On-Line Supplement to the Paper “Revenue Management of Callable Products” *

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In this on-line supplement to the paper “Revenue Management of Callable Products”, we shall provide proofs of Lemma 1, Lemma 2, Lemma 3, Lemma 4, and Proposition 6 as well as Equation 11 rewritten for fast computation.

Proof of Lemma 1.

On the set \( \{D_L \leq a\} \), \( S_L(a+1) = S_L(a) = D_L \), which means that \( V_L(a+1) \) and \( V_L(a) \) have the same distribution. Thus, \( E[R(a+1) - R(a)|D_L \leq a] = 0 \), from which we have

\[
\Delta r(a, p) = E[R(a+1, p) - R(a, p)|D_L \geq a+1]P(D_L \geq a+1). \tag{1}
\]

On the set \( \{D_L \geq a+1\} \), \( S_L(a+1) = a+1 \) and \( S_L(a) = a \). Since \( E(X - Y) \) only depends on the marginal distribution of \( X \) and \( Y \), and does not depend on the dependent structure of \( X \) and \( Y \), we have

\[
E[\{\min(D_H, c - \tilde{V}_L(a+1)) - \min(D_H, c - \tilde{V}_L(a))\}|D_L \geq a+1] = E[\min(D_H, c - \tilde{\xi}_i) - \min(D_H, c - \sum_{i=1}^a \tilde{\xi}_i)\}].
\]

via the independence of \( D_L \) and \( D_H \), where \( \tilde{\xi}_i, i \geq 1 \), are independent Bernoulli random variables with a success probability \( \tilde{q} \). Since \( D_H \) is also an integer valued random variable and the values of

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\( c - \sum_{i=1}^{a+1} \xi_i \) and \( c - \sum_{i=1}^{a} \xi_i \) can only differ by at most 1, we have

\[
\min(D_H, c - \sum_{i=1}^{a+1} \xi_i) - \min(D_H, c - \sum_{i=1}^{a} \xi_i) = \{ c - \sum_{i=1}^{a+1} \xi_i - (c - \sum_{i=1}^{a} \xi_i) \}_{1(D_H \geq c - \sum_{i=1}^{a} \xi_i)} = -\xi_{a+1} 1_{(D_H \geq c - \sum_{i=1}^{a} \xi_i)}.
\]

Therefore,

\[
E[\{ \min(D_H, c - \bar{V}_L(a+1)) - \min(D_H, c - \bar{V}_L(a))\} | D_L \geq a+1] = -\bar{q} P \left\{ D_H \geq c - \sum_{i=1}^{a} \xi_i \right\}. 
\tag{2}
\]

Similarly,

\[
E[\{ \min((S_L(a+1) + D_H - c)^+, V_L(a+1)) - \min((S_L(a) + D_H - c)^+, V_L(a))\} | D_L \geq a+1]
\]

\[
= E[\{ \min((a+1 + D_H - c)^+, \sum_{i=1}^{a+1} \xi_i) - \min((a+1 + D_H - c)^+, \sum_{i=1}^{a} \xi_i)\}]
\]

\[
= P \left\{ \xi_{a+1} = 0, (a+1 + D_H - c)^+ > (a+1 + D_H - c)^+, \sum_{i=1}^{a} \xi_i > (a+1 + D_H - c)^+ \right\} 
+ P \left\{ \xi_{a+1} = 1, (a+1 + D_H - c)^+ = (a+1 + D_H - c)^+, (a+1 + D_H - c)^+ \geq \sum_{i=1}^{a+1} \xi_i \right\} 
+ P \{ \xi_{a+1} = 1, (a+1 + D_H - c)^+ = (a+1 + D_H - c)^+ + 1 \}
\]

\[
= \bar{q} P \left\{ \sum_{i=1}^{a} \xi_i > a + D_H - c \geq 0 \right\} + \bar{q} P \left\{ D_H \geq c - a \right\}
\]

\[
= \bar{q} \left\{ P \left\{ \sum_{i=1}^{a} \xi_i > a + D_H - c \geq 0 \right\} - P \left\{ a + D_H - c \geq 0 \right\} \right\} + P \{ D_H \geq c - a \}
\]

\[
= -\bar{q} P \left\{ \sum_{i=1}^{a} \xi_i \leq a + D_H - c \right\} + P \{ D_H \geq c - a \}.
\]

