the LT method.

5.3 Barrier options under the Brownian model

To obtain the Laplace transform for the up-and-out put barrier option presented in the previous section we need to compute $C(-\zeta, 1-\xi; t)$ in (5) and Corollary 3. Assuming the stock price follows equation (19), the first factor of the product in (5) is given by (20). The second factor in (5) can be computed using the recursive formula (9) from Theorem 1. We therefore need to find the coefficients $\hat{a}_k$, which require the computation of $\mathsf{E}^*[e^{(\xi' + \zeta')B^+_{0,k}}]$ and $\mathsf{E}^*[e^{-\xi' B^-_{0,k}}]$, with $\xi' = -(\xi - 1)$ and $\zeta' = -\zeta$. The first term is obtained following the same approach as in (21). The second term is given by

$$\mathrm{E}^*\left[e^{-\xi' B_{0,k}^-}\right] = \Phi\left(\frac{\mu\sqrt{t(k)}}{\sigma}\right) + \frac{1}{2} e^{-\xi'\mu t(k) + \frac{1}{2}(\xi'\sigma)^2 t(k)} \operatorname{Erfc}\left(\frac{\mu + \xi'\sigma^2}{\sigma\sqrt{2}}\sqrt{t(k)}\right)$$

We can therefore easily compute the coefficients $\hat{a}_k$, using the Error Function (or Faddeeva Function) as for the lookback option, thus leading to a Laplace transform for barrier options.

In Table 4, we compare the LT method with MC. For the gamma values we cannot use the MC results, given the high standard deviation of such estimates in this case. Instead, the gamma is computed using a finite difference approach. More precisely, $\Gamma = (P(S + \delta) + P(S - \delta) - 2 \times P(S))/\delta^2$, where $P(S)$ is the option value for an initial asset value of S obtained using the LT method. In our examples we set $\delta = 0.05$. Of course the same approach can be used to check the value of delta

TABLE 4 Up-and-out put under the Brownian model.

Points (m)	Price LT	Price MC (Std err)	Δ: LT	Δ: MC (Std err)	Γ: LT	Γ: Finite difference	Time (sec)
Barrier level H = 110							
5	6.010	6.004 (0.0038)	−0.4541	−0.4514 (0.0026)	0.0213	0.0212	1.01
10	4.682	4.677 (0.0035)	−0.4890	−0.4887 (0.0031)	0.0289	0.0292	2.01
20	3.611	3.605 (0.0032)	−0.5202	−0.5251 (0.0035)	0.0391	0.0392	4.25
40	2.789	2.786 (0.0029)	−0.5497	−0.5553 (0.0038)	0.0522	0.0520	9.98
80	2.180	2.182 (0.0026)	−0.5798	−0.5811 (0.0040)	0.0677	0.0676	25.24
160	1.738	1.741 (0.0023)	−0.6120	−0.6111 (0.0042)	0.0832	0.0828	73.89
Barrier level H = 105							
5	6.985	6.978 (0.0040)	−0.4598	−0.4608 (0.0023)	0.0172	0.0172	1.02
10	6.008	6.001 (0.0038)	−0.5084	−0.5096 (0.0029)	0.0198	0.0204	2.00
20	5.231	5.226 (0.0036)	−0.5555	−0.5620 (0.0033)	0.0208	0.0204	4.25
40	4.657	4.653 (0.0035)	−0.5957	−0.5985 (0.0036)	0.0180	0.0180	9.82
80	4.249	4.248 (0.0034)	−0.6227	−0.6177 (0.0037)	0.0112	0.0112	25.36
160	3.957	3.956 (0.0033)	−0.6349	−0.6410 (0.0039)	0.0063	0.0068	74.18

The parameters are $S = 100$, $K = 100$, $\sigma = 0.3$, $r = 0.05$, $T = 1.0$.

by estimating $\Delta = (P(S + \delta) + P(\delta))/\delta$, although it is not used here. In Tables 5 and 6, we further compare the prices and deltas with those reported in Broadie *et al* (1997). The results seem to be consistent. Additional comparison with results in Broadie and Yamamoto (2003) in Section 5.5 further confirms the accuracy of our method.

TABLE 5 Down-and-out call under the Brownian model: comparison of the LT and BGK method for computing the prices and delta at different barrier levels.

H	Price LT	Price BGK	Delta: LT	Delta BGK
85	6.322	6.322	0.591	0.591
86	6.306	6.306	0.594	0.594
87	6.281	6.281	0.600	0.600
88	6.242	6.242	0.607	0.607
89	6.184	6.184	0.618	0.618
90	6.098	6.098	0.633	0.633
91	5.977	5.977	0.653	0.653
92	5.810	5.810	0.678	0.678
93	5.584	5.584	0.710	0.711
94	5.288	5.288	0.750	0.750
95	4.907	4.907	0.798	0.798
96	4.427	4.427	0.854	0.854
97	3.834	3.824	0.917	0.917
98	3.127	3.126	0.967	0.966
99	2.336	2.337	0.958	0.958

The parameters are $S = 100$, $K = 100$, $\sigma = 0.3$, $r = 0.1$, $m = 50$, $T = 0.2$. For the 50 monitoring points it always takes less than 15 seconds on a Pentium IV 1.8 Ghz to compute the price and delta using the LT method.

TABLE 6 Down-and-out call under the Brownian model: comparison of the LT and BGK method for computing the prices across different strikes, maturities, and volatilities.

	K=100, σ=0.6, T=0.2		K=100, σ=0.3, T=2.0		K=110, σ=0.3, T=0.2	
Barrier	Price LT	Price BGK	Price LT	Price BGK	Price LT	Price BGK
85	10.505	10.505	20.819	20.819	2.496	2.496
87	10.019	10.02	19.571	19.571	2.491	2.491
89	9.383	9.383	18.114	18.114	2.475	2.475
91	8.572	8.572	16.435	16.436	2.433	2.433
93	7.563	7.563	14.537	14.537	2.336	2.336
95	6.344	6.344	12.451	12.451	2.135	2.135
97	4.942	4.941	10.254	10.254	1.756	1.756
99	3.475	3.475	8.063	8.061	1.136	1.136

The parameters are $S = 100$, $r = 0.10$, $m = 50$.

TABLE 7 Up-and-out put under the Brownian model: extreme case comparison of the LT method vs. MC using 50 million simulation runs.

Additional parameters	$n = 5$ Price LT	$n = 5$ Price MC	$n = 25$ Price LT	$n = 25$ Price MC	$n = 50$ Price LT	$n = 50$ Price MC
$H = 100.05$, $\sigma = 30\%$	4.44271	4.44260	2.26220	2.26184	1.65087	1.65038
(very close to barrier)		(0.0013)		(0.0010)		(0.0009)
$H = 105.0$, $\sigma = 5\%$	0.49237	0.49259	0.49204	0.49223	0.49188	0.49209
(very low volatility)		(0.00017)		(0.00017)		(0.000010)
$H = 105.0$, $\sigma = 100\%$	17.98788	17.98710	11.01063	11.01450	9.05224	9.05428
(very high volatility)		(0.0038)		(0.0033)		(0.0030)

The parameters are $S = 100$, $K = 100$, $T = 0.5$, $r = 0.05$. MC standard errors are reported in parentheses.

