Finance Stochast. 3, 55–82 (1999)

Finance and Stochastics

Connecting discrete and continuous path-dependent options

Mark Broadie[1], **Paul Glasserman**[1], **S.G. Kou**[2,3]

[1] Graduate School of Business, Columbia University, New York, NY 10027, USA
[2] Department of Statistics, University of Michigan, Ann Arbor, MI 48109–1027, USA
(e-mail: kou@umich.edu)
[3] Department of IEOR, Columbia University, New York, NY 10027, USA

Abstract. This paper develops methods for relating the prices of discrete- and continuous-time versions of path-dependent options sensitive to extremal values of the underlying asset, including lookback, barrier, and hindsight options. The relationships take the form of correction terms that can be interpreted as shifting a barrier, a strike, or an extremal price. These correction terms enable us to use closed-form solutions for continuous option prices to approximate their discrete counterparts. We also develop discrete-time discrete-state lattice methods for determining accurate prices of discrete and continuous path-dependent options. In several cases, the lattice methods use correction terms based on the connection between discrete- and continuous-time prices which dramatically improve convergence to the accurate price.

Key words: Barrier options, lookback options, continuity corrections, trinomial trees

JEL classification: G13, C63, G12

Mathematics Subject Classification (1991): 90A09, 60J15, 65N06

1 Introduction

This paper develops methods for relating the prices of discrete- and continuous-time versions of path-dependent options sensitive to extremal values of the underlying asset, including lookback, barrier, and hindsight options and, by extension, others that can be constructed from these, such as ladder options. The payoffs in the continuous-time versions depend on the price of the underlying asset throughout the life of the option, whereas the payoffs in the discrete-time versions are determined by underlying prices at a finite set of times. The payoffs of lookback

Manuscript received: December 1996; final version received: December 1997

and barrier options, for example, depend on the maximum or minimum underlying price; the terms of the contract dictate whether the maximum or minimum is evaluated in continuous or discrete time. Questions concerning the relation between discrete- and continuous-time prices arise in at least three ways:

- Nearly all closed-form expressions available for pricing path-dependent options are based on continuous-time paths, but many traded options are based on discrete price fixings. In this setting, the question becomes how best to use a continuous formula to approximate the price of a discrete option.
- Numerical methods are necessary for precise evaluation of discrete option prices. These are themselves based on a discretization of time, but typically a much finer one than that specified in the terms of an option. Thus, numerically pricing a discrete option involves two discrete time increments – the intervals between price fixings that determine the option payoff and the time step in the numerical method. The problem in this setting is one of analyzing the relation between two discrete-time processes related to a common continuous-time process.
- Even if the option of interest is based on continuous monitoring of the underlying asset price, a discrete numerical method is often required for valuation – for example, if the option is American. Improving the quality of the numerical method involves analyzing how a discrete-time, discrete-valued process approximates a continuous-time, continuous-valued process.

These issues arise quite transparently in the pricing of literal options, but not only in that context. For example, Longstaff [46] approximates the value of marketability of a security over a fixed horizon with a type of continuous-time lookback option and gives a closed-form expression for the value; the discrete version would also be relevant in his setting. Merton [49], Black and Cox [7], and more recently Leland [42], Longstaff and Schwartz [47], and Rich [52] among others, have used barrier models for valuing debt and contingent claims with endogenous default. For tractability, this line of work typically assumes continuous monitoring of a reorganization boundary;[1] but to the extent that default can be modeled as a barrier crossing, it is arguably one that can be triggered only at specific dates – e.g., coupon payment dates.

The impact on option values of discrete price fixings has been noted frequently in the literature. Chance [18], Flesaker [26], Heynen and Kat [34], and Kat and Verdonk [39] discuss the implications of ignoring the discrete-continuous distinction in the context of specific market instruments and give numerical illustrations showing substantial mispricings when, e.g., daily fixings are approximated by continuous monitoring. Computational issues arising in the pricing of discrete path-dependent options are discussed in Andersen and Brotherton-Ratcliffe [3], Babbs [6], Cheuk and Vorst [21, 20], Kat [38], and Levy [43].

By further developing a line of work initiated in Broadie, Glasserman, and Kou [13], in this paper we develop a general approach to moving between discrete and continuous prices in the three classes of problems described earlier. Specifically, we present the following:

- Correction terms that dramatically improve the approximation of discrete-time prices using continuous-time formulas. For barrier options, the correction shifts the barrier to price a discrete option using the continuous formula. For lookbacks, the correction shifts the expected maximum or minimum price.
- Lattice methods that use correction terms to improve convergence to the exact discrete-time price. For example, in the case of a discrete barrier option we use a trinomial method that puts a row of nodes at the level of the *shifted* barrier.
- Lattice methods using different but related correction terms to improve convergence to continuous-time prices.

A first-order correction term was introduced for barrier options in Broadie, Glasserman, and Kou [13]. That correction is based on a constant $\beta_1 = -\zeta(1/2)/\sqrt{2\pi}$, with ζ the Riemann zeta function. Rather remarkably, the same constant enters in the first-order correction for lookback options, but by an entirely different route: in both cases, β_1 arises as the mean of a limiting distribution, but the distributions in the two cases are different. For the price of a lookback at inception we derive a second-order approximation based on the second moment of one of these limiting distributions. The role of β_1 in discrete barrier options is rooted in work of Chernoff [17][2] and Siegmund and Yuh [58] on diffusion approximations to random walks; its role in discrete lookback options is rooted in the analysis of Calvin [15] and Asmussen, Glynn, and Pitman [5] on the maximum of Brownian motion.

The rest of this paper is organized as follows. Section 2 presents corrections for continuous-time formulas to price discrete-time options approximately. Section 3 develops numerical methods for pricing discrete-time and continuous-time options accurately. Proofs are deferred to an appendix.

2 Continuity corrections

We assume throughout that the continuous-time price is determined by the assumptions of Black and Scholes [8]. There is a single risky asset whose price $\{S_t, t \geq 0\}$ evolves according to

$$dS_t = \nu S_t\, dt + \sigma S_t\, dZ_t, \tag{1}$$

with Z a standard Wiener process, and ν and $\sigma > 0$ constants. The term structure is flat, with r denoting the constant continuously compounded risk-free interest rate. The price of a claim contingent on $\{S_t, 0 \leq t \leq T\}$ is the expected present value of its payoff, the expectation taken with $\nu = r$.[3] Since we are interested in probabilities and expectations only under the risk-neutral measure, henceforth we take $\nu = r$.

To specify the payoffs of some path-dependent options, we need additional notation. With $\nu = r$, write the solution to (1) as

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma Z_t} \equiv S_0 e^{B_t}, \tag{2}$$

with B_t a Wiener process having drift $r - \frac{1}{2}\sigma^2$ and variance parameter σ^2. For the discrete options, let m be the number of price-fixing dates and $\Delta t = T/m$ the interval between fixings. Set

$$\begin{aligned}
\tau = \tau_H &= \text{first } t \text{ at which } S_t \text{ reaches level } H; \\
\tilde{\tau} = \tilde{\tau}_H &= \text{first } k \text{ at which } S_{k\Delta t} \text{ crosses level } H; \\
M &= \max_{0 \le t \le T} B_t \\
\tilde{M}_m &= \max_{0 \le k \le m} B_{kT/m}.
\end{aligned}$$

To allow the barrier level H to be either above or below the initial asset price S_0 without introducing any ambiguity, $\tilde{\tau}$ is more precisely defined to be $\inf\{k \ge 0 : S_{k\Delta t} > H\}$ if $H > S_0$ and $\inf\{k \ge 0 : S_{k\Delta t} < H\}$ if $H < S_0$. We exclude the case $H = S_0$.

Letting $\mathbf{1}_{\{\cdot\}}$ denote the indicator of the event $\{\cdot\}$ and writing x^+ for $\max\{x, 0\}$, the payoff of a continuous knock-out call with strike K and barrier H is given by

$$(S_T - K)^+ \mathbf{1}_{\{\tau_H > T\}};$$

that of a knock-in call is given by

$$(S_T - K)^+ \mathbf{1}_{\{\tau_H \le T\}};$$

and corresponding put payoffs are obtained by replacing $S_T - K$ with $K - S_T$. The discrete-barrier counterparts to the payoffs above are

$$(S_{m\Delta t} - K)^+ \mathbf{1}_{\{\tilde{\tau}_H > m\}}$$

and

$$(S_{m\Delta t} - K)^+ \mathbf{1}_{\{\tilde{\tau}_H \le m\}},$$

respectively. The payoffs on continuous and discrete lookback puts are

$$S_0 e^M - S_T \quad \text{and} \quad S_0 e^{\tilde{M}_m} - S_T; \tag{3}$$

for lookback calls, replace the max in the definitions of M and $\tilde{M}_m$ with a min and multiply by -1 in (3). Finally, the payoffs on hindsight calls (sometimes referred to as fixed-strike lookbacks) are

$$(S_0 e^M - K)^+ \quad \text{and} \quad (S_0 e^{\tilde{M}_m} - K)^+$$

with continuous and discrete fixings, respectively. Hindsight puts are formed in the obvious way. Some related variants of the options above are two-dimensional barrier options, in which one asset determines the barrier crossing and the other the terminal payoff; partial barrier options, in which the barrier is in effect only in some subinterval of $[0, T]$; and percentage lookbacks,[4] in which the minimum or maximum price of the underlying is multiplied by a constant in the usual lookback payoff. The continuous versions of all of these options can be priced in closed form. See, in particular, Merton [48], Rubinstein and Reiner [54], Chance

[18], Boyle and Lau [10], Rich [51], Carr [16], and Heynen and Kat [31, 32] for various kinds of barrier options, and for various kinds of lookbacks see Conze and Viswanathan [23], Garman [27], Goldman, Sosin, and Gatto [29], Goldman, Sosin, and Shepp [30], and Heynan and Kat [33].

Closed-form expressions for discrete versions of the options above typically involve values of m-dimensional multivariate cumulative normal distributions and are therefore of little value for more than about $m = 5$ fixing dates. The following results show, however, that with appropriate corrections the continuous formulas can be used to price discrete options quite accurately for moderate to large values of m. We begin by quoting a result from Broadie, Glasserman, and Kou [13]. As in Sect. 1, set $\beta_1 = -\zeta(1/2)/\sqrt{2\pi} \approx 0.5826$, with ζ the Riemann zeta function (see, e.g., [1] for background on ζ).