In other words,

\[
E[\{ \min((S_L(a+1) + D_H - c)^+, V_L(a+1)) - \min((S_L(a) + D_H - c)^+, V_L(a))\} | D_L \geq a+1]
\]

\[
= -\bar{q} P \{ c - \text{bino}(a, \bar{q}) \leq D_H \} + P \{ D_H \geq c - a \}. 
\tag{3}
\]

Combining (1), (2), and (3) yields

\[
r(a+1, p) - r(a, p)
= p_L - \bar{q} p H P \{ D_H \geq c - \text{bino}(a, \bar{q}) \} + \bar{q} \bar{q} P \{ c - \text{bino}(a, \bar{q}) \leq D_H \} - p P \{ D_H \geq c - a \},
\]

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Thus, by the definition of formal we observe that if units of capacity should be made available for sale at the low fare. To make this intuition more need to protect more than $\bar{a}$ because $\bar{q} = 1 - g(p) \leq 1$ and $P\{D_H \geq c - \text{bino}(a, \bar{q})\} \leq P\{D_H \geq c - a\}$. Intuitively, we do not need to protect more than $b$ units of capacity for high fare customers and therefore at least $(c - b)^{+}$ units of capacity should be made available for sale at the low fare. To make this intuition more formal we observe that if $b < c$, then at $a = c - b$ we have $p_H P\{D_H \geq c - a\} = p_H P\{D_H \geq b\} = 0$. Thus, by the definition of $a_T^*$, we have $a_T^* \geq c - b$, completing the proof. \(\square\)

\textbf{Proof of Lemma 2.}

\emph{Proof.} We will first show that $a_T^* \leq a(p)$. It is enough to prove that $\psi(a, p) \leq p_H P\{D_H \geq c - a\}$. But this is equivalent to $(p_H - p)\bar{q}P\{D_H \geq c - \text{bino}(a, \bar{q})\} \leq (p_H - p)P\{D_H \geq c - a\}$, which holds because $\bar{q} = 1 - g(p) \leq 1$ and $P\{D_H \geq c - \text{bino}(a, \bar{q})\} \leq P\{D_H \geq c - a\}$.\n
We first note the following properties of the incomplete beta function:

\begin{align*}
I_q(a, n - a + 1) &= \sum_{j=a}^{n} \binom{n}{j} q^j (1 - q)^{n-j}, \quad 0 \leq a \leq n + 1; \quad I_x(a, b) = 1 - I_{1-x}(b, a);
\end{align*}

\begin{equation}
\frac{d}{dq} I_q(a, b) = \frac{q^{a-1}(1 - q)^{b-1}}{B(a, b)}, \quad B(a + 1, b) = \frac{a}{a + b} B(a, b), \quad a, b > 0. \tag{4}
\end{equation}

We shall only study the case when $n \geq 2$ and $1 \leq y \leq n - 1$, as the other two cases hold automatically. In this case, when $y \geq 1$,

\begin{align*}
E[\min(X, y)] &= \sum_{i=1}^{y} \binom{n}{i} q^i (1 - q)^{n-i} + y P\{X \geq y + 1\}
= nq \sum_{i=1}^{y} \binom{n - 1}{i - 1} q^{i-1} (1 - q)^{n-i} + y P\{X \geq y + 1\}
= nq \sum_{j=0}^{y-1} \binom{n - 1}{j} q^{j} (1 - q)^{n-1-j} + y P\{X \geq y + 1\}
= nq(1 - I_q(y, n - y)) + y I_q(y + 1, n - y).
\end{align*}