To check whether the LT algorithm works well in some extreme cases, we also report in Table 7 the results for very high/low volatility, and when the asset price is very close the barrier. The results show that the LT algorithm performs well even in these extreme cases.

5.4 Barrier option under Merton's and double exponential jump-diffusion models

Assuming the stock price follows the process described in (22), we can easily compute the coefficients $\hat{a}_k$, using the approach in Section 5.2, where the quantities $E^*[e^{\xi' X_{t,\,t(l)}}]$ and $E^*[e^{(\xi'+\zeta)B^+_{0,\,k}}]$ have already been derived. Therefore, we just need to compute $E^*[e^{-\xi' B^-_{0,\,k}}]$. For MJD, at any time $t \in [t(l-1), t(l))$, conditioning on the number of jump in $[t(l), t(k))$, we have:

$$\mathsf{E}^*\left[e^{-\xi' B^-_{0,k}}\right] = \sum_{n=0}^{\infty}\left\{\Phi\left(\frac{\mu_k^n}{\sigma_k^n}\right) + \frac{1}{2}e^{-\xi'\mu_k^n + \frac{1}{2}(\xi'\sigma_k^n)^2}\,\mathrm{Erfc}\left(\frac{\mu_k^n - \xi'(\sigma_k^n)^2}{\sigma_k^n\sqrt{2}}\right)\right\}\pi_{t(l),t(k)}(n)$$

where, as before, $\mu_k^n = \mu^* + n\mu_J$ and

$$\sigma_k^n = \sqrt{n\sigma_J^2 + \sigma^2 t(k)}$$

For DEJD, $\mathsf{E}^*[e^{-\xi' B^-_{0,\,k}}]$ can be easily computed following the same methodology in Kou (2002) for the pricing of call/put; the results are omitted for brevity. In summary, we can easily find the coefficients $\hat{a}_k$, and, therefore, the Laplace transform for the barrier option under both MJD and DEJD. In Tables 8–11 we compare prices and sensitivities for an up-and-out put option under the two models. Again, the algorithm provides consistent results even for values of the barrier very close to the original asset price.

TABLE 8 Up-and-out put under Merton's model: accuracy test of the LT method with different monitoring frequencies under Merton's model.

Points (m)	Price LT	Price MC (Std err)	Δ: LT	Δ: MC (Std err)	Γ: LT	Γ: Finite difference	Time (sec)
H = 101							
5	5.801	5.802	–0.4776	–0.4882	0.0244	0.0244	5.56
		(0.0038)		(0.0039)			
10	4.507	4.506	–0.5350	–0.5346	0.0349	0.0349	10.58
		(0.0035)		(0.0047)			
20	3.489	3.490	–0.5931	–0.5913	0.0492	0.0492	20.43
		(0.0031)		(0.0054)			
40	2.727	2.729	–0.6509	–0.6533	0.0666	0.0666	41.02
		(0.0028)		(0.0059)			
80	2.175	2.172	–0.7088	–0.7194	0.0842	0.0842	84.30
		(0.0026)		(0.0064)			
160	1.784	1.786	–0.7660	–0.7714	0.0945	0.0945	179.41
		(0.0023)		(0.0067)			
H = 105							
5	6.861	6.863	–0.4817	–0.4814	0.0172	0.0172	5.55
		(0.0039)		(0.0034)			
10	5.993	5.993	–0.5479	–0.5475	0.0169	0.0169	10.55
		(0.0038)		(0.0043)			
20	5.349	5.351	–0.6081	–0.6128	0.0115	0.0115	20.61
		(0.0037)		(0.0049)			
40	4.898	4.904	–0.6507	–0.6407	0.0011	0.0011	41.34
		(0.0036)		(0.0052)			
80	4.579	4.581	–0.6725	–0.6649	–0.0066	–0.0066	84.96
		(0.0035)		(0.0054)			
160	4.348	4.348	–0.6849	–0.6864	–0.0071	–0.0071	183.82
		(0.0034)		(0.0056)			

The parameters are $S = 100$, $K = 100$, $\mu_J = -0.01$, $\sigma_J = 0.141$, $\sigma_{TOT} = 0.3$, $r = 0.05$, $T = 1.0$.

5.5 Comparison with Broadie–Yamamoto algorithm

Recently, Broadie and Yamamoto (2003) have proposed a very efficient algorithm based on the fast Gauss transform to price various discrete path-dependent options, including lookback and barrier options. If the objective is to compute the option prices when the return distribution is Gaussian or a mixture of independent Gaussian random variables, then the Broadie–Yamamoto algorithm is preferable, since it is extremely fast. On the other hand, our Laplace transform method aims at a more *general* setting; more precisely, we can compute the prices for a broader class of return processes, including non-Gaussian distributions (eg, the double

TABLE 9 Up-and-out put under Merton's model: accuracy test of the LT method with different barrier levels under Merton's model.

Barrier	Price LT	Price MC (Std err)	Δ: LT	Δ: MC (Std err)	Γ: LT	Γ: Finite difference
$\sigma_D = 0.212$, $\lambda = 2.24$, ($\sigma_{TOT} = 0.3$)						
101	1.664	1.666	−0.8331	−0.8451	0.0960	0.0960
		(0.0016)		(0.0048)		
103	2.874	2.877	−0.7296	−0.7327	−0.0080	−0.0080
		(0.0020)		(0.0038)		
105	3.620	3.625	−0.6125	−0.6122	0.0064	0.0064
		(0.0021)		(0.0029)		
107	4.043	4.048	−0.5376	−0.5410	0.0167	0.0167
		(0.0022)		(0.0021)		
109	4.271	4.277	−0.4928	−0.4952	0.0238	0.0239
		(0.0022)		(0.0016)		
111	4.391	4.396	−0.4684	−0.4690	0.0286	0.0286
		(0.0022)		(0.0011)		
113	4.454	4.459	−0.4552	−0.4565	0.0313	0.0313
		(0.0022)		(0.0008)		
115	4.487	4.492	−0.4483	−0.4493	0.0328	0.0328
		(0.0022)		(0.0006)		
$\sigma_D = 0.353$, $\lambda = 6.22$, ($\sigma_{TOT} = 0.5$)						
101	2.528	2.529	−0.8547	−0.8601	0.0960	0.0960
		(0.0025)		(0.0062)		
103	3.973	3.975	−0.8453	−0.8521	−0.0020	−0.0020
		(0.0030)		(0.0058)		
105	5.130	5.135	−0.7457	−0.7385	−0.0116	−0.0116
		(0.0032)		(0.0048)		
107	5.984	5.991	−0.6623	−0.6615	−0.0033	−0.0033
		(0.0034)		(0.0041)		
109	6.604	6.611	−0.6004	−0.5979	0.0025	0.0025
		(0.0034)		(0.0034)		
111	7.050	7.056	−0.5545	−0.5553	0.0068	0.0069
		(0.0035)		(0.0029)		
113	7.368	7.375	−0.5210	−0.5232	0.0102	0.0102
		(0.0035)		(0.0024)		
115	7.594	7.599	−0.4969	−0.4960	0.0127	0.0127
		(0.0035)		(0.0020)		

The parameters are $S = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.212$, $\mu_J = -0.01$, $\sigma_J = 0.141$, $T = 0.2$, $m = 50$. The CPU times on a Pentium IV 1.8 Ghz for the LP method, with 50 monitoring points, are approximately 35 and 46 seconds for total volatility levels $\sigma_{TOT} = 0.3$ and $\sigma_{TOT} = 0.5$, respectively.