Theorem 1 ([13]) *Let $V(H)$ be the price of a continuous down-and-in call, down-and-out call, up-and-in put, or up-and-out put. Let $V_m(H)$ be the price of an otherwise identical discrete barrier option. Then $V_m(H) = V(He^{\pm\beta_1\sigma\sqrt{T/m}}) + o(1/\sqrt{m})$, with + for an up option and − for a down option.*

One interprets this result as saying that to price a discretely monitored barrier option using the continuous formula, one should first shift the barrier away from S_0 by a factor $e^{\beta_1\sigma\sqrt{T/m}}$. This corrects for the fact that when the discrete-time process $\{S_{k\Delta t}, k = 0, 1, 2, \ldots\}$ breaches the barrier, it overshoots it. The constant $\beta_1\sigma\sqrt{T/m}$ should be viewed as an approximation to the overshoot in the logarithm of the price of the underlying. Indeed, in the driftless case $r - \frac{1}{2}\sigma^2 = 0$, with $H > S_0$, we have (as a consequence of Lemma 10.11 and Theorem 10.55 of Siegmund [57])

$$\sqrt{m}E[B_{\tilde{\tau}_H\Delta t} - \log(H/S_0)] \to \beta_1\sigma\sqrt{T} \tag{4}$$

so $\beta_1\sigma\sqrt{T/m}$ approximates $E[B_{\tilde{\tau}_H\Delta t}] - \log(H/S_0)$. Taylor expansion further suggests the approximations

$$\begin{aligned} E[S_{\tilde{\tau}_H\Delta t}] = S_0E[\exp(B_{\tilde{\tau}_H\Delta t})] &\approx S_0E[1 + B_{\tilde{\tau}_H\Delta t}] \\ &\approx S_0(1 + \log(H/S_0) + \beta_1\sigma\sqrt{T/m}) \\ &\approx He^{\beta_1\sigma\sqrt{T/m}}. \end{aligned}$$

This interpretation is also reflected in the precise result

$$P(\tilde{\tau}_H \leq m) = P(\tau_{H\exp(\beta_1\sigma\sqrt{\Delta t})} \leq T) + o(1/\sqrt{m}). \tag{5}$$

See [13] for a more detailed treatment in this context and see Siegmund [57] and Siegmund and Yuh [58] for mathematical underpinnings.[5]

In light of (3) and the fact that $E[S_T] = e^{rT}S_0$, pricing a discrete lookback put at the inception of the contract entails the evaluation of $E[e^{\tilde{M}_m}]$. Here is a heuristic argument from the barrier correction to an approximation for lookbacks. For any $x > S_0$, we have

$$\{\tilde{\tau}_x \le m\} = \{S_0 e^{\tilde{M}_m} > x\} \quad \text{and} \quad \{\tau_x \le T\} = \{S_0 e^M \ge x\}.$$

We may therefore rewrite (5) as

$$P(S_0 e^{\tilde{M}_m} > x) \approx P(S_0 e^M > x e^{\beta_1 \sigma \sqrt{T/m}}).$$

By integrating, we get

$$\begin{aligned} E[S_0 e^{\tilde{M}_m}] &= \int_0^\infty P(S_0 e^{\tilde{M}_m} > x)\, dx \\ &\approx \int_0^\infty P(S_0 e^M > x e^{\beta_1 \sigma \sqrt{T/m}})\, dx \\ &= e^{-\beta_1 \sigma \sqrt{T/m}} \int_0^\infty P(S_0 e^M > y)\, dy \\ &= e^{-\beta_1 \sigma \sqrt{T/m}} E[S_0 e^M], \end{aligned} \tag{6}$$

which suggests a correction mechanism for relating discrete and continuous lookbacks. However, it appears to be impossible to turn this heuristic sketch into a valid argument, because (5) holds only for $x > S_0$ and this approach requires integrating down to $x = S_0$. The correction can be justified by appealing to rather different results of Asmussen, Glynn, and Pitman [5]; by following and extending their approach, we in fact arrive at a second-order correction.

It follows from Theorem 2 and Lemma 6 of Asmussen, Glynn, and Pitman [5] that

$$\beta_1 = \lim_{m \to \infty} \frac{\sqrt{m} E[M - \tilde{M}_m]}{\sigma \sqrt{T}} \tag{7}$$

and that

$$\beta_2 \triangleq \lim_{m \to \infty} \frac{m[(M - \tilde{M}_m)^2]}{\sigma^2 T}$$

exists. Though no simpler expression is available for β_2, we have found numerically that $\beta_2 \approx 0.425$. We now have

Theorem 2 *Let V denote the price of a continuous lookback put or call at the inception of the contract, and let V_m be the price of an otherwise identical discrete lookback. Define*

$$\gamma_\pm = \frac{1}{2} \left[\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\mu^2 T}{2\sigma^2}} \pm \left(\mu \sqrt{T} \Phi(\mu \sqrt{T}/\sigma) - \frac{\mu \sqrt{T}}{2} \right) \right],$$

where $\mu = r - \frac{1}{2}\sigma^2$ and Φ denotes the cumulative standard normal distribution. Then

$$\begin{aligned} V_m &= (V + S_0) \left(1 - \frac{\beta_1 \sigma \sqrt{T}}{\sqrt{m}} + \frac{\gamma_+ \sqrt{T} + \frac{1}{2}\beta_2 \sigma^2 T}{m} \right) \\ &\quad - S_0 + \mathrm{Cov}[e^M, \tilde{M}_m - M] + o(1/m) \end{aligned} \tag{8}$$

for puts and

$$V_m = (V - S_0)\left(1 + \frac{\beta_1\sigma\sqrt{T}}{\sqrt{m}} + \frac{\gamma_-\sqrt{T} + \frac{1}{2}\beta_2\sigma^2 T}{m}\right)$$
$$+S_0 - \mathrm{Cov}[e^{-M'}, \tilde{M}'_m - M'] + o(1/m) \qquad (9)$$

for calls, where $M' = \max_{0\le t\le T}(-B_t)$ *and* $M'_m = \max_{0\le k\le m}(-B_{kT/m})$.

If we expand only up to terms of order $1/\sqrt{m}$, we get

$$V_m = (V + S_0)\left(1 - \frac{\beta_1\sigma\sqrt{T}}{\sqrt{m}}\right) - S_0 + o(1/\sqrt{m})$$

for a lookback put. In view of (3) and the fact that $e^{-rT}E[S_T] = S_0$, this is equivalent to the statement

$$S_0 E[e^{\tilde{M}_m}] = S_0 E[e^M]\left(1 - \frac{\beta_1\sigma\sqrt{T}}{\sqrt{m}}\right) + o(1/\sqrt{m});$$

and this in turn is equivalent to

$$S_0 E[e^{\tilde{M}_m}] = S_0 E[e^M] e^{-\beta_1\sigma\sqrt{T/m}} + o(1/\sqrt{m}),$$

by Taylor's theorem. Thus, Theorem 2 is consistent with – and indeed refines – the approximation arrived at heuristically in (6).[6]

Asmussen, Glynn, and Pitman [5] show that $\sqrt{m}(\tilde{M}_m - M)$ and M are asymptotically independent. It follows through a uniform integrability argument that

$$\sqrt{m}\mathrm{Cov}[e^M, \tilde{M}_m - M] \to 0,$$

so there is no covariance term when we expand to terms of order $1/\sqrt{m}$. However, we may not have convergence to zero when the covariance is scaled by m; indeed, it appears that $m\mathrm{Cov}[e^M, \tilde{M}_m - M]$ converges to a constant. It is not clear how this constant should depend on the drift and variance parameter of B_t, so it seems difficult to include this term in a practical approximation. In numerical examples we find that this term is very small and therefore omit it. We refer to the expressions in (8) and (9) – even without the covariance terms – as "second-order approximations," though the error is strictly $o(1/m)$ only if the covariance terms are included.

Theorem 2 applies only at the inception of the contract. At an arbitrary time $0 < t < T$ in the life of a continuous lookback put, its value is

$$V(S_+) = e^{-r(T-t)}E[\max\{S_+, \max_{t\le u\le T} S_u\}] - S_t, \qquad (10)$$

where $S_+ = \max_{0\le u\le t} S_u$. The price of a continuous lookback call similarly depends on $S_- = \min_{0\le u\le t} S_u$ and $\min_{t\le u\le T} S_u$. The price of a discrete lookback put at the k^{th} fixing is

$$V_m(S_+) = e^{-r(m-k)\Delta t} E[\max\{S_+, \max_{k \leq j \leq m} S_{j\Delta t}\}] - S_{k\Delta t}, \tag{11}$$

with $S_+ = \max_{0 \leq j \leq k} S_{j\Delta t}$. The value of discrete lookback put at the k^{th} fixing admits an analogous expression involving $S_- = \min_{0 \leq j \leq k} S_{j\Delta t}$ and $\min_{k \leq j \leq m} S_{j\Delta t}$. We refer to S_- and S_+ as the *predetermined min* and *max* respectively. The fact that these are calculated differently for discrete and continuous options is inconsequential: when an option is valued the corresponding $S_\pm$ is known and acts as a parameter of the payoff function, just like the strike in a standard option. We now have

Theorem 3 *The price of a discrete lookback at the* k^{th} *fixing date and the price of a continuous lookback at time* $t = k\Delta t$ *satisfy*

$$V_m(S_\pm) = \pm \left[e^{\mp \beta_1 \sigma \sqrt{T/m}} V(S_\pm e^{\pm \beta_1 \sigma \sqrt{T/m}}) + (e^{\mp \beta_1 \sigma \sqrt{T/m}} - 1) S_t \right] + o(1/\sqrt{m}), \tag{12}$$

where, in $\pm$ *and* $\mp$*, the top case applies for puts and the bottom for calls.*

In the case of a put, some algebra shows that (12) is equivalent to

$$E[\max\{\max_{k \leq j \leq m} S_{j\Delta t}, S_+\}] = e^{-\beta_1 \sigma \sqrt{(T-m)/m}} E[\max\{\max_{t \leq u \leq T} S_t, e^{\beta_1 \sigma \sqrt{T/m}} S_+\}]$$
$$+ o(1/\sqrt{m}).$$

Theorem 3 may therefore be interpreted as follows: to value a discrete lookback put using the continuous price, first inflate the predetermined max by a factor of $e^{\beta_1 \sigma \sqrt{T/m}}$, then deflate the expected maximum over $[0, T]$ by the same factor. For a lookback call, first deflate the predetermined min, then inflate the expected minimum.