We also have

\begin{align*}
\frac{d}{dq} E[\min(X, y)] &= n\{1 - I_q(y, n - y)\} + nq\left(-\frac{q^{y-1}(1 - q)^{n-y-1}}{B(y, n - y)}\right) + y \frac{q^{y}(1 - q)^{n-y-1}}{B(y + 1, n - y)}
= n\{1 - I_q(y, n - y)\} + nq\left(-\frac{q^{y-1}(1 - q)^{n-y-1}}{B(y, n - y)}\right) + n \frac{q^{y}(1 - q)^{n-y-1}}{B(y, n - y)}
= n\{1 - I_q(y, n - y)\},
\end{align*}

which completes the proof.
via (4). Finally, when \( y \geq x, x \in [0,n], \) and \( n \geq 1, \) we have

\[
P\{\min(X,y) > x\} = P\{X > x\} = \sum_{j=[x]+1}^{n} \binom{n}{j} q^{j} (1-q)^{n-j} = I_{q}([x]+1, n-[x]),
\]

which completes the proof. □

**Proof of Lemma 4.**

Proof. By Assumption 2, it is sufficient to show that \( \frac{d}{dq} E[\min(G(a),V_{L}(a))] \) is positive and strictly decreasing in \( q, q \in [0, \bar{q}_{H}], \) because then \( E[\min(G(a),V_{L}(a))] \) must be strictly increasing in \( q. \) To do this, note that by Lemma 3

\[
\frac{d}{dq} E[\min(G(a),V_{L}(a))] = E[S_{L}(a)\{1-I_{q}(G(a),S_{L}(a)-G(a))\}; S_{L}(a)+D_{H} > c, D_{H} \leq c-1]
\]

\[
+ E[S_{L}(a); D_{H} \geq c, S_{L}(a) \geq 1].
\]

On the set \( \{S_{L}(a)+D_{H} > c, D_{H} \leq c-1\}, \) by the definition of \( G(a), \) we must have \( G(a) > 0, S_{L}(a) < G(a). \) Thus, \( 0 < I_{q}(G(a),S_{L}(a)-G(a)) < 1, \) and \( I_{q}(G(a),S_{L}(a)-G(a)) \) is strictly increasing in \( q, \) for \( 0 < q < 1, \) by (10). Since \( E[X] > 0 \) for any random variable \( X \geq 0 \) with \( P\{X > 0\} > 0, \) Assumption 1 implies that \( \frac{d}{dq} E[\min(G(a),V_{L}(a))] \) must be positive and strictly decreasing in \( q, q \in [0, \bar{q}_{H}]. \) To check for uniqueness, notice that the left side of (9) is a strictly increasing function of \( q, \) the right side is a strictly decreasing function of \( q, \) and when \( q = 0 \) (at the recall price \( p_{L} \)) the left hand side is zero (because \( V_{L}(a) = 0). \) If at \( q = \bar{q}_{H} \) the right side of (9) is not greater than the left side, then there must be a unique root within \([0, \bar{q}_{H}]; \) otherwise, \( \frac{d}{dq} r(a,p) > 0 \) for all \( p \in [p_{L}, \bar{q}_{H}], \) and \( \bar{q}_{H} \) must be optimal. □

**Proof of Proposition 6.**

Consider the case of a risk-neutral first-period customer with maximum willingness-to-pay of \( R_{L} \) who is faced with the choice between purchasing a low-fare standard product for \( p_{L} \) or a pure callable with a price of \( p_{C} \) and a call price of \( \hat{p}. \) He will purchase the standard product if \( R_{L}-p_{L} \geq \max[(1-q)(R_{L}-p_{c})+q(p-p_{c}),0] \) and the callable product if \( (1-q)(R_{L}-p_{c})+q(p-p_{c}) > \max[R_{L}-p_{L},0], \) where \( q > 0 \) is his ex ante probability that the airline will exercise the call for his product. Assume that, instead of offering callables at a price of \( p_{C} \) and a strike price of \( p \) along with the standard low-price products, the airline offers both callables and standard products at the
price $p_L$ with a free callable option at $\hat{p}$. Then, a customer with valuation $R_L$ would purchase the call option in both cases if

$$(1 - q)(R_L - p_C) + q(p - p_C) = (1 - q)(R_L - p_L) + q(\hat{p} - p_L)$$

or $\hat{p} = p + (p_L - p_C)/q$. Note that $\hat{p}$ is independent of $R_L$. This means that offering both standard and callable products at $p_L$ with a strike price of $\hat{p}$ will result in the same demand for any distribution of $R_L$ as offering standard products at $p_L$ and callable products at $p_C$ with a strike price of $p$ assuming only that customers are risk neutral and share a common \textit{ex ante} probability $q$.