TABLE 10 Up-and-out put under the DE model: accuracy test of the LT method with different monitoring frequencies under the DE model.

Points (m)	Price LT	Price MC (Std err)	Δ: LT	Δ: MC (Std err)	Γ: LT	Γ: Finite difference	Time (sec)
H = 101							
5	6.476	6.485	–0.4961	–0.4950	0.0246	0.0246	12.93
		(0.0044)		(0.0045)			
10	4.984	4.986	–0.5659	–0.5799	0.0356	0.0356	25.61
		(0.0040)		(0.0055)			
20	3.837	3.835	–0.6341	–0.6347	0.0506	0.0506	49.60
		(0.0036)		(0.0061)			
40	2.989	2.989	–0.7004	–0.7002	0.0689	0.0689	98.25
		(0.0032)		(0.0066)			
80	2.380	2.383	–0.7656	–0.7648	0.0873	0.0873	197.03
		(0.0029)		(0.0072)			
160	1.951	1.954	–0.8296	–0.8354	0.0978	0.0978	396.31
		(0.0026)		(0.0076)			
H = 105							
5	7.659	7.673	–0.5019	–0.5010	0.0163	0.0163	13.07
		(0.0046)		(0.0041)			
10	6.633	6.640	–0.5802	–0.5801	0.0155	0.0155	25.48
		(0.0044)		(0.0050)			
20	5.892	5.890	–0.6490	–0.6481	0.0090	0.0090	50.41
		(0.0042)		(0.0056)			
40	5.381	5.382	–0.6969	–0.7110	–0.0028	–0.0028	99.85
		(0.0041)		(0.0062)			
80	5.022	5.027	–0.7211	–0.7206	–0.0114	–0.0167	199.89
		(0.0039)		(0.0063)			
160	4.763	4.770	–0.7350	–0.7480	–0.0120	–0.0120	402.75
		(0.0039)		(0.0065)			

The parameters are $S = 100$, $K = 100$, $\eta_1 = 10.0$, $\eta_2 = 5.712$, $\sigma_{TOT} = 0.3$, $r = 0.05$, $T = 1.0$.

exponential jump-diffusion model), which may not be easily written as a mixture of independent Gaussian random variables. Furthermore, it is very easy to compute, almost at no additional computational cost, hedging parameters (the Greeks) using the proposed Laplace transform method.

In the cases of the Brownian model and Merton's model, both the Laplace transform method and Broadie–Yamamoto method are applicable to compute option prices. Tables 12 and 13 confirm that both methods lead to almost identical numerical results for the prices.

TABLE 11 Up-and-out put under the DE model: accuracy test of the LT method with different barrier levels under the DE model.

Barrier	Price LT	Price MC (Std err)	Δ: LT	Δ: MC (Std err)	Γ: LT	Γ: Finite difference
σ_D = 0.212, λ = 2.29, (σ_{TOT} = 0.3)						
101	1.755	1.757 (0.0018)	−0.8705	−0.8729 (0.0055)	0.0945	0.0946
103	3.037	3.039 (0.0023)	−0.7578	−0.7577 (0.0045)	−0.0142	−0.0142
105	3.839	3.844 (0.0025)	−0.6316	−0.6339 (0.0035)	0.0018	0.0018
107	4.305	4.312 (0.0026)	−0.5498	−0.5480 (0.0026)	0.0131	0.0131
109	4.566	4.572 (0.0026)	−0.4999	−0.4974 (0.0020)	0.0211	0.0211
111	4.712	4.718 (0.0027)	−0.4708	−0.4737 (0.0017)	0.0263	0.0263
113	4.794	4.800 (0.0027)	−0.4542	−0.4529 (0.0012)	0.0295	0.0295
115	4.841	4.847 (0.0027)	−0.4447	−0.4474 (0.0011)	0.0314	0.0314
σ_D = 0.353, λ = 6.37, (σ_{TOT} = 0.5)						
101	2.704	2.708 (0.0028)	−0.9039	−0.9153 (0.0070)	0.0974	0.0974
103	4.256	4.259 (0.0034)	−0.8913	−0.8818 (0.0064)	−0.0072	−0.0071
105	5.510	5.511 (0.0037)	−0.7830	−0.7722 (0.0056)	−0.0168	−0.0168
107	6.447	6.450 (0.0039)	−0.6920	−0.6878 (0.0048)	−0.0078	−0.0078
109	7.140	7.142 (0.0040)	−0.6238	−0.6247 (0.0041)	−0.0014	−0.0014
111	7.650	7.653 (0.0041)	−0.5723	−0.5770 (0.0036)	0.0033	0.0033
113	8.023	8.026 (0.0041)	−0.5338	−0.5294 (0.0030)	0.0070	0.0070
115	8.295	8.299 (0.0041)	−0.5053	−0.5042 (0.0026)	0.0098	0.0098

The parameters are $S = 100$, $K = 100$, $r = 0.05$, $\eta_1 = 10$, $\eta_2 = 5.712$, $p = 0.6$, $T = 0.2$, $m = 50$. The CPU times on a Pentium IV 1.8 Ghz for the LP method, with 50 monitoring points, are approximately 80 and 110 seconds for total volatility levels $\sigma_{TOT} = 0.3$ and $\sigma_{TOT} = 0.5$, respectively.

TABLE 12 Down-and-out call under the Brownian model: accuracy comparison of the prices given by the Laplace transform (LT) method and Broadie and Yamamoto (BY) method under the Brownian model.

	$n = 5$		$n = 25$		$n = 50$	
Barrier	**Price LT**	**Price BY**	**Price LT**	**Price BY**	**Price LT**	**Price BY**
91	6.18729	6.18729	6.03202	6.03203	5.97705	5.97707
93	5.99968	5.99976	5.68752	5.68753	5.58434	5.58434
95	5.67129	5.67111	5.08147	5.08142	4.90681	4.90679
97	5.16694	5.16725	4.11573	4.11582	3.83393	3.83398
99	4.48965	4.48917	2.81255	2.81244	2.33645	2.33639

The parameters are $S = 100$, $K = 100$, $\sigma = 0.2$, $T = 0.2$, $r = 0.1$.

TABLE 13 Down-and-out call under Merton's model: accuracy comparison of the prices given by the Laplace transform (LT) method and Broadie and Yamamoto (BY) method under Merton's model.