The last case we consider in detail is the pricing of a hindsight option. The price of a continuous hindsight call at time t with predetermined max S_+ and strike K is

$$V(S_+, K) = e^{-r(T-t)} E[(\max\{S_+, \max_{t \leq u \leq T} S_u\} - K)^+];$$

similarly,

$$V(S_-, K) = e^{-r(T-t)} E[(K - \min\{S_-, \min_{t \leq u \leq T} S_u\})^+]$$

is the price of a continuous hindsight put. The discrete counterparts at the k^{th} fixing date are

$$V_m(S_+, K) = e^{-r(m-k)\Delta t} E[(\max\{S_+, \max_{k \leq j \leq m} S_{j\Delta t}\} - K)^+]$$

and

$$V_m(S_-, K) = e^{-r(m-k)\Delta t} E[(K - \min\{S_-, \min_{k \leq j \leq m} S_{j\Delta t}\})^+].$$

Conze and Viswanathan [23] provide explicit formulas for the continuous prices. The next result shows how to adjust these formulas to price the discrete versions:

Theorem 4 *For hindsight options,*

$$V_m(S_\pm, K) = V(S_\pm e^{\pm\beta_1\sigma\sqrt{T/m}}, Ke^{\pm\beta_1\sigma\sqrt{T/m}})e^{\mp\beta_1\sigma\sqrt{T/m}} + o(1/\sqrt{m}),$$

where, in $\pm$ and $\mp$, the top case applies for calls and the bottom for puts.

In fact, there is a simple relationship between hindsight calls (puts) and lookback puts (calls) at an arbitrary time $t \in [0, T]$. Using the identity $(x-y)^+ = \max(x, y) - y$, we have

$$\begin{aligned} V^c(S_+, K) &= e^{-r(T-t)}E[(\max\{S_+, \max_{t\le u\le T} S_u\} - K)^+] \\ &= e^{-r(T-t)}E[\max\{S_+, \max_{t\le u\le T} S_u, K\} - K] \\ &= e^{-r(T-t)}E[\max\{S_+, \max_{t\le u\le T} S_u, K\} - S_T + S_T - K] \\ &= e^{-r(T-t)}E[\max\{S_+, \max_{t\le u\le T} S_u, K\} - S_T] + S_t - e^{-r(T-t)}K \\ &= V^p(\max(S_+, K)) + S_t - e^{-r(T-t)}K, \end{aligned} \tag{13}$$

where $V^c(S_+, K)$ is the price of a continuous hindsight call with predetermined max S_+ and strike price K, and $V^p(\max(S_+, K))$ is the price of a continuous lookback put with predetermined max given by $\max(S_+, K)$. Similarly, the discrete versions of these options satisfy

$$V_m^c(S_+, K) = V_m^p(\max(S_+, K)) + S_{k\Delta t} - e^{-r(m-k)\Delta t}K, \tag{14}$$

at $k = 1, \ldots, m$. The result in equation (13) is useful because it simplifies some formulas and derivations in the existing literature. Equation (14) is similarly useful since it shows that any numerical method for pricing lookbacks can be used to price hindsight options.

In the next section, we give some numerical examples illustrating the accuracy of the corrected pricing formulas in Theorems 2–4. This is not entirely straightforward because computing very accurate prices (accurate enough to reliably measure the error in our approximations) for the options we consider is difficult using standard implementations of binomial or trinomial trees. So, we first we develop specially tailored numerical procedures for pricing these options accurately; several of these exploit corrections analogous to those in our approximate formulas. Extensive numerical results supporting Theorem 1, based on one of these methods, are reported in Broadie, Glasserman, and Kou [13].

3 Lattice methods for discrete options

In this section, we develop numerical procedures which can compute the prices of the options considered in Sect. 2 to a high degree of accuracy in a reasonable amount of computing time. The terms "high degree of accuracy" and "reasonable amount of computing time" will be quantified shortly. Accurate numerical methods are necessary to test the effectiveness of approximation methods and also to price American-style versions of the options. We begin with discrete barrier options, then consider discrete lookback options, and finally examine discrete numerical methods for pricing continuous lookback options.

In several cases, the numerical procedures use shifting techniques analogous to those in Theorems 1–4. To see what sort of adjustment one should expect, recall that β_1 arose in (4) as the limiting expected overshoot over a boundary for a random walk with normal increments. Lattice methods approximate Brownian motion by a random walk with increments of the form $\pm a$ (in the binomial case) or $\pm a$ and 0 (in the trinomial case). The "average" overshoot over a boundary for such a random walk is $a/2$, so in correcting lattice methods we should expect to see a factor of $1/2$ rather than β_1. Also, whereas previously the relevant time increment was T/m, with m the number of monitoring dates, now the relevant time increment is T/n, with n the number of steps in a tree.

3.1 Lattice methods for discrete barrier options

In order to accurately price discrete barrier options we propose a trinomial lattice procedure with several important modifications to speed convergence. Before describing our method, we show that straightforward implementations of several alternative procedures are inadequate for computing highly accurate prices.[7] Suppose we wish to compute the true price of a discrete down-and-out barrier option with parameters $S_0 = K = 100$, $\sigma = 0.6$, $r = 0.1$, $T = 0.2$, $H = 95$, and $m = 50$, and we would like to estimate the true price to within about \$0.001, i.e., one-tenth of a cent accuracy. This is a particularly difficult test case because of the high volatility and the close proximity of H to S_0. Monte Carlo simulation is an obvious numerical procedure to test. Running one million simulation trials gave a 95% confidence interval of [6.339, 6.404], which has a width of about six cents. To achieve a confidence interval width of 0.1 cents by simulation would require approximately 4.2 billion simulation trials, requiring approximately 10 days of computing time on an Intel Pentium 133MHz processor. The situation can be improved slightly using the European option value as a control variate. A one million trial simulation gave a 95% confidence interval of [6.342, 6.384], with a width of about four cents. To achieve a width of one-tenth of one cent would still require approximately 1.8 billion simulation trials.[8]

Next we review the standard trinomial approach proposed in Boyle [9] and extended in Kamrad and Ritchken [37], and then discuss our modifications to price discrete barrier options more accurately. The trinomial approach approximates the continuous-time continuous-state lognormal process for S by a tree which has three outcomes at each node. Beginning from state S at time t, the trinomial process moves to state uS, S, or dS at time $t+h$ with probabilities p_u, p, and p_d, respectively. Here, $h = T/n$ represents an arbitrary time step parameter. The asset price multipliers u and d are given by

$$u = e^{\lambda\sigma\sqrt{h}} \quad \text{and} \quad d = e^{-\lambda\sigma\sqrt{h}} = 1/u,$$

where λ is a "stretch" parameter which will be discussed shortly. The probabilities are set to

$$p_u = \frac{1}{2\lambda^2} + \frac{\mu\sqrt{h}}{2\lambda\sigma}, \; p = 1 - \frac{1}{\lambda^2}, \; \text{and} \; p_d = 1 - p_u - p,$$

where $\mu \equiv r - \sigma^2/2$. With these choices of parameters, the mean and variance of the discrete trinomial process match the first two moments of the lognormal asset price process given in (1).

For $\lambda = 1$, $p = 0$, and the trinomial method specializes to the binomial approach of Cox, Ross, and Rubinstein [24]. For $\lambda^* \triangleq \sqrt{3/2} \approx 1.225$, $p = 1/3$ and the probabilities p_u and p_d converge to 1/3 as $h \to 0$. Boyle [9] recommends this particular choice for λ and Omberg [50] provides additional motivation for this choice. The trinomial method requires approximately 50% more computation time compared to the binomial method for the same number of time steps. Broadie and Detemple [14] show that for standard American options without barriers, the trinomial method's increased accuracy (compared to the binomial approach) is very nearly offset by the increased computational time. This indicates that the additional branching does not, by itself, represent a significant improvement. We will see that the benefit from using a trinomial method lies instead in the additional degree of freedom it provides in building a tree.

In the barrier option context, a particular advantage of the trinomial approach is the flexibility provided by the stretch parameter λ. For the binomial method, Boyle and Lau [10] found that the placement of the nodes in the tree relative to the barrier radically affects the convergence of the binomial method for pricing continuous barrier options. If a layer of nodes is just beyond the barrier, then the option prices are significantly closer to the true value. In the binomial method, the proper alignment of the nodes and the barrier only happens for certain values of n which can be computed in advance. For discrete barriers, if we also require the number of time steps n to be divisible by the number of discrete barrier points m, the resulting "good" values of n may be unreasonably large. Ritchken [53] suggests a trinomial approach for pricing continuous barrier options because the stretch parameter λ can be chosen so that the barrier and a layer of nodes coincide for any number of time steps n.[9,10]

To price a discrete barrier option using the trinomial method with approximately n time steps, first adjust n to the nearest integer which is divisible by the number of monitoring points m. In order to choose λ, Ritchken [53] suggests tentatively setting $\lambda = 1$, determining where the barrier falls relative to the node layers, and then increasing λ until a layer of nodes coincides with the barrier. This is the *basic trinomial method* for pricing discrete barrier options. Even though the trinomial method works well for pricing continuous barrier options, we will see that this straightforward extension to pricing discrete barrier options does not work well.

To this basic method, we recommend several improvements. First, the discrete barrier at level H is replaced by a discrete shifted barrier at level $H' = He^{\pm 0.5\lambda^*\sigma\sqrt{h}}$ (with + for an up option and − for a down option). The factor of $1/2$ is, as previously noted, the analog of β_1 for a trinomial random walk. This shift is also analogous to the continuity correction applied to the normal distribution as an approximation to the binomial distribution. See, e.g., Feller [25] for a discussion of this correction.[11] Second, the number of time steps n and the stretch parameter λ are determined as described below. Third, we begin the option price calculation at the $(m-1)^{\text{st}}$ barrier point. At this time the barrier option corresponds to a simple European-type option, which can be priced using the Black-Scholes formula or a simple variation. In order to choose n and λ, we tentatively set $\lambda = \lambda^*$ and define λ_1 to be the smallest value larger than λ so that a layer of nodes coincides with the shifted barrier and λ_2 to be the largest value smaller than λ so that a layer of nodes coincides with the shifted barrier. Now consider various time steps $n = km$, for $k = 0, 1, \ldots, k'$, where where k' corresponds to the first time a layer of nodes crosses the shifted barrier (i.e., the first decrease in $\lambda_1(k)$). From this set, choose the number of time steps which minimizes $|\lambda_i(k) - \lambda^*|$ for $i = 1, 2$ and $k = 0, \ldots, k'$. In short, this procedure produces an n which is divisible by m, a stretch parameter λ which is close to λ^*,[12,13] and a layer of nodes which coincides with the shifted barrier. We refer to the combination of these techniques as the *enhanced trinomial method.*[14]