We now need to show that the provider will achieve the same expected revenue in both cases. Let $R(\alpha)$ be revenue in the case when callables cost $p_C$ and the exercise price is $p$ and $\hat{R}(\alpha)$ be the case when callables cost $p_L$ and the exercise price is $\hat{p} = p + (p_L - p_C)/q$. Then, using the notation of Equation (1):

$R(\alpha) = p_CV_L(\alpha) + p_L[S_L(\alpha) - V_L(\alpha)] + p_H \min(D_H, c - \hat{V}_L(\alpha)) - \hat{p} \min[(S_L(\alpha) + D_H - c)^+, V_L(\alpha)]$

$\hat{R}(\alpha) = p_SL(\alpha) + p_H \min[D_H, c - \hat{V}_L(\alpha)] - \hat{p} \min[(S_L(\alpha) + D_H - c)^+, V_L(\alpha)]$

so that

$$R(\alpha) - \hat{R}(\alpha) = (p_C - p_L)V_L(\alpha) + (\hat{p} - p) \min[S_L(\alpha) + D_H - c, V_L(\alpha)],$$

$$= (p_C - p_L)V_L(\alpha) + \frac{p_L - p_C}{q} \min[S_L(\alpha) + D_H - c, V_L(\alpha)].$$

If each customer correctly anticipates the fraction of products that will be called, then

$$q = \frac{\min[S_L(\alpha) + D_H - c, V_L(\alpha)]}{V_L(\alpha)}$$

and $R(\alpha) - \hat{R}(\alpha) = 0$.

\textbf{Rewriting the Terms in Equation 11 for Programming Purposes.}

$$E[S_L(a)I q(S_L(a) - G(a), G(a)); S_L(a) + D_H > c, D_H \leq c - 1]$$

$$= \sum_{i=2}^{a} \sum_{j=0}^{c-1} P\{D_L = i, D_H = j\} i q(c - j, i + j - c) 1_{i + j \geq c + 1}$$

$$+ \sum_{j=0}^{c-1} P\{D_L \geq a + 1, D_H = j\} a q(c - j, a + j - c) 1_{a + j \geq c + 1},$$
\[
E[G(a)I_q(G(a) + 1, S_L(a) - G(a)); S_L(a) + D_H > c, D_H \leq c - 1]
\]
\[
= \sum_{i=2}^{a} \sum_{j=0}^{c-1} P\{D_L = i, D_H = j\}(i + j - c)I_q((i + j - c + 1, c - j)1_{i+j\geq c+1})
\]
\[
+ \sum_{j=0}^{c-1} P\{D_L \geq a + 1, D_H = j\}(a + j - c)I_q(a + j - c + 1, c - j)1_{a+j\geq c+1},
\]
\[
E[S_L(a); D_H \geq c, S_L(a) \geq 1] = \sum_{i=0}^{a} P\{D_L = i, D_H \geq c\}i + a \sum_{i=a+1}^{\infty} P\{D_L = i, D_H \geq c\}.
\]

Also, for traditional revenue management, we have
\[
r(a) = p_L E[S_L(a)] + p_H E[\min(c - S_L(a), D_H)] \\
= p_L \left\{ \sum_{i=0}^{a} P\{D_L = i\}i + aP\{D_L \geq a + 1\} \right\} \\
+ p_H \left\{ \sum_{i=0}^{a} \sum_{j=0}^{c-i} P\{D_L = i, D_H = j\}j + \sum_{i=0}^{a} (c - i)P\{D_L = i, D_H \geq c - i + 1\} \\
+ P\{D_L \geq a + 1\} \left[ \sum_{j=0}^{c-a} P\{D_H = j\}j + (c - a)P\{D_H \geq c - a + 1\} \right] \right\}.
\]