	$n = 5$		$n = 25$		$n = 50$	
Barrier	**Price LT**	**Price BY**	**Price LT**	**Price BY**	**Price LT**	**Price BY**
91	8.63048	8.63049	8.28428	8.28430	8.17962	8.17963
93	8.28833	8.28838	7.71612	7.71613	7.54699	7.54700
95	7.77087	7.77072	6.82050	6.82045	6.56072	6.56070
97	7.05569	7.05593	5.48764	5.48771	5.09158	5.09162
99	6.16435	6.16397	3.76274	3.76265	3.10787	3.10782

The parameters are $S = 100$, $K = 100$, $\sigma = 0.2$, $\lambda = 2.0$, $\mu_J = 0.045$, $\sigma_J = 0.3$, $T = 0.2$, $r = 0.1$.

6 Conclusion

In this paper, we proposed a new methodology based on Laplace transforms to compute the price and hedging parameters of discretely monitored lookback and barrier options. The proposed approach is appealing because it is valid in a quite general framework, including both the classical Brownian model and jump diffusion models. It is also easily implementable given the availability of standard routines for the inversion of Laplace transforms. Our numerical analysis demonstrates that the method is accurate and fast.

One limitation of the method is that the European call and put prices have to be computed accurately and fast, preferably by using analytical formulae to reduce errors and to increase the speed in computing the recursions. However, analytical formulae may not be available for general Lévy-process models.

The method could also be extended to price other derivatives, whose values are a function of the joint distribution of the terminal asset value and its discretely monitored maximum (or minimum) throughout the lifetime of the option, such as partial lookback options.

Appendix A – Extension of the results

A1 Extension of the results to fixed and floating strike lookback calls

Let

$$\hat{M}_{l,k} = \min_{l \le j \le k} \sum_{i=l+1}^{j} X_i, \hat{Y}_{l,m}^{t} = X_{t,t(l)} + \hat{M}_{l,m}, \hat{A}(u;t) := \mathsf{E}^*\left[\mathrm{e}^{u\hat{Y}_{l,m}^{t}}\right] \text{ and}$$

$$m_{t_{(l)},T} := \min_{l \le j \le m} S_j = S(t)\exp\left(X_{t,t(l)} + \min_{l \le j \le m} \sum_{i=1}^{j} X_i\right)$$

$$= S(t)\mathrm{e}^{X_{t,t(l)} + \hat{M}_{l,m}} = S(t)\mathrm{e}^{\hat{Y}_{l.m}^{t}}$$

Then $\hat{A}(u, t)$ can be computed via the recursion in Lemma 1 by changing X_i to $-X_i$.

We can express the value of a floating strike lookback call, at a generic time $t \in [t(l-1), t(l))$, with $1 \le l \le m$, as:

$$LC(t,T) = S(t) - \mathrm{e}^{-r(T-t)}\mathsf{E}^*\left[m_{0,T} \mid \mathcal{F}_t\right] = S(t) - \mathrm{e}^{-r(T-t)}$$
$$\times\left(\mathsf{E}^*\left[m_{t(l),T} \mid \mathcal{F}_t\right] - \mathsf{E}^*\left[\left(m_{t(l),T} - m_{0,t(l-1)}\right)\mathbf{1}_{\{m_{0,t(l-1)} \le m_{t(l),T}\}} \mid \mathcal{F}_t\right]\right) \tag{24}$$

Since $\mathsf{E}^*[(m_{t(l),T} \mid \mathcal{F}_t] = S(t)\hat{A}(1;t)$, only the second term in the parenthesis needs to be computed.

THEOREM A1 *At any time $t \in [t(l-1), t(l))$, $m \ge l \ge 1$, the Laplace transform with respect to x of*

$$g\left(x, S(t)\right) = \mathsf{E}^*\left[\left(m_{t(l),T} - \mathrm{e}^{-x}\right)\mathbf{1}_{\{\mathrm{e}^{-x} \le m_{t(l),T}\}} \mid \mathcal{F}_t\right] \tag{25}$$

is given by

$$\hat{g}(\xi) = (S(t))^{\xi+1} \cdot \frac{\mathsf{E}^*\left[\mathrm{e}^{(\xi+1)(X_{t,t(l)} + \hat{M}_{l,m})}\right]}{\xi(\xi+1)} = (S(t))^{\xi+1} \cdot \frac{\mathsf{E}^*\left[\mathrm{e}^{(\xi+1)\hat{Y}_{l,m}^{t}}\right]}{\xi(\xi+1)} \tag{26}$$

with $\xi \in R$, $\xi > -1$.

PROOF Letting the risk-neutral density of $\hat{Y}_{l,m}^{t}$ be $\varphi(\hat{Y}_{l,m}^{t}; y)$, we can rewrite (25) as

$$g(x; S(t)) = \int_{x-\log(S(t))}^{\infty} (S(t)\mathrm{e}^{y} - \mathrm{e}^{-x})\varphi\left(\hat{Y}_{l,m}^{t}; y\right)\mathrm{d}y$$

The Laplace transform is then given by

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} e^{-\xi x} \left[\int_{-x-\log(S(t))}^{\infty} (S(t)e^{y} - e^{-x}) \varphi\left(\hat{Y}_{l,m}^{t}; y\right) dy \right] dx$$

Fubini's theorem yields

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} \left[\int_{-y-\log S(t)}^{\infty} e^{-\xi x} dx \right] S(t) e^{y} \varphi\left(\hat{Y}_{l,m}^{t}; y\right) dy$$

$$- \int_{-\infty}^{\infty} \left[\int_{-y-\log S(t)}^{\infty} e^{-(\xi+1)x} dx \right] \varphi\left(\hat{Y}_{l,m}^{t}; y\right) dy$$

$$= \frac{(S(t))^{\xi+1}}{\xi} - \int_{-\infty}^{\infty} e^{(\xi+1)y} \varphi\left(\hat{Y}_{l,m}^{t}; y\right) dy - \frac{(S(t))^{(\xi+1)}}{\xi+1} \int_{-\infty}^{\infty} e^{(\xi+1)y} \varphi\left(\hat{Y}_{l,m}^{t}; y\right) dy$$

$$= \frac{(S(t))^{\xi+1}}{\xi(\xi+1)} \mathsf{E}^{*}\left[e^{(\xi+1)\hat{Y}_{l,m}^{t}} \right]$$

which proves the theorem. □

Corollary A1 *At any time* $t \in [t(l-1), t(l))$, *with* $m \geq l \geq 1$, *we have*

$$LC(t,T)$$
$$= S(t) - e^{-r(T-t)} \left(S(t)\hat{A}(1;t) - \mathcal{L}_{\xi}^{-1}\left(\frac{(S(t))^{(\xi+1)}}{\xi(\xi+1)} \hat{A}(\xi+1;t) \right)\Bigg|_{-\log m_{0,t(l-1)}} \right),$$