In order to evaluate these methods, we first compare them in special cases when the discrete barrier option price can be computed exactly. Formulas for discrete barrier options in terms of cumulative multivariate normal distributions are given in Heynan and Kat [35]. The algorithm in Schervish [55, **?**] can be used to evaluate these distributions when the number of monitoring points is small (e.g., five or less). Table 1 shows how the basic and enhanced trinomial methods perform when pricing a particular down-and-out call option. The basic trinomial method has a large 13.7 cent error with $n = 248$ steps and still has an unacceptably large error of 3.7 cents with $n = 8000$ steps. The enhanced trinomial method already achieves penny accuracy at 256 times steps. The monotonicity of the prices as the number of steps increases suggests that Richardson extrapolation (see Geske and Johnson [28]) can be used to further improve convergence. The enhanced trinomial method appears to have linear convergence, so the relevant two-point Richardson extrapolation formula is $P = aP_k + bP_n$, where P_n is the price with n steps, P_k the price with $k > n$ steps, $a = k/(k-n)$ and $b = 1 - a$. When $k = 2n$, the formula is simply

$$P = 2P_{2n} - P_n. \tag{15}$$

The basic trinomial method appears to have slower square-root convergence. In this case, the relevant two-point Richardson extrapolation formula is $P = aP_k + bP_n$, with $k > n$, $a = \sqrt{k}/(\sqrt{k} - \sqrt{n})$ and $b = 1 - a$. When $k = 2n$, the formula is simply

$$P = 3.414\,P_{2n} - 2.414\,P_n. \tag{16}$$

In order to get a more meaningful comparison of the methods, we priced a random sample of 500 options. The distribution of parameters for the test is: $S = 100$, σ is uniform on [0.1,0.6], r is uniform on [0, 0.1], T is uniform on [0.1,

Table 1. The discrete down-and-out call option parameters are: $S = K = 100$, $H = 95$, $T = 0.2$, $\sigma = 0.6$, $r = 0.1$, and $m = 4$. The basic method uses square-root exptrapolation, the enhanced uses linear extrapolation. The true price is 9.49052.

Enhanced trinomial				Basic trinomial			
n	λ	Price	2-pt Extrap.	n	λ	Price	2-pt Extrap.
256	1.22365	9.49690		248	1.00346	9.35373	
504	1.22598	9.49349	9.4899	500	1.06861	9.33210	9.27990
1240	1.22397	9.49189	9.4907	1000	1.00750	9.38524	9.51352
2308	1.22450	9.49124	9.4905	2000	1.06861	9.41166	9.47541
4524	1.22454	9.49090	9.4905	4000	1.00750	9.43805	9.50177
8632	1.22485	9.49072	9.4905	8000	1.00575	9.45352	9.49086

1.0], H is uniform on [70, 95], and K is uniform on $[1.1H, 130]$ (conditional on H), and $m = 3$. The methods were compared based on RMS-relative error over all options in the test set with a true price of at least \$0.50. The results are shown in Fig. 1. The error of the enhanced trinomial method is more than one order of magnitude smaller than the basic method for comparable work. When both methods are improved by two-point Richardson extrapolation, the enhanced method dominates by about two orders of magnitude for comparable work. The linear convergence rate of the enhanced method is also evident in Fig. 1.

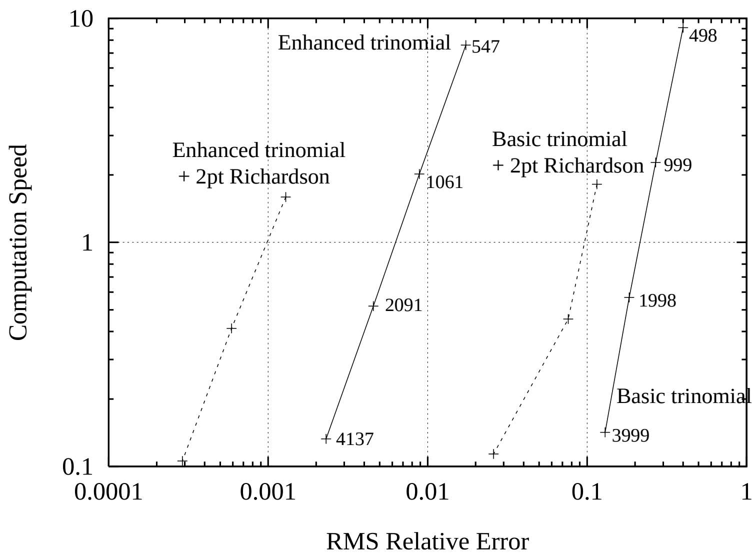

Fig. 1. Comparison of trinomial methods for pricing discrete down-and-out call options with $m = 3$ barrier points. Computation speed is measured in option prices computed per second on a 133MHz Pentium processor. RMS relative error is given in percent. Preferred methods are in the upper-left corner. The numbers next to the methods indicate the average number of time steps over the sample of 500 options. Square-root extrapolation is used with the basic method; linear extrapolation is used with the enhanced method.

Next we compare the enhanced method, with and without extrapolation, and the approximation in Theorem 1 for pricing discrete barrier options with different monitoring frequencies. The results for 500 options with the same parameter distribution are shown in Fig. 2. For daily and weekly monitoring of the barrier the true option price is not known, so we use the price generated by the enhanced trinomial method with 10,000 and 20,000 steps and two-point extrapolation. The enhanced trinomial method performs better with less frequent monitoring of the barrier; equivalently, the error is reduced as the number of trinomial time steps between monitoring points increases. In contrast, the error of the approximation in Theorem 1 decreases quickly as the number of monitoring points increases. Going from weekly to daily monitoring increases the number of barrier points by a factor of 4.8 (=250/52), while the error of the approximation in Theorem 1 decreases by a factor of 23 (from 0.158% to 0.007%). The 0.007% error of the approximation with daily monitoring is roughly equivalent in accuracy to 11,000 steps with the enhanced trinomial method (and is roughly five orders of magnitude faster to compute). Although not illustrated in Fig. 1, the accuracy of the approximation improves very fast as the barrier H moves away from S, even for a small number of monitoring points.

3.2 Lattice methods for discrete lookback options

Lattice methods for pricing lookback options have been proposed in Babbs [6], Cheuk and Vorst [21], and Hull and White [36], and compared in Kat [38]. The previous work has not explored convergence rates nor examined the modifications necessary to incorporate a predetermined min or max into the algorithm. (These require some care.) In this subsection we focus on discrete lookbacks; continuous lookbacks are treated in the next subsection.

The main computational difficulty in pricing lookback options, compared to standard options, is path dependence. The Hull and White [36] algorithm accounts for path dependence by keeping track of the current asset value and the current minimum (or maximum) value achieved. This leads to a very flexible method, but the extra dimension causes a great sacrifice in computational speed. Babbs [6] and Cheuk and Vorst [21] propose a clever transformation that eliminates the added dimension. They do this by constructing a binomial tree for the state variable defined by the ratio of the current minimum (or maximum) price to the current asset price. We follow this approach, but use trinomial trees instead of binomial trees. The added flexibility of the trinomial method becomes important after the initiation of the option contract, when the predetermined max or min may differ from the current asset price.

Briefly, the trinomial method applied to lookback calls builds a tree for the state variable $R = S_+/S$. From state R at time t, the process moves to state Ru, R, or Rd, at time $t+h$. (Exceptions occur at monitoring times for discrete lookbacks or at the boundary $R = 1$ for continuous lookbacks.) Pricing is done using pseudo-probabilities $p_u' = p_u u$, $p' = p$, and $p_d' = p_d d$. Details of the binomial version of this procedure are given in [6], [21], and [38].

The convergence of the trinomial method for pricing discrete lookbacks is illustrated in Table 2. For a small number of monitoring points, the true value of

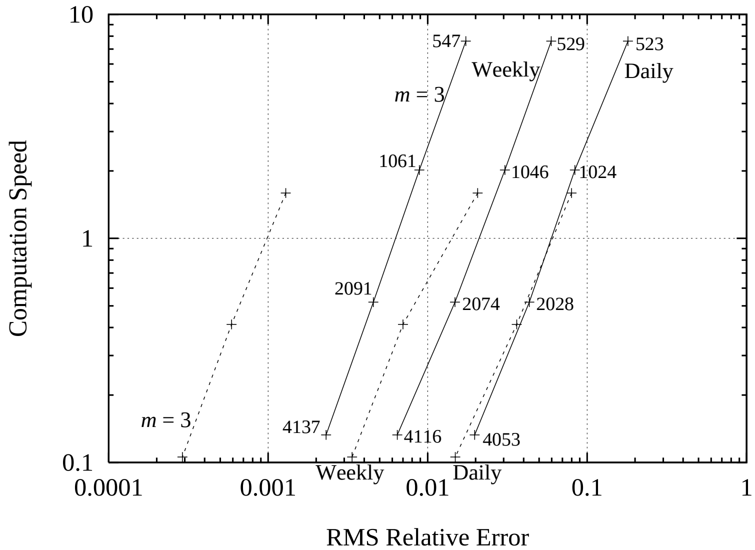

Fig. 2. Performance of the enhanced trinomial method for pricing discrete down-and-out call options with daily, weekly, and three monitoring times of the barrier. The numbers next to the methods indicate the average number of time steps. RMS relative error is given in percent. Preferred methods are in the upper-left corner. *Dashed lines* indicate 2-point linear extrapolation results. The approximation in Theorem 1 has errors of 1.500%, 0.158% and 0.007% for $m = 3$, weekly, and daily monitoring, respectively, and a computation speed of 25,000 options per second. The modified correction partial barrier approximation of [13] has errors of 0.273%, 0.046%, and 0.010% for the respective monitoring frequencies, and a computation speed of 2,000 options per second.

the option can be determined from the formula in [34] and numerically evaluated using the algorithm in [55, 56]. Convergence of the trinomial method appears to be linear in the number of time steps. Two-point Richardson extrapolation using (15) further improves matters.

Next we examine the error of the first-order approximation

$$V_m \approx (V + S_0)e^{-\beta_1 \sigma \sqrt{T/m}} - S_0 \qquad (17)$$

and the second-order approximation in Theorem 2 (omitting the covariance terms). Table 3 shows results for a particular discrete lookback put option as the number of monitoring points increases. Note that monthly, weekly, and daily monitoring correspond to $m = 6$, 26, and 125, respectively. In Table 3, the "true" value is approximated by the trinomial method with 200 and 400 time steps between monitoring times and two-point extrapolation. For example, for $m = 160$ the true price is given by the extrapolation of the 32,000 and 64,000 step trinomial tree values. The first-order approximation exhibits approximately linear convergence: doubling the number of monitoring points cuts the error by about half. The second-order approximation is much more accurate than the first-order approximation and has a faster convergence rate as well. The second-order approximation achieves "penny accuracy" at $m = 40$ monitoring points in this example.