$$\frac{\partial}{\partial S(t)} LC(t,T)$$
$$= 1 - e^{-r(T-t)} \cdot \left(\hat{A}(1;t) - \mathcal{L}_{\xi}^{-1}\left(\frac{(S(t))^{\xi}}{\xi} \hat{A}(\xi+1;t) \right)\Bigg|_{-\log m_{0,t(l-1)}} \right),$$

$$\frac{\partial^2}{\partial^2 S(t)} LC(t,T) = e^{-r(T-t)} \cdot \mathcal{L}_{\xi}^{-1}\left((S(t))^{(\xi-1)} \hat{A}(\xi+1;t) \,\Big|_{-\log m_{0,t(l-1)}} \right),$$

$$\frac{\partial}{\partial \sigma} LC(t,T) = e^{-r(T-t)}$$
$$\times \left(S(t) \frac{\partial}{\partial \sigma} \hat{A}(\xi+1;t) - \mathcal{L}_{\xi}^{-1}\left(\frac{(S(t))^{(\xi+1)}}{\xi(\xi+1)} \frac{\partial}{\partial \sigma} \hat{A}(\xi+1;t) \right)\Bigg|_{-\log m_{0,t(l-1)}} \right) \quad (27)$$

PROOF Equation (27) is an immediate consequence of (24) and (26). All other results follow from the interchange of derivative and integral, justified by Theorem A. 12 on pp. 203-204 in Schiff (1999). □

COROLLARY A2 *At any time* $t \in [t(l-1), t(l))$, $l \geq 1$, *the price of a fixed strike lookback put is given by*

$$FP(t,T) = \begin{cases} Ke^{-r(T-t)} + LC(t,T) - S(t); & \text{if } m_{0,\,t(l-1)} \leq K \\ e^{-r(T-t)}\left[K - S(t)\hat{A}(1;t) + g(-\log(K), S(t))\right] & \text{if } m_{0,\,t(l-1)} > K \end{cases}$$

PROOF If $m_{0,\,t(l-1)} \leq K$, then clearly $m_{0,\,T} \leq K$. Thus, $FP(t,T) = e^{-r(T-t)} \cdot (K - \mathsf{E}^*[m_{0,\,T} \mid \mathcal{F}_t])$. If $m_{0,\,t(l-1)} > K$, then

$$\begin{aligned} FP(t,T) &= e^{-r(T-t)} \cdot \mathsf{E}^*\left[(K - m_{t(l),T})^+ \mid \mathcal{F}_t\right] \\ &= e^{-r(T-t)}\left(K - \mathsf{E}^*[m_{t(l),T} \mid \mathcal{F}_t] + \mathsf{E}^*\left[(m_{t(l),T} - K)1_{\{K \leq m_{t(l),T}\}} \mid \mathcal{F}_t\right]\right) \end{aligned}$$

from which the conclusion follows. □

We omit for brevity the results for the greeks, which are of immediate derivation.

A2 Extension of the results to other barrier options

There are eight types of barrier options, four "up" types (e.g. up-and-in call) and four "down" types (e.g. down-and-in call). In this section we will extend the results of Section 3.3 to all other types of barrier options.

(a) "Up" type barrier options

First we will find the two-dimensional Laplace transform for an up-and-out call option, following the approach for up-and-in put option in Subsection 3.3.

THEOREM A2 *Let* $\xi > -1$ *and* $\zeta > 0$ *and assume that* $C(-\zeta, \xi+1; t) = \mathsf{E}^*[e^{(\xi+1)X_{t,T} - \zeta Y^t_{l,m}}] < \infty$. *Define* $\tilde{\kappa} = -\log(K)$ *and* $h = \log(H)$. *At any generic time* $t \in [t(l-1), t(l))$, $m \geq l \geq 1$, *the double Laplace Transform of* $g(\tilde{\kappa}, h; S(t)) = \mathsf{E}^*\left[(S(T) - e^{-\tilde{\kappa}})^+ \mathbf{1}_{\{M_{0,T} > e^h\}} \mid \mathcal{F}_t\right]$ *is given by:*

$$\hat{g}(\xi,\zeta) = \frac{(S(t))^{\xi-\zeta+1}}{\xi(\xi+1)\zeta}\mathsf{E}^*\left[e^{(\xi+1)X_{t,T} - \zeta Y^t_{l,m}}\right] = \frac{(S(t))^{\xi-\zeta+1}}{\xi(\xi+1)\zeta}C(-\zeta, \xi+1; t) \quad (28)$$

PROOF The proof follows the one of Theorem 2 and is therefore omitted. □

COROLLARY A3 At any time $t \in [t(l-1), t(l))$, $m \geq l \geq 1$, *we have*

$$UOC(t,T) = e^{-r(T-t)}\,\mathcal{L}^{-1}_{\xi,\zeta}\left(\frac{(S(t))^{\xi-\zeta+1}}{\xi(\xi+1)\zeta}C(-\zeta,\xi+1;t)\right)\Bigg|_{-\log(K),\log(H)},$$

$$\frac{\partial}{\partial S(t)} UOC(t,T) = e^{-r(T-t)} \mathcal{L}^{-1}_{\xi,\zeta}\left(\frac{(\xi-\zeta+1)(S(t))^{\xi-\zeta}}{\xi(\xi+1)\zeta} C(-\zeta,\xi+1;t) \right)\Bigg|_{-\log(K),\log(H)},$$

$$\frac{\partial^2}{\partial^2 S(t)} LP(t,T) = e^{-r(T-t)} \mathcal{L}^{-1}_{\xi,\zeta}\left(\frac{(\xi-\zeta+1)(\xi+\zeta)(S(t))^{\xi-\zeta-1}}{\xi(\xi+1)\zeta} C(-\zeta,\xi+1;t) \right)\Bigg|_{-\log(K),\log(H)},$$

$$\frac{\partial}{\partial \sigma} UOC(t,T) = e^{-r(T-t)} \mathcal{L}^{-1}_{\xi,\zeta}\left(\frac{(S(t))^{\xi-\zeta+1}}{\xi(\xi+1)\zeta} \frac{\partial}{\partial \sigma} C(-\zeta,\xi+1;t) \right)\Bigg|_{-\log(K),\log(H)} \tag{29}$$

Knowledge of the up-and-out put and call prices and greeks allows us to easily compute the value and hedging parameters of down-and-out and up-and-in put options, because

$$UIC(t, T) = BSC(t, T) - UOC(t, T), \quad UIP(t, T) = BSP(t, T) - UOP(t, T)$$

where $BSC(t, T)$ and $BSP(t, T)$ are the value of a standard call and put options with the same strike K of the two barrier options, respectively.

(b) "Down" type barrier options

By symmetry (see, among others, Haug 1999), "down" type barrier options can be deduced from the "up" type barrier options. More precisely, consider first the case of down-and-out call

$$\begin{aligned} DOC(t,T) &= e^{-r(T-t)} \mathsf{E}^*\left[(S(T)-K)^+ \mathbf{1}_{\{m_{0,T} \geq H\}} \,|\, \mathcal{F}_t\right] \\ &= e^{-r(T-t)} \mathsf{E}^*\left[S(T)K\left(\frac{1}{S(T)} - \frac{1}{K}\right)^+ \mathbf{1}_{\{m_{0,T} \geq H\}} \,|\, \mathcal{F}_t \right] \\ &= S(T)K \cdot \mathsf{E}^*\left[Z(T)\left(\frac{1}{S(T)} - \frac{1}{K}\right)^+ \mathbf{1}_{\{1/m_{0,T} \leq 1/H\}} \,|\, \mathcal{F}_t \right] \end{aligned}$$

where $Z(T) = e^{-r(T-t)} S(T)/S(t)$ is a martingale under P^*. It is fairly easy to find the dynamics of the stochastic process $\bar{S}(T) = 1/S(T)$ under the probability measure $\tilde{P}$, such that $d\tilde{P}/dP^* = Z(T)$, in a Brownian or a jump-diffusion framework (Schroder 1999). Therefore, defining $\bar{M}_{0,T} = \max_{0 \leq j \leq T/\Delta T} \bar{S}_j = 1/m_{0,T}$, we can rewrite:

$$DOC(t,T) = S(T)K \cdot \tilde{\mathsf{E}}\left[\left(\frac{1}{K} - \bar{S}_T\right)^+ \mathbf{1}_{\{\bar{M}_{0,T} \le 1/H\}} \,\middle|\, \mathcal{F}_t\right]$$

Hence, we have reduced the problem to pricing an up-and-out put for the asset $\bar{S}_T$, with strike $1/K$ and barrier $1/H$.

In particular, for the Brownian case, at any time $t < T$, we have

$$DOC(S(t), K, H, \sigma, r, q, T-t) = S(t)K \cdot UOP\left(\frac{1}{S(t)}, \frac{1}{K}, \frac{1}{H}, \sigma, q, r, T-t\right)$$

$$DOP(S(t), K, H, \sigma, r, q, T-t) = S(t)K \cdot UOC\left(\frac{1}{S(t)}, \frac{1}{K}, \frac{1}{H}, \sigma, q, r, T-t\right)$$

Therefore, the two options can be priced directly using the Laplace transforms previously described. Under the MJD model, we have at any $t < T$:

$$\begin{aligned} &DOC(S(t), K, H, \sigma, r, q, \lambda, \mu_J, \sigma_J, T-t) \\ &= S(t)K \cdot UOP\left(\frac{1}{S(t)}, \frac{1}{K}, \frac{1}{H}, \sigma, q, r, \tilde{\lambda}, \widetilde{\mu_J}, \sigma_J, T-t\right) \end{aligned}$$

$$\begin{aligned} &DOP(S(t), K, H, \sigma, r, q, \lambda, \mu_J, \sigma_J, T-t) \\ &= S(t)K \cdot UOC\left(\frac{1}{S(t)}, \frac{1}{K}, \frac{1}{H}, \sigma, q, r, \tilde{\lambda}, \widetilde{\mu_J}, \sigma_J, T-t\right) \end{aligned}$$

$$\text{where} \quad \tilde{\lambda} = \lambda e^{\mu_J + \frac{1}{2}\sigma_J^2} \quad \text{and} \quad \widetilde{\mu_J} = -\mu_J - \sigma_J^2$$

For the DEJD, instead, we have

$$\begin{aligned} &DOC(S(t), K, H, \sigma, r, q, \lambda, p, \eta_1, \eta_2, T-t) \\ &= S(t)K \cdot UOP\left(\frac{1}{S(t)}, \frac{1}{K}, \frac{1}{H}, q, r, \tilde{\lambda}, 1-\tilde{p}, \tilde{\eta}_2, \tilde{\eta}_1, T-t\right) \end{aligned}$$

$$\begin{aligned} &DOP(S(t), K, H, \sigma, r, q, \lambda, p, \eta_1, \eta_2, T-t) \\ &= S(t)K \cdot UOC\left(\frac{1}{S(t)}, \frac{1}{K}, \frac{1}{H}, q, r, \tilde{\lambda}, 1-\tilde{p}, \tilde{\eta}_2, \tilde{\eta}_1, T-t\right) \end{aligned}$$

$$\text{with} \quad \tilde{\lambda} = \lambda\varsigma, \; \tilde{p} = \frac{p\eta_1}{\varsigma(\eta_1 - 1)}, \; \tilde{\eta}_1 = \eta_1 - 1 \text{ and } \tilde{\eta}_2 = \eta_2 + 1,$$

$$\text{where } \varsigma = \frac{p\eta_1}{\eta_1 - 1} + \frac{p\eta_2}{\eta_2 + 1}$$

The other two remaining barrier call option values, *DIP* and *DIC*, can be easily derived via the parities:

$$DIP(t, T) = BSP(t, T) - DOP(t, T), \quad DIC(t, T) = BSC(t, T) - DOC(t, T)$$

Appendix B – Laplace transform inversion via the two-sided Euler algorithm

As mentioned above, we invert the Laplace transforms previously derived using the two-sided extension of the Euler algorithm as described in Petrella (2004). The (one-sided) Euler algorithm has gained a lot of popularity in queueing and network analysis due to its simplicity of implementation, speed and high accuracy. In finance, Fu, Madan and Wang (1997) use it to price continuous Asian options by inverting the Geman and Yor (1993) Laplace transform, and Davydov and Linetsky (2001) implement the algorithm to price continuous double-barrier step options and lookback options under the CEV model. For a survey of different Laplace inversion algorithms and their performances in pricing derivatives we refer the reader to Craddock, Heath and Platen (2000).

In this appendix we will verify the technical conditions given in Petrella (2004) to bound the inversion errors (on both sides of the real line) and specify our choice of parameters in the inversion algorithm.

B1 The Euler inversion for lookback options

(a) Floating lookback put

In this case we need to invert the Laplace transform of

$$f(M_{0,t(l-1)}; S(t)) = \mathsf{E}^*\left[(M_{0,t(l-1)} - M_{t(l),T})\mathbf{1}_{\{M_{0,t(l-1)} \geq M_{t(l),T}\}} \,|\, \mathcal{F}_t\right]$$

Rescaling by a constant $C \leq \min(S(t), M_{0,\,t(l-1)})$, we can write:

$$f(M_{0,t(l-1)}; S(t)) = C\mathsf{E}^*\left[\left(\frac{M_{0,t(l-1)}}{C} - \frac{S(t)}{C}\mathrm{e}^{Y^t_{l,m}}\right)\mathbf{1}_{\{Y^t_{l,m} \leq \log(M_{0,t(l-1)}) - \log(S(t))\}} \,|\, \mathcal{F}_t\right]$$

Although the choice of C is somewhat arbitrary, the purpose of introducing C is to make sure that the Laplace inversion will not be conducted at extreme points for a wide range of model parameters.