Table 2. Convergence of the trinomial method for pricing a discrete lookback put option. The parameters are: $S_0 = 100$, $r = 0.1$, $\sigma = 0.3$, $T = 0.2$, with the number of monitoring points m varying as indicated. With $m = 4$ monitoring points the true option price is 6.574365. The trinomial method uses $\lambda^* = 1.22474$. Linear extrapolation is used in the 2-pt Extr. columns.

	$m = 4$			$m = 50$	
n	Price	2-pt Extr.	n	Price	2-pt Extr.
200	6.56845		200	8.91972	
400	6.57140	6.57435	400	8.93387	8.94801
800	6.57288	6.57436	800	8.94104	8.94821
1600	6.57362	6.57436	1600	8.94471	8.94838
3200	6.57399	6.57437	3200	8.94657	8.94842
6400	6.57418	6.57437	6400	8.94750	8.94843

Table 3. Performance of the first- and second-order approximations for pricing a discrete lookback put option. The parameters are: $S_0 = 100$, $r = 0.1$, $\sigma = 0.3$, $T = 0.5$, with the number of monitoring points m varying as indicated. The continuously monitored option price is 15.35256.

m	True	1^{st}-Order	2^{nd}-Order	Error1	Error2
5	10.06425	9.15000	10.18203	−0.91424	0.11779
10	11.39775	10.93133	11.44688	−0.46642	0.04913
20	12.44463	12.20843	12.46604	−0.23620	0.02141
40	13.23942	13.12034	13.24909	−0.11908	0.00967
80	13.82950	13.76963	13.83398	−0.05986	0.00449
160	14.26104	14.23100	14.26317	−0.03004	0.00213

In order to more systematically test the convergence of the trinomial method and the error of the first and second-order approximations, we priced a random sample of 500 options. The distribution of parameters for the test is: $S = 100$, σ is uniform on [0.1,0.6], r is uniform on [0, 0.1] and T is uniform on [0.1, 1.0]. Finally, each parameter is selected independently of the others. Error is measured by RMS-relative error over all options in the test set with a true price of at least \$0.50. The results are shown in Fig. 3. For $m = 3$ the true price is computed analytically. For weekly and daily monitoring, the true price is approximated by the trinomial method with 10,000 and 20,000 steps and two-point extrapolation. The error of the trinomial method appears to decrease linearly with the number of time steps. The trinomial error increases with the monitoring frequency, indicating that the number of steps between monitoring points is an important variable. Two-point extrapolation appears to increase the order of convergence. (In fact, with $m = 3$ monitoring points, the error of the trinomial method plus extrapolation is difficult to measure, so that only a single point is shown on the graph.) For daily monitoring, the second-order approximation has an error comparable to a 2,000 step trinomial tree (without extrapolation) and is four orders of magnitude faster to compute.

After the inception of a lookback contract, the value of the option depends on the current asset price as well as the current (i.e., predetermined) maximum or minimum price. In the trinomial lookback method where the state variable is the ratio of the current max (or min) to the current asset price, layers of nodes correspond to the ratios $S_+/S = e^{j\lambda\sigma\sqrt{h}}$ for $j = 0, 1, \ldots$. In order to have this equation hold exactly for arbitrarily specified S and S_+, we first solve

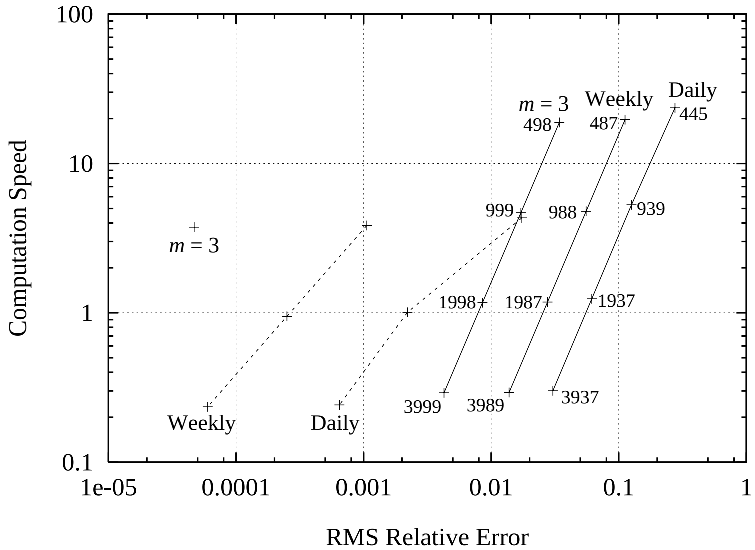

Fig. 3. Performance of the trinomial method to price discrete lookback put options with daily, weekly, and three monitoring times of the maximum. Computation speed is measured in option prices computed per second on a 133MHz Pentium processor. RMS relative error is given in percent. The numbers next to the methods indicate the average number of time steps. *Dashed lines* indicate 2-point linear extrapolation results. The first-order approximation in (17) has errors of 15.620%, 2.014%, and 0.394% for $m = 3$, weekly, and daily monitoring, respectively, and a computation speed of 62,500 options per second. The second-order approximation of Theorem 2 (omitting the covariance term) has errors of 3.475%, 0.228%, and 0.035% for the respective monitoring frequencies, and a computation speed of 45,000 options per second.

$S_+/S = e^{j\lambda^*\sigma\sqrt{h}}$ for j (this corresponds to finding which layer of nodes the current ratio falls between using the stretch parameter λ^*), then we round j to the nearest integer, and finally we adjust λ so that equality holds exactly. In short, we find the λ closest to λ^* so that the current ratio falls exactly on a layer of nodes in the trinomial tree. For a small number of monitoring points we compared analytical option values using the formula in [34] with the values generated from the trinomial method. The trinomial values converged as expected, and the results were comparable to those in Table 2 (and so are not reported).

In order to assess the effectiveness of the approximation in Theorem 3, we priced discrete lookback put options with different monitoring frequencies and two values for the predetermined max. The results are presented in Table 4. The true values were estimated from the trinomial method with 200 and 400 time steps between monitoring times and two-point extrapolation. The results show errors which decrease approximately linearly with the number of monitoring points. Smaller errors are obtained when the predetermined max S_+ is further from the current asset price S. As mentioned earlier, these results are exactly comparable for the corresponding discrete hindsight call option.

Table 4. Performance of the approximation of Theorem 3 for pricing a discrete lookback put option with a predetermined maximum. The parameters are: $S = 100$, $r = 0.1$, $\sigma = 0.3$, $T = 0.5$, with the number of monitoring points m and the predetermined maximum S_+ varying as indicated. The option in the left panel has a continuously monitored option price of 16.84677, the right panel is 21.06454.

	$S_+ = 110$				$S_+ = 120$		
m	True	Approx.	Error	m	True	Approx.	Error
5	13.29955	12.79091	−0.50864	5	18.83723	18.44999	−0.38724
10	14.12285	13.85570	−0.26715	10	19.32291	19.11622	−0.20669
20	14.80601	14.66876	−0.13725	20	19.74330	19.63509	−0.10821
40	15.34459	15.27470	−0.06990	40	20.08297	20.02718	−0.05579
80	15.75452	15.71899	−0.03553	80	20.34598	20.31747	−0.02851
160	16.05908	16.04117	−0.01791	160	20.54389	20.52942	−0.01447

3.3 Lattice methods for continuous lookback options

When the maximum or minimum asset price is monitored continuously, analytical formulas are available for European-style lookback options. However, numerical procedures are necessary to price American lookback options with continuous monitoring. In this subsection we explore various modifications to the discrete trinomial method to price continuous lookbacks. We treat lookbacks with and without predetermined extrema.

Next we test the performance of the standard trinomial procedure of the previous subsection for pricing a continuous lookback call option. The left panel of Table 5 shows the slow convergence of the standard trinomial method. The convergence is approximately square-root in the number of time steps and two-point extrapolation using equation (16) significantly accelerates convergence. (These are observations are consistent with those of Asmussen, Glynn, and Pitman [5] in a simulation setting.) Babbs [6] proposed an alternative approach based on a “reflected barrier” and is related to the numerical scheme independently proposed by Liu [45] and summarized in [5]. Adapting this procedure to the trinomial method gives improved convergence as shown in the middle panel on Table 5.[15] Two-point extrapolation using equation (15) further improves matters. Alternatively, we can adapt the first-order approximation in equation (17) to adjust the output of the discrete trinomial method for approximating the continuous lookback price. More precisely, for lookback calls we define the *corrected trinomial* price V_n' by

$$V_n' = (V_n - S_0)e^{-0.5\lambda\sigma\sqrt{T/n}} + S_0 \ , \tag{18}$$

where V_n is the standard trinomial price using n time steps and λ is the trinomial stretch parameter. For example, using $n = 25$ steps, the price in Table 5 with the standard method is 9.36106, but the corrected value using equation (18) is 10.83824. Complete results using equation (18) and two-point extrapolation using equation (15) are given in the right panel of Table 5. The convergence of Babbs’ “reflected trinomial” method and of the “corrected trinomial” method (with and without extrapolation) are essentially indistinguishable and both are clearly superior to the standard method.[16]

Table 5. Performance of three trinomial methods for pricing a continuously monitored lookback call option. The parameters are: S_0 = 100, r = 0.05, σ = 0.3, T = 0.2, with the number of trinomial steps indicated by n. The left panel is the standard trinomial method; the middle panel is Babbs' reflection method; the right panel does a post-pricing correction to the price. All methods use the trinomial parameter λ^* = 1.22474. The standard method uses square-root exptrapolation, the other two use linear extrapolation. The analytical value of the option is 10.71902.

	Standard		Babbs		Corrected	
n	Trinom	2-pt Extr.	Trinom	2-pt Extr.	Trinom	2-pt Extr.
25	9.36106		10.66695		10.83824	
50	9.73612	10.64150	10.69286	10.71878	10.77882	10.71939
100	10.01267	10.68028	10.70591	10.71896	10.74896	10.71911
200	10.21391	10.69968	10.71246	10.71900	10.73400	10.71904
400	10.35903	10.70937	10.71574	10.71901	10.72651	10.71902
800	10.46306	10.71420	10.71738	10.71902	10.72277	10.71902
1600	10.53733	10.71661	10.71820	10.71902	10.72089	10.71902
3200	10.59019	10.71781	10.71861	10.71902	10.71996	10.71902

Next we digress to illustrate the importance of keeping the trinomial stretch parameter constant when used in conjunction with extrapolation. Table 6 shows results for pricing a continuous lookback call option using Babbs' reflected barrier trinomial method. The left panel holds λ constant at 1.02, the middle panel holds λ constant at 1.42, and in the right panel λ alternates between the two values. When λ is constant at either 1.02 or 1.42, convergence of the trinomial price is monotonic, and extremely rapid convergence is obtained using the two-point extrapolation formula (15). However, when λ alternates the trinomial values are not monotonic, and extrapolation actually slows convergence. That is why λ is kept close to constant in the enhanced trinomial method for pricing discrete barrier options and in the method described next.