When t is a monitoring point, we can set $C = \min(S(t), 0.99M_{0,\,t(l-1)})$. Thus, the function $f(M'; S(t))$ is confined on the positive real line, because $M' := \log(M_{0,\,t(l-1)}/C) \geq 0$. We can therefore refer to the one-sided version of the Euler algorithm. We find that using $N_1 = N_2 = 40$ iterations (by symmetry, on both sides of the real line) to compute the partial sums ensures extremely accurate results.

If t is not a monitoring point, with $t(l-1) \le t < t(l)$, since $Y^t_{l,m} \ge X_{t,t(l)}$ and $\log(S(t)/C) \ge 0$, we have $f(M_{0,t(l-1)}; S(t)) \le C e^{M'} P(X_{t,t(l)} \le M')$, with $M' = \log(M_{0,t(l-1)}/C)$. We can then apply the results in Petrella (2004) for a plain vanilla put option. Accordingly, following the notation in Petrella (2004), we choose $M' = \alpha A$ with

$$\alpha = \min\left(\sigma\sqrt{t(l)-t}, \frac{1}{4}\right)$$

and $A = 18.4$, if the resulting C satisfies $C \le S(t)$, which holds in all our numerical cases. In the DEJD model we also need to make sure the characteristic function of $X_{t,t(l)}$ has no discontinuities. To this end, we fix

$$\alpha = \max\left(\frac{1}{\eta_2}, \min\left(\sigma\sqrt{t(l)-t}, \frac{1}{4}\right)\right)$$

In all cases we set the number of iterations to $N_1 = N_2 = 150$, to guarantee high accuracy in the inversion.

(b) Floating lookback call

In this case we need to invert the Laplace transform of

$$g(m_{0,t(l-1)}; S(t)) = \mathsf{E}^*\left[(m_{t(l),T} - m_{0,t(l-1)})\mathbf{1}_{\{m_{0,t(l-1)} \le m_{t(l),T}\}} \mid \mathcal{F}_t\right]$$

Following Petrella (2004), rescaling by a constant C, we can write

$$g(m_{0,t(l-1)}; S(t)) = \frac{1}{C}\mathsf{E}^*\left[\left(CS(t)e^{\hat{Y}^t_{l,m}} - Cm_{0,t(l-1)}\right)\mathbf{1}_{\{\hat{Y}^t_{l,m} \ge \log(m_{0,t(l-1)}) - \log(S(t))\}} \mid \mathcal{F}_t\right]$$

When t is a monitoring point, we can set $C = \min(1/S(t), 0.99/m_{0,t(l-1)})$ and $m' = -\log(C \cdot m_{0,t(l-1)}) \ge 0$. Again, the function $g(m'; S(t))$ will only be defined on the positive real line. We can therefore refer to the one-sided version of the Euler algorithm. We use $N_1 = N_2 = 40$ iterations (again, by symmetry).

When not at a monitoring point, defining $m' = -\ln(C \cdot m_{0,t(l-1)})$, we can write

$$g(m'; S(t)) \le S(t)\mathsf{E}^*\left[\left(e^{X_{t,t(l)}} - e^{-m'}\right)\mathbf{1}_{\{X_{t,t(l)} - m'\}}\right]$$

because $\hat{Y}^t_{l,m} < X_{t,t(l)}$. We are then in the framework described in Petrella (2004) for the pricing of plain vanilla call options and choose the inversion parameters accordingly. Specifically, we set $m' = \alpha A$, with

$$\alpha = \min\left(\sigma\sqrt{t(l)-t}, \frac{1}{4}\right)$$

and $A = 18.4$. In the DEJD implementation, we fix

$$\alpha = \max\left(\frac{1}{(\eta_1 - 2)},\ \min\left(\sigma\sqrt{t(l) - t}, \frac{1}{4}\right)\right)$$

to avoid discontinuities of the characteristic function of $X_{t,\,t(l)}$. As before, we set $N_1 = N_2 = 150$ for all cases.

B2 The Euler inversion for barrier options

(a) Up-and-out put

By introducing two rescaling parameters C_1 and C_2, the up-and-out put price can be expressed as

$$f(K,H;S(t)) = C_1 \mathsf{E}^* \left[\left(\frac{K}{C_1} - \frac{S(T)}{C_1}\right)^+ \mathbf{1}_{\left\{\frac{M_{0,T}}{C_2} < \frac{H}{C_2}\right\}} \,|\, \mathcal{F}_t\right]$$

When t is a monitoring point, we can choose $C_2 = S(t)$ so that the option price is only defined for positive values of the variable $h' = \ln(H/S(t))$. Hence we only need to study the discretization error on the negative real axis for the random variable $k' = \ln(K/C_1)$. But since

$$f(K,H;S(t)) \le C_1 \mathsf{E}^* \left[\left(\frac{K}{C_1} - \frac{S(T)}{C_1}\right)^+ \,|\, \mathcal{F}_t\right]$$

we can use the results in Petrella (2004) for a plain vanilla put option. Hence, we set $k' = \alpha_1 A_1$, with

$$\alpha_1 = \min\left(\sigma\sqrt{T - t}, \frac{1}{4}\right)$$

and $A_1 = 40$. Further we choose $A_2 = 18.4$ and number of iterations $N_{11} = N_{12} = 200$, $N_{21} = N_{22} = 50$ for the pure-diffusion case and $N_{11} = N_{12} = 100$, $N_{21} = N_{22} = 30$ in the jump-diffusion cases.

If t is not a monitoring point, we must carefully choose C_2 as well. We can easily find exponential bounds in k' and h' for the price of an up-and-out put:

$$f(k', h'; S(t)) \le \begin{cases} e^{k'} & \text{if } k' \ge 0,\ h' \ge 0 \\ e^{k'}\mathsf{P}\left(X_{t,T} \le k'\right) & \text{if } k' \le 0,\ h' \ge 0 \\ \mathsf{P}\left(X_{t,t(l)} < h'\right) & \text{if } k' \ge 0,\ h' < 0 \\ \mathsf{P}\left(X_{t,T} < k', X_{t,t(l)} < h'\right) & \text{if } k' < 0,\ h' < 0 \end{cases} \tag{30}$$

In fact, for any $\vartheta > 0$ and $k' < 0$, we have $\mathsf{P}(X_{t,\,T} < k') = \mathsf{P}(e^{-\vartheta X_{t,T}} > e^{-\vartheta k'}) < e^{\vartheta k'}\mathsf{E}(e^{-\vartheta X_{t,T}})$, by Markov's inequality. In the same way, $\mathsf{P}(X_{t,t,l} < h') < e^{\vartheta h'}E(e^{-\vartheta X_{t,t,l}})$ for $\vartheta > 0$, $h' < 0$. Also, for $k' < 0$, $h' < 0$,

$$\mathsf{E}\left(e^{-\vartheta X_{t,T}-\zeta X_{t,t(l)}}\right) \geq \int_{-\infty}^{h'}\left(\int_{-\infty}^{h'} e^{-\vartheta X_{t,T}-\zeta X_{t,t(l)}} f\left(X_{t,T};X_{t,t(l)}\right)\mathrm{d}X_{t,T}\right)\mathrm{d}X_{t,t(l)}$$
$$\geq e^{-\vartheta k'-\zeta h'}\mathsf{P}\left(X_{t,T}<k', X_{t,t(l)}<h'\right)$$

with $\vartheta, \zeta > 0$. We can therefore bound all the probabilities in (30). As outlined in Petrella (2004), by using appropriate parameters, we can obtain exponential bounds on the discretization errors as well. In our implementation we have chosen

$$k'=\sigma_1 A_1 \ \text{ with } \ A_1 = 40 \ \text{ and } \ \alpha_1 = \min\left(\sigma\sqrt{T-t}, \tfrac{1}{4}\right)$$
$$\text{and } \ h'=\sigma_2 A_2 \ \text{ with } \ \alpha_2 = \min\left(\sigma\sqrt{t(l)-t}, \tfrac{1}{4}\right) \ \text{ and } \ A_2 = 18.4$$