Table 6. Impact of the choice of the trinomial λ for pricing a continuously monitored lookback call option. The parameters are: S_0 = 100, r = 0.05, σ = 0.3, T = 1.0, with the number of trinomial steps indicated by n. All trinomial prices use Babbs' reflection method. Linear extrapolation is used in the 2-pt Extr. columns. The left panel has λ = 1.02; the middle panel has λ = 1.42; the right panel has λ alternating between 1.02 and 1.42. The analytical value of the option is 23.78844.

	λ = 1.02		λ = 1.42		λ Alternates	
n	Trinom	2-pt Extr.	Trinom	2-pt Extr.	Trinom	2-pt Extr.
100	23.76633		23.72183		23.76633	
200	23.77737	23.78841	23.75512	23.78841	23.75512	23.74390
400	23.78290	23.78843	23.77177	23.78843	23.78290	23.81068
800	23.78567	23.78843	23.78010	23.78843	23.78010	23.77731
1600	23.78705	23.78844	23.78427	23.78844	23.78705	23.79400
3200	23.78774	23.78844	23.78635	23.78844	23.78635	23.78566

We have seen that the standard trinomial method converges slowly for pricing continuous lookback options, but significant speed-ups are possible by adding a "reflecting barrier" or by "correcting" the price. When there is a predetermined max or min, the situation changes again. In this case, both methods converge slowly, but significant improvement is possible by "correcting" the predetermined min or max before applying either method. For the reflected barrier method, the corrected min S'_- is defined by

$$S'_{-} = (S_{-})e^{2(0.5)\lambda\sigma\sqrt{h}}. \tag{19}$$

The factor of two in (19) appears because of the reflecting boundary, as suggested by Theorem 10.6 of Siegmund [57].

For the reflected barrier method, λ is chosen the same as in the discrete lookback case, i.e., λ is close to λ^* and a layer of nodes falls exactly on the current ratio of the predetermined min to the current asset price. For the "corrected min" method, the min is shifted as in equation (19), with λ initially set at λ^*. But in order to have the current ratio fall exactly on a layer of nodes, λ must be adjusted slightly as previously described. Fortunately, this fixed point problem is easily solved by iterating this procedure, and convergence occurs in a few iterations.

Numerical results for pricing a continuous lookback call option with a predetermined min are given in Table 7. The results show the slow convergence of Babbs' reflected barrier trinomial method, with some improvement using two-point square-root extrapolation. The "corrected min" method converges much faster, and two-point linear extrapolation works even better. For example, using 800 steps, the reflected barrier method has an error of 31 cents, while the corrected min method has achieved penny accuracy. Through equation (13), the results in Table 7 apply as well to the corresponding hindsight put option.

A summary of the key modifications to the trinomial method for pricing the options considered in this section appears in Table 8.

Table 7. Performance of two trinomial methods for pricing a continuously monitored lookback call option. The parameters are: $S_0 = 110$, $S_{-} = 100$, $r = 0.05$, $\sigma = 0.3$, $T = 0.2$, with the number of trinomial steps indicated by n. The left panel is Babbs' reflection method; the right panel is Babbs' method plus a shifting of the min as in equation (19). Both methods use the trinomial parameters close to $\lambda^* = 1.22474$, but with S_0/S_{-} lining up on a row of nodes. The reflected barrier method uses square-root exptrapolation, the corrected min method uses linear extrapolation. The analytical value of the option is 14.45970.

	Babbs		Babbs + corrected min	
n	Trinom	2-pt Extr.	Trinom	2-pt Extr.
25	16.10640		14.23807	
50	15.70332	14.73030	14.34392	14.44977
100	15.28256	14.26683	14.40389	14.46386
200	15.07963	14.58975	14.43086	14.45782
400	14.87165	14.36960	14.44576	14.46066
800	14.76944	14.52269	14.45249	14.45923
1600	14.67506	14.44724	14.45613	14.45977
3200	14.60977	14.45216	14.45793	14.45973
6400	14.56739	14.46508	14.45881	14.45968

4 Conclusions

We have addressed the problem of pricing path-dependent options depending on extremal values of the underlying asset when the extremal values are determined over a discrete set of dates rather than in continuous time. We have introduced

Table 8. Summary of key modifications for trinomial pricing of discrete and continuous barrier and lookback options. Details and several additional enhancements discussed in the text are omitted for brevity. The convergence of each of the methods appears to be $O(1/n)$, so linear extrapolation further enhances the methods. Without the indicated modifications, convergence for discrete barriers and continuous lookbacks appears to be $O(1/\sqrt{n})$.

	Discrete	Continuous
Barrier option	Layer of nodes on shifted barrier	Layer of nodes on original barrier
Lookback option	No adjustment	Shift expected extremum or add reflection (Babbs)
Lookback with predetermined min or max	Layer of nodes on predetermined ratio	Layer of nodes on shifted predetermined ratio; shift expected extremum or add add reflection

correction terms that allow the discrete versions to be quite accurately using formulas for the continuous versions. The correction terms involve shifting a barrier or a value of the underlying. We have also developed specially tailored numerical methods to evaluate the approximations and to price the options to even higher accuracy. These methods entail shifting values in a trinomial tree, analogous to the shifts used in the approximations.

Appendix: Proofs

A.1. Proof of Theorem 2

In proving Theorem 2, we detail the case of a lookback put. The argument for calls is symmetric. Since $V = e^{-rT}E[e^M] - S_0$ and $V_m = e^{-rT}E[e^{\tilde{M}_m}] - S_0$, (8) is equivalent to the claim that

$$E[e^{\tilde{M}_m}] = E[e^M]\left(1 - \frac{\beta_1\sigma\sqrt{T}}{\sqrt{m}} + \frac{\gamma_+\sqrt{T} + \frac{1}{2}\beta_2\sigma^2 T}{m}\right) + \mathrm{Cov}[e^M, \tilde{M}_m - M] + o(1/m); \qquad (20)$$

the proof of (20) relies on three lemmas.

Lemma 1 *For any function f twice continuously differentiable on* $[0,1]$ *we have*

$$\int_0^1 \frac{f(x)}{\sqrt{x}}\,dx = \frac{1}{m}\sum_{k=1}^m \frac{f(k/m)}{\sqrt{k/m}} - \zeta(1/2)f(0)\frac{1}{\sqrt{m}} - \frac{1}{2}f(1)\frac{1}{m} + O(m^{-3/2}).$$

Proof. Define $g(t) = [f(t^2) - f(0)]/t$ for $0 < t \le 1$ and $g(0) = \lim_{t\downarrow 0} g(t) = 0$. Then g is twice continuously differentiable on $[0,1]$, and therefore by Lemma 5(b) of Asmussen, Glynn, and Pitman [5],

$$\int_0^1 g(\sqrt{x})\,dx = \frac{1}{m}\sum_{k=1}^m g(\sqrt{k/m}) + \frac{g(0)-g(1)}{2m} + O(m^{-3/2}).$$

Substituting for g we get

$$\int_0^1 \frac{f(x)}{\sqrt{x}}\,dx - \int_0^1 \frac{f(0)}{\sqrt{x}}\,dx = \frac{1}{m}\sum_{k=1}^{m} \frac{f(k/m)-f(0)}{\sqrt{k/m}} + \frac{f(0)-f(1)}{2m} + O(m^{-3/2}),$$

and thus

$$\int_0^1 \frac{f(x)}{\sqrt{x}}\,dx = \frac{1}{m}\sum_{k=1}^{m} \frac{f(k/m)}{\sqrt{k/m}} + f(0)\left(\int_0^1 \frac{1}{\sqrt{x}}\,dx - \frac{1}{m}\sum_{k=1}^{m}\frac{1}{\sqrt{k/m}}\right) + \frac{f(0)-f(1)}{2m} + O(m^{-3/2}).$$

It follows from Knopp [40, p.538] that

$$\left(\int_0^1 \frac{1}{\sqrt{x}}\,dx - \frac{1}{m}\sum_{k=1}^{m}\frac{1}{\sqrt{k/m}} + \frac{\zeta(1/2)}{\sqrt{m}} + \frac{1}{2m}\right) = O(m^{-2}),$$

so

$$\begin{aligned}\int_0^1 \frac{f(x)}{\sqrt{x}}\,dx &= \frac{1}{m}\sum_{k=1}^{m} \frac{f(k/m)}{\sqrt{k/m}} + f(0)\left(\frac{-\zeta(1/2)}{\sqrt{m}} - \frac{1}{2m} + O(m^{-2})\right) \\ &\quad + \frac{f(0)-f(1)}{2m} + O(m^{-3/2}) \\ &= \frac{1}{m}\sum_{k=1}^{m} \frac{f(k/m)}{\sqrt{k/m}} - \zeta(1/2)f(0)\frac{1}{\sqrt{m}} - \frac{1}{2}f(1)\frac{1}{m} + O(m^{-3/2}).\end{aligned}$$

□

Lemma 2 *As* $m \to \infty$,

$$E[e^{\tilde{M}_m}] = E[e^M]\left(1 + E[\tilde{M}_m - M] + \frac{\beta_2\sigma^2 T}{2m}\right) + \mathrm{Cov}[e^M, \tilde{M}_m - M] + o(1/m).$$

Proof. By Taylor expansion we get

$$e^{\tilde{M}_m} - e^M = e^M(\tilde{M}_m - M) + \frac{1}{2}e^{\bar{M}_m}(\tilde{M}_m - M)^2,$$

for some $\bar{M}_m$ between $\tilde{M}_m$ and M. As m increases, $\tilde{M}_m \to M$, and from Theorem 1 of [5] we know that $\sqrt{m}(M - \tilde{M}_m)$ converges in distribution to a random variable $W\sigma\sqrt{T}$ independent of M. Thus,

$$m(e^{\tilde{M}_m} - e^M - e^M(\tilde{M}_m - M)) \tag{21}$$

converges in distribution to $\frac{1}{2}e^M W^2\sigma^2 T$. If $e^{\bar{M}_m}\cdot m(\tilde{M}_m - M)^2$ is uniformly integrable, we can interchange limit and expectation to conclude that

$$m(E[e^{\tilde{M}_m}] - E[e^M] - E[e^M(\tilde{M}_m - M)]) \to \frac{1}{2}E[e^M]\beta_2\sigma^2 T; \tag{22}$$

i.e., that

$$
\begin{aligned}
E[e^{\tilde{M}_m}] - E[e^M] &= E[e^M(\tilde{M}_m - M)] + \frac{1}{2m}E[e^M]\beta_2\sigma^2 T + o(1/m) \\
&= E[e^M]E[\tilde{M}_m - M] + \mathrm{Cov}[e^M, \tilde{M}_m - M] \\
&\quad + \frac{1}{2m}E[e^M]\beta_2\sigma^2 T + o(1/m),
\end{aligned}
$$

which is equivalent to the statement in the lemma.

It remains to justify the interchange of limit and expectation used to go from (21) to (22). For this, we use the Cauchy-Schwarz inequality to get

$$
E\left[\left(e^{\bar{M}_m} \cdot m(\tilde{M}_m - M)^2\right)^2\right] \leq \sqrt{E[e^{4\bar{M}_m}]} \cdot \sqrt{E[(m(\tilde{M}_m - M)^2)^4]}.
$$

But

$$
\sup_{m \geq 1} E[e^{p\bar{M}_m}] \leq E[e^{p \max_{0 \leq t \leq T} B_t}] < \infty,
$$

for any $p > 0$, and

$$
\sup_{m \geq 1} E|\sqrt{m}(\tilde{M}_m - M)|^p < \infty
$$

for any $p > 0$, by Lemma 6 of [5]. These conditions imply uniform integrability and thus justify the interchange of limit and expectation. □

Lemma 3 *As* $m \to \infty$,

$$
E[\tilde{M}_m - M] = -\frac{\beta_1\sigma\sqrt{T}}{\sqrt{m}} + \frac{\gamma_+\sqrt{T}}{m} + O(m^{-3/2}).
$$

Proof. By Spitzer's identity (see, e.g., Asmussen [4], especially p.174 and p.177) and the argument in the proof of Theorem 2 of [5], we have

$$
E[\tilde{M}_m - M] = \frac{1}{m}\sum_{k=1}^{m} g(k/n) - \int_0^1 g(x)\,dx, \tag{23}
$$

where

$$
\begin{aligned}
g(x) &= \mu T\Phi(\mu\sqrt{xT}/\sigma) + \sigma\sqrt{T}e^{-\frac{\mu^2 xT}{2\sigma^2}}\frac{1}{\sqrt{2\pi x}} \\
&\equiv g_1(x) + \frac{g_2(x)}{\sqrt{x}}.
\end{aligned}
$$

By Lemma 5(b) of [5], the contribution of g_1 to (23) is

$$
\frac{g_1(1) - g_1(0)}{2m} + O(m^{-3/2}) = \frac{1}{2m}\left(\mu T\Phi(\mu\sqrt{T}/\sigma) - \frac{\mu T}{2}\right) + O(m^{-3/2}),
$$

and by Lemma 1 the second term contributes

$$
\frac{\zeta(1/2)g_2(0)}{\sqrt{m}} + \frac{g_2(1)}{2m} + O(m^{-3/2}) = \frac{-\beta_1\sigma\sqrt{T}}{\sqrt{m}} + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\frac{e^{-\frac{\mu^2 T}{2\sigma^2}}}{2m} + O(m^{-3/2}).
$$

The lemma now follows from combining the contributions of the two terms. □

We can now complete the proof of Theorem 2. By substituting the expression in Lemma 3 for $E[\tilde{M}_n - M]$ in the expansion of Lemma 2, we get (20), as required.

A.2. Proof of Theorems 3 and 4

We need the following preliminary result:

Lemma 4 *For any $x > 1$,*

$$E[(e^{\tilde{M}_m} - x)^+] = e^{-\beta_1\sigma\sqrt{\Delta t}}E[(e^M - e^{\beta_1\sigma\sqrt{\Delta t}}x)^+] + o(1/\sqrt{m}).$$

Proof. For any random variable X and event F, the notation $E[X;F]$ means $E[X\mathbf{1}_F]$, with $\mathbf{1}_F$ the indicator of F. Now we have

$$\begin{aligned} &E[(e^{\tilde{M}_m} - x)^+] \\ &\quad = E[(e^M - x)^+] - E[e^M - e^{\tilde{M}_m}; e^{\tilde{M}_m} > x] - E[e^M - x; e^{\tilde{M}_m} \le x < e^M] \\ &\quad \equiv A - B - C. \end{aligned} \tag{24}$$

We analyze these terms in reverse order. First observe that

$$C \le E[e^M - e^{\tilde{M}_m}; e^{\tilde{M}_m} \le x < e^M].$$

From the uniform integrability argument in Lemma 3 we know that $E[\sqrt{m}(e^M - e^{\tilde{M}_m})]$ converges as $m \to \infty$. Furthermore, $P(e^{\tilde{M}_m} \le x < e^M) \to 0$, so $\sqrt{m}E[e^M - e^{\tilde{M}_m}; e^{\tilde{M}_m} \le x < e^M] \to 0$ by the dominated convergence theorem; i.e., C is $o(1/\sqrt{m})$. Next,

$$\begin{aligned} B &= E[e^M - e^{\tilde{M}_m}; e^M > x] - E[e^M - e^{\tilde{M}_m}; e^{\tilde{M}_m} \le x < e^M] \\ &= E[e^M - e^{\tilde{M}_m}; e^M > x] + o(1/\sqrt{m}), \quad \text{by the argument used for } C \\ &= E[e^M\sigma\beta_1\sqrt{\Delta t}; e^M > x] + o(1/\sqrt{m}), \quad \text{by the argument used in Lemma 2.} \end{aligned}$$

Thus, (24) becomes

$$\begin{aligned} E[(e^{\tilde{M}_m} - x)^+] &= E[(e^M - x)^+] - E[e^M\sigma\beta_1\sqrt{\Delta t}; e^M > x] + o(1/\sqrt{m}) \\ &= E[e^M(1 - \sigma\beta_1\sqrt{\Delta t}) - x; e^M > x] + o(1/\sqrt{m}) \\ &= E[e^{M - \sigma\beta_1\sqrt{\Delta t}} - x; e^M > x] + o(1/\sqrt{m}) \\ &= e^{-\sigma\beta_1\sqrt{\Delta t}}E[(e^M - e^{\sigma\beta_1\sqrt{\Delta t}}x)^+] \\ &\quad -E[e^M - e^{\sigma\beta_1\sqrt{\Delta t}}x; x \le e^M < xe^{\sigma\beta_1\sqrt{\Delta t}}] + o(1/\sqrt{m}), \end{aligned}$$

and

$$\begin{aligned} &|E[e^M - e^{\sigma\beta_1\sqrt{\Delta t}}x; x \le e^M < xe^{\sigma\beta_1\sqrt{\Delta t}}]| \\ &\quad \le x(e^{\sigma\beta_1\sqrt{\Delta t}} - 1)P(\log x \le M < \log x + \sigma\beta_1\sqrt{\Delta t}) \\ &\quad = O(1/\sqrt{m}) \cdot O(1/\sqrt{m}) = o(1/\sqrt{m}), \end{aligned}$$

which concludes the proof. □

We can now prove Theorem 3. We detail the case of a put; a call works similarly. Starting from the discrete price we have

$$
\begin{aligned}
V_m(S_+) &= e^{-r(T-t)}E[\max(S_+, e^{\tilde{M}_m}) - S_T] \\
&= e^{-r(T-t)}(S_+ + E[(e^{\tilde{M}_m} - S_+)^+]) - S_t \\
&= e^{-r(T-t)}(S_+ + e^{-\beta_1\sigma\sqrt{\Delta t}}E[(e^M - e^{\beta_1\sigma\sqrt{\Delta t}}S_+)^+]) - S_t + o(1/\sqrt{m}) \\
&= e^{-r(T-t)}e^{-\beta_1\sigma\sqrt{\Delta t}}(e^{\beta_1\sigma\sqrt{\Delta t}}S_+ + E[(e^M - e^{\beta_1\sigma\sqrt{\Delta t}}S_+)^+]) \\
&\qquad -S_t + o(1/\sqrt{m}) \\
&= e^{-r(T-t)}e^{-\beta_1\sigma\sqrt{\Delta t}}(E[\max(e^M, e^{\beta_1\sigma\sqrt{\Delta t}}S_+) - S_T]) \\
&\qquad +(e^{-\beta_1\sigma\sqrt{\Delta t}} - 1)S_t + o(1/\sqrt{m}) \\
&= e^{-\beta_1\sigma\sqrt{\Delta t}}V(e^{\beta_1\sigma\sqrt{\Delta t}}S_+) + (e^{-\beta_1\sigma\sqrt{\Delta t}} - 1)S_t + o(1/\sqrt{m}).
\end{aligned}
$$

The third equality – the key step – is just Lemma 4. The proof of Theorem 4 follows essentially the same steps. Alternatively, Theorem 4 can be derived from Theorem 3 by using the relation in (13).

Acknowledgement. This research was supported in part by National Science Foundation grant DMI-94-57189.

Endnotes

[1] Black and Cox [7, p.354] and Leland [42, p.1221] briefly touch on the distinction between discrete and continuous monitoring.

[2] Ait-Sahlia [2] builds on Chernoff [17] to develop a continuity correction for the exercise boundary of American options.

[3] A continuous dividend yield δ is easily accommodated by setting $\nu = r - \delta$; to lighten notation, we do not treat this case explicitly.

[4] Conze and Viswanathan [23] call these partial lookbacks; they allow the payoff on ordinary lookbacks to depend on the minimum or maximum of the underlying over a subinterval of the life of the option.

[5] We recently became aware of Chuang [22] which includes a remark (p.86) independently suggesting the possibility of using Siegmund's correction for discrete barrier options. However, the suggestion is not pursued there.

[6] There is a fortuitous element to the consistency of the two approaches. The constant β_1 arises as the mean of two limiting distributions: the overshoot distribution for a normal random walk (see (4)) and the difference between the discrete and continuous maximum along a Brownian path (see (7)). Though they have the same mean, the two distributions are different; in particular, β_2 is not the second moment of the limiting overshoot. So, although the first-order corrections for barrier and lookback options appear to be quite analogous, they arise in different ways. For an interesting discussion of β_1 and related quantities, see Chang and Peres [19].