In the DEJD case, we set

$$\alpha_1 = \max\left(\frac{1}{\eta_2}, \min\left(\sigma\sqrt{T-t}, \tfrac{1}{4}\right)\right)$$
$$\text{and } \ \alpha_2 = \max\left(\frac{1}{(2\eta_2-\frac{1}{\alpha_1})}, \min\left(\sigma\sqrt{t(l)-t}, \tfrac{1}{4}\right)\right)$$

to avoid discontinuities of the characteristic function of $X_{t,t(l)}$. We also fix $N_{11} = N_{12} = 200$, $N_{21} = N_{22} = 100$ for all models.

(b) Up-and-out call

Using two rescaling parameters C_1 and C_2, we express the up-and-out call price as

$$f(K,H;S(t)) = \frac{1}{C_1}\mathsf{E}^*\left[\left(S(T)C_1 - KC_1\right)^+ \mathbf{1}_{\left\{\frac{M_{0,T}}{C_2}<\frac{H}{C_2}\right\}} \mid \mathcal{F}_t\right]$$

with $C_1 < \min(1/S(t), 1/K)$.

If t is a monitoring point, we can choose $C_2 = S(t)$ so that the option price is only defined for positive values of the variable $h' = \ln(H/S(t))$. Hence we only need to worry about the discretization error on the negative real axis for the random variable $\tilde{k}' = \ln(1/(KC_1))$. But since

$$f(K,H;S(t)) \leq \frac{1}{C_1}\mathsf{E}^*\left[\left(S(T)C_1 - KC_1\right)^+ \mid \mathcal{F}_t\right]$$

we can use the results in Petrella (2004) for a plain vanilla call option. Hence, we set $\tilde{k}' = \alpha_1 A_1$, with

$$\alpha = \min\left(\sigma\sqrt{t(l)-t}, \tfrac{1}{4}\right)$$

and $A_1 = 40$. Further we choose $A_2 = 18.4$ and number of iterations $N_{11} = N_{12} = 200$, $N_{21} = N_{22} = 50$ for the pure-diffusion case and $N_{11} = N_{12} = 100$, $N_{21} = N_{22} = 30$ in the jump-diffusion cases.

If t is not a monitoring point, we must carefully choose C_2, as well. Exponential bounds in $\tilde{k}$ and h', can also be derived for an up-and-out call. In fact,

$$f(\tilde{k}', h', S(t)) \leq \begin{cases} S(t) & \text{if } \tilde{k}' \geq 0,\ h' \geq 0 \\ S(t)\bar{\mathsf{P}}\left(X_{t,T} > -\tilde{k}'\right) & \text{if } \tilde{k}' \leq 0,\ h' \geq 0 \\ S(t)\bar{\mathsf{P}}\left(X_{t,t(l)} < h'\right) & \text{if } \tilde{k}' \geq 0,\ h' < 0 \\ S(t)\bar{\mathsf{P}}\left(X_{t,T} > -\tilde{k}', X_{t,t(l)} < h'\right) & \text{if } \tilde{k}' < 0,\ h' < 0 \end{cases}$$

where $\bar{\mathsf{P}}$ is the probability measure under which the discounted asset price $S(T)\mathrm{e}^{-rT}$ is the numeraire. Further, using Markov's inequality, for any $\vartheta > 0$ and $\tilde{k}' < 0$, we have $\bar{\mathsf{P}}(X_{t,T} > -\tilde{k}') = \bar{\mathsf{P}}\mathrm{e}^{\vartheta X_{t,T}} > \mathrm{e}^{-\vartheta\tilde{k}'}) < \mathrm{e}^{\vartheta\tilde{k}'}\bar{\mathsf{E}}(\mathrm{e}^{-\vartheta X_{t,T}})$, with $\bar{\mathsf{E}}$ representing the expectation under $\bar{\mathsf{P}}$. In the same way, $\bar{\mathsf{P}}(X_{t,t(l)} < h') < \mathrm{e}^{\vartheta h'}\bar{\mathsf{E}}(\mathrm{e}^{-\vartheta X_{t,t(l)}})$ for $\vartheta > 0$, $h' < 0$. We also have, for $\tilde{k}' < 0$, $h' < 0$,

$$\begin{aligned} \bar{\mathsf{E}}\left(\mathrm{e}^{\vartheta X_{t,T} - \zeta X_{t,t(l)}}\right) &\geq \int_{-\infty}^{h'} \left(\int_{-\tilde{k}'}^{\infty} \mathrm{e}^{\vartheta X_{t,T} - \zeta X_{t,t(l)}} f\left(X_{t,T}; X_{t,t(l)}\right) \mathrm{d}X_{t,T} \right) \mathrm{d}X_{t,t(l)} \\ &\geq \mathrm{e}^{-\vartheta\tilde{k}' - \zeta h'} \bar{\mathsf{P}}\left(X_{t,T} > -\tilde{k}', X_{t,t(l)} < h'\right) \end{aligned}$$

with ϑ, $\zeta > 0$. Following Petrella (2004), we can then find exponential bounds for the discretization errors as well. In our implementation we have chosen

$$\tilde{k}' = \sigma_1 A_1 \text{ with } A_1 = 40 \text{ and } \alpha_1 = \min\left(\sigma\sqrt{T-t}, \tfrac{1}{4}\right)$$
$$\text{and } h' = \sigma_2 A_2 \text{ with } \alpha_2 = \min\left(\sigma\sqrt{t(l)-t}, \tfrac{1}{4}\right) \text{ and } A_2 = 18.4$$

In the DEJD case, we set

$$\alpha_1 = \max\left(\frac{1}{\eta_2}, \min\left(\sigma\sqrt{T-t}, \tfrac{1}{4}\right)\right)$$
$$\text{and } \alpha_2 = \max\left(\frac{1}{(2\eta_2 + \frac{1}{\alpha_1})}, \min\left(\sigma\sqrt{t(l)-t}, \tfrac{1}{4}\right)\right)$$

to avoid discontinuities of the characteristic function of $X_{t,t(l)}$. We also fix $N_{11} = N_{12} = 200$, $N_{21} = N_{22} = 100$ for all models.

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