[7] Of course significant improvements may be possible over straightforward implementations. For some results along these lines, see Andersen and Brotherton-Ratcliffe [3].

[8] We also tested low discrepancy (quasi-Monte Carlo) methods using Faure and Sobol' sequences; see Boyle, Broadie, and Glasserman [11] for an introduction and references on low discrepancy methods. These methods do not provide simple error estimates and, in this application, converged erratically. Using the Faure sequence, a rough estimate of the uncertainty after 10 million points (which took 70 minutes on an Intel Pentium 133MHz) is 0.5 cents. Results using Sobol' points were generally similar. Low discrepancy methods do not, therefore, solve the problem.

[9] More precisely, a value of λ between one and two can be so chosen as long as the barrier H is not too close to the initial asset price S. For example, if H lies just below S, then the barrier will fall between S and dS. Then the only feasible stretch parameter will be less than one, which leads to negative probabilities. In this case, the number of times steps can be increased until Sd coincides with H. Alternatively, Cheuk and Vorst [20] propose a modification of the trinomial method where the nodes are shifted to line up with the barrier. Their method works well even if the barrier is very close to the initial asset price.

[10] See Brenner [12] and Li and Lu [44] for related trinomial approaches.

[11] Cheuk and Vorst [20] independently proposed an analogous procedure where the tree is constructed so that the barrier falls between a layer of nodes. They chose to set the tree so that the barrier falls exactly in the middle of a layer of nodes based on numerical experimentation.

[12] Once n and λ have been determined, the process could be repeated so that the shifted barrier H' exactly equals $He^{\pm 0.5\lambda\sigma\sqrt{h}}$. We did not implement this slight improvement.

[13] While it is desirable to have λ close to λ^*, it is more important that λ be nearly constant as n is varied for extrapolation purposes. This point is illustrated later in Table 6 in a slightly different context.

[14] This method was used to produce the accurate values in [13] for comparison with the approximation developed there, though the method itself was not described.

[15] At the reflecting boundary $R = 1$, we take the pseudo-probability of an upmove to be $p'_u + p'_d$ and the pseudo-probability of a horizontal move to be p'.

[16] Babbs' reflection method and our correction can both be interpreted as accounting for the fact that reflected Brownian motion does not spend time at the origin, though a reflected random walk does.

References

1. Abramowitz, M., Stegun, I.: Handbook of mathematical functions, 10th edn. New York: Dover 1972
2. Ait-Sahlia, F.: Optimal stopping and weak convergence methods for some problems in financial economics, Doctoral dissertation, Department of Operations Research, Stanford University, Stanford, CA (1995)
3. Andersen, L., Brotherton-Ratcliffe, R.: Exact exotics. Risk **9**, 85–89 (1996)
4. Asmussen, S.: Applied probability and queues. Chichester, England: Wiley 1987
5. Asmussen, S., Glynn, P., Pitman, J.: Discretization error in simulation of one-dimensional reflecting brownian motion. Ann. Appl. Probab. **5**, 875–896 (1995)
6. Babbs, S.: Binomial valuation of lookback options, working paper. Midland Global Markets, London (1992)
7. Black, F., Cox, J.: Valuing corporate debt: Some effects of bond indenture provisions. J. Finance **31**, 351–367 (1976)
8. Black, F., Scholes, M.: The pricing of options and corporate liabilities. J. Polit. Econ. **81**, 637–654 (1973)
9. Boyle, P.: A lattice framework for option pricing with two state variables. J. Financial and Quantitat. Anal. **23**, 1–12 (1988)
10. Boyle, P.P., Lau, S.H.: Bumping up against the barrier with the binomial method. J. Derivat. **1**, 6–14 (Summer, 1994)
11. Boyle, P., Broadie, M., Glasserman P.: Monte carlo methods for security pricing. J. Econ. Dynamics Control **21**, 1267–1321 (1997)
12. Brenner, R.: Volatility is not constant. In: Nelken, I. (ed.) Handbook of Exotic Options: Instruments, Analysis, and Applications. Chicago, Illinois: Irwin 1995, pp. 293–315
13. Broadie, M., Glasserman, P., S. Kou.: A continuity correction for discrete barrier Options. Math. Finance **7**, 325–349 (1997)
14. Broadie, M., Detemple, J.: American option valuation: new bounds, approximations, and a comparison of existing methods. Rev. Financial Studies **9**, 1211–1250 (1996)

15. Calvin, J.M.: Average performance of passive algorithms for global optimization. J. Math. Anal. Appl. **191**, 608–617 (1995)
16. Carr, P.: Two extensions to barrier option valuation. Appl. Math. Finance **2**, 173–209 (1995)
17. Chernoff, H.: Sequential tests for the mean of a normal distribution IV. Ann. Math. Statist. **36**, 55–68 (1965)
18. Chance, D.M.: The pricing and hedging of limited exercise caps and spreads. J. Financial Res. **17**, 561–584 (1995)
19. Chang, J.T., Peres, Y.: Ladder heights, gaussian random walks and the Riemann zeta function. Working paper, Yale University, 1995
20. Cheuk, T., Vorst, T.: Complex barrier options, J. Derivat. **4**, 8–22 (Fall, 1996)
21. Cheuk, T., Vorst, T.: Currency lookback options and the observation frequency: A binomial approach. J. Int. Money Finance **16**, 173–187 (1997)
22. Chuang, C.-S.: Joint distribution of brownian motion and its maximum, with a generalization to correlated BM and applications to barrier options. Statist. Probab. Lett. **28**, 81–90 (1996)
23. Conze, A., Viswanathan, R.: Path dependent options: The case of lookback options. J. Finance **46**, 1893–1907 (1991)
24. Cox, J.C., Ross, S.A., Rubinstein, M.: Option pricing: A simplified approach. J. Financial Econ. **7**, 229–263 (1997)
25. Feller, W.: An introduction to probability theory and its applications. New York: Wiley 1968
26. Flesaker, B.: The design and valuation of capped stock index options. Working paper, Department of Finance, University of Illinois, Champaign, IL, 1992
27. Garman, M.: Recollection in tranquillity. Risk **2**, 16–18 (March 1989). (Reprinted in: From black-scholes to black holes: New frontiers in options. Risk/Finex 1992, 171–175)
28. Geske, R., Johnson, H.E.: The American put options valued analytically. J. Finance **39**, 1511–1524 (1984)
29. Goldman, M.B., Sosin, H., Gatto, M.: Path dependent options: Buy at the low, sell at the high. J. Finance **34**, 1111–1127 (1979)
30. Goldman, M.B., Sosin, H., Shepp, L.: On contingent claims that insure ex-post optimal stock market timing. J. Finance **34**, 401–414 (1979)
31. Heynen, R.C., Kat, H.M.: Crossing barriers. Risk **7**, 46–49 (June, 1994), Correction Risk **8**, 18 (March, 1995) (Reprinted in: R. Jarrow (ed.) Over the rainbow: developments in exotic options and complex swaps. Risk/FCMC 179–182)
32. Heynen, R.C., Kat, H.M.: partial barrier options. J. Financial Eng. **3**, 253–274 (1994)
33. Heynen, R.C., Kat, H.M.: Selective memory. Risk **7**, 73–76 (November, 1994). (Reprinted in: R. Jarrow (ed.) Over the rainbow: developments in exotic options and complex swaps. Risk/FCMC, 197–202.)
34. Heynen, R.C., Kat, H.M.: Lookback options with discrete and partial monitoring of the underlying price. Appl. Math. Finance **2**, 273–284 (1995)
35. Heynen, R.C., Kat, H.M.: Discrete partial barrier options with a moving barrier. Working paper, First National Bank of Chicago, Tokyo, Japan, 1995
36. Hull, J., White, A.: Efficient procedures for valuing european and american path-dependent options. J. Derivat. **1**, 21–31 (Fall, 1993)
37. Kamrad, B., Ritchken, P.: Multinomial approximating models for options with k state variables. Manag. Sci. **37**, 1640–1652 (1991)
38. Kat, H.M.: Pricing lookback options using binomial trees: An evaluation. J. Financial Eng. **4**, 375–397 (1995)
39. Kat, H.M., Verdonk, L.: Tree surgery. Risk **8**, 53–56 (February, 1995)
40. Knopp, K.: Theory and applications of infinite series. New York: Dover 1990
41. Kunitomo, N., Ikeda, M.: Pricing options with curved boundaries. Math. Finance **2**, 275–298 (1992)
42. Leland, H.E.: Corporate debt value, bond covenants, and optimal capital structure. J. Finance **49**, 1213–1252 (1994)
43. Levy, E., Mantion, F.: Discrete by nature. Risk **10**, 74–75 (January, 1997)
44. Li, A., Lu, S.: Lattice methods for barrier options: A probabilistic approach. Working paper, First National Bank of Chicago, 1995
45. Liu, Y.: Numerical approaches to reflected diffusion processes. Working paper, Purdue University, 1993

46. Longstaff, F.A.: How much can marketability affect security values? J. Finance **50**, 1767–1774 (1995)
47. Longstaff, F.A., Schwartz, E.S.: A simple approach to valuing risky fixed and floating rate debt. J. Finance **50**, 789–819 (1995)
48. Merton, R.C.: Theory of rational option pricing. Bell J. Econ. Manag. Sci. **4**, 141–183 (1973)
49. Merton, R.C.: On the pricing of corporate debt: The risk structure of interest rates. J. Finance **29**, 449–469 (1974)
50. Omberg, E.: Efficient discrete time jump process models in option pricing. J. Financial Quantitat. Anal. **23**, 161–174 (1988)
51. Rich, D.: The mathematical foundations of barrier option pricing theory. Adv. Futures Options Res. **7**, 267–312 (1994)
52. Rich, D.: The valuation of black-scholes options subject to intertemporal default risk. Rev. Derivat. Res. **1**, 25–59 (1996)
53. Ritchken, P.: On pricing barrier options. J. Derivat. **3**, 19–28 (Winter, 1995)
54. Rubinstein, M., Reiner, E.: Breaking down the barriers. Risk **4**, 28–35 (September, 1991)
55. Schervish, M.: Algorithm AS195: Multivariate normal probabilities with error bound. Appl. Statist. **33**, 81–94 (1984)
56. Schervish, M.: Correction to multivariate normal probabilities with error bound. Appl. Statist. **34**, 103–104 (1984)
57. Siegmund, D.: Sequential analysis: Tests and confidence intervals. Berlin, Heidelberg, New York: Springer 1985
58. Siegmund, D., Yuh, Y-S.: Brownian approximations for first passage probabilities. Z. Wahrscheinlichkeitstheorie verw. Gebiete **59**, 239–248 (1982)