# PROFIT SHARING IN HEDGE FUNDS

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**Abstract**

In a new scheme for hedge fund managerial compensation known as the first-loss scheme, a fund manager uses her investment in the fund to cover any fund losses first; by contrast, in the traditional scheme currently used in most U.S. funds, the manager does not cover investors' losses in the fund. We propose a framework based on cumulative prospect theory to compute and compare the trading strategies, fund risk, and managers' and investors' utilities in these two schemes analytically. The model is calibrated to the historical attrition rates of U.S. hedge funds. We find that with reasonable parameter values both fund managers' and investors' utilities can be improved and fund risk can be reduced simultaneously by replacing the traditional scheme (with 10% internal capital and 20% performance fee) with a first-loss scheme (with 10% first-loss capital and 30% performance fee). When the performance fee in the first-loss scheme is 40% (a current market practice), however, such substitution renders investors worse off.

KEY WORDS: cumulative prospect theory, portfolio selection, hedge funds, managerial incentive, first-loss scheme

## 1. INTRODUCTION

Traditionally, hedge fund managers and investors are responsible for the losses in their own capital invested in the fund, but the managers take a portion (about 20%) of the investors' profits as a performance fee. Although the connection between risk taking and performance fees in hedge funds has been extensively studied (e.g., Carpenter 2000, Hodder and Jackwerth 2007, Kouwenberg and Ziemba 2007, and Bichuch and Sturm 2014), there has been little investigation of profit sharing in hedge funds between the investors and fund managers. More precisely, how do we compare different profit sharing schemes?[1]

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[1] Investors choose hedge funds for various reasons, such as increasing returns, reducing risk, and increasing diversification; see for instance McCrary (2004, Chapter 1) and Lhabitant (2011, Chapter 23). In this paper, we do not test the validity of these reasons. Instead, we focus on how investors and managers share profit and risk in hedge funds.

A scheme known as the *first-loss* scheme has recently become popular in China and is also emerging in the United States.[2] Under this scheme, fund managers typically put up about 10% of the fund capital from their own money as "first-loss" capital (or deposit). The losses from the fund will be offset by the first-loss capital first before investors take a hit. As compensation for covering the losses first, managers can take a higher percentage of profits, typically 40%. A natural question arises as to which scheme is better, the traditional one or the first-loss scheme. A further issue is whether it is possible to improve one or both of these schemes.

In this paper, we provide an analytical framework to compare the traditional and first-loss schemes. The framework consists of two parts. The first uses cumulative prospect theory (Tversky and Kahneman, 1992) to compare the two schemes analytically. The second uses the historical attrition rates of U.S. hedge funds, which are between 10% to 20%, to calibrate utility parameters in the model. There are three reasons why cumulative prospect theory (CPT) is useful for our problem: 1) CPT is descriptively better than expected utility theory (EUT) and has been applied in many areas of finance; see e.g., Barberis and Thaler (2003) and see Section 2.3 below; 2) the use of CPT leads to analytical tractability; and 3) the model based on CPT seems to fit the data better than that based on EUT. Indeed, within the framework of EUT, Carpenter (2000) shows that the risk of a hedge fund is decreasing with respect to the performance fee when the manager has a power utility function, while Kouwenberg and Ziemba (2007) show that the opposite is true if the manager's preferences are modeled by CPT. The empirical study in Kouwenberg and Ziemba (2007) supports the latter conclusion.

The contribution of this paper is threefold. First, we obtain the optimal trading strategies in closed form in both schemes. In this regard, we find that the optimal strategies depend only on a loss-gain ratio that measures the ratio of loss impediment and gain incentive for the manager, regardless of the scheme being used. In addition, we show that in our setting one can compare the fund risk (e.g., using either asset volatility or value-at-risk) in the two schemes easily by comparing the loss-gain ratios. Finally, we obtain closed-form formulae for the utilities of both managers and investors when the managers use the optimal strategies.

Second, we use the historical attrition rates of U.S. hedge funds to calibrate a critical parameter in CPT preferences: the diminishing sensitivity of the utility function. Although this parameter has been widely estimated using experimental data, it has rarely been estimated using financial data. We find that the diminishing sensitivity estimated using the hedge fund data is smaller than the experimental estimates in the literature.

Finally, we compare the traditional and first-loss schemes from the perspectives of fund risk, managers, and investors. Using the ranges of the calibrated utility parameters and reasonable market parameter values, we find that in most cases switching from the traditional to the first-loss scheme can reduce the fund risk and improve the well being of both managers and investors simultaneously, if the incentive rate in the first-loss scheme is set at about 30%. However, our numerical result also suggests that investors will be worse off with the switch if the incentive rate in the first-loss scheme is 40%, which is the current market practice.

A closely related paper is Chang, Cvitanić, and Zhou (2015), who study a principal-agent

[2]See `http://www.marketwatch.com/story/more-hedge-funds-lured-to-new-source-of-capital-2011-05-23?pagenumber=1` for a report from CBS Marketwatch on May 23, 2011 regarding the implementation of the first-loss scheme in some U.S. funds; see the citations in the online Baidu encyclopedia `http://baike.baidu.com/view/134402.htm` on its use in privately held funds in China.

problem with moral hazard in which the agent's preferences are modeled by CPT. In their paper, asset volatility is controlled by the firm's principal: its shareholders. In our model, however, the investment portfolio, which determines the volatility of the fund's asset value, is controlled by the agent: the fund managers. To our knowledge, the only other paper that has used CPT in the study of hedge fund investment is Kouwenberg and Ziemba (2007), which focuses on fund risk, whereas we focus on profit sharing. Also, they neither computed managers' and investors' utilities explicitly nor investigated the first-loss scheme.

By contrast, studies of hedge fund risk taking using an EUT framework are numerous. Bichuch and Sturm (2014), for instance, consider hedge fund investment in a general setting for a risk averse manager with expected utility (EU) preferences under the traditional scheme, whereas Guasoni and Obłój (2016) find the optimal trading strategy of a risk averse manager with a power utility function under the traditional scheme with high-water mark provisions. However, these papers do not consider profit sharing, which is the main focus of our work. A detailed comparison of our model setting and theirs is provided in Section 2.4.

To further illustrate the difference in optimal trading strategies with CPT and EU managers, we compute the optimal asset value of the fund when the manager has EU preferences. We find that with CPT managers, the loss distribution of the fund is binary: the fund either has no loss or is liquidated. With EU managers, however, the fund can suffer a nonzero loss without being liquidated. Furthermore, with CPT managers, the optimal asset value of the fund depends only on the loss-gain ratio, whether the traditional or the first-loss scheme is used. With EU managers, however, the loss-gain ratio is not the only determinant of the fund's asset value. Moreover, with EU managers, the fund can suffer small losses but not large ones without being liquidated under the first-loss scheme, but the opposite is true under the traditional scheme.

Our study of profit sharing in hedge funds takes a different approach from the classical risk sharing literature. In this literature, agents' preferences are assumed to be known and optimal risk sharing is found among a feasible set of risk sharing contracts. Therefore, the optimal risk sharing depends on the agents' preferences and on other model parameters, which are difficult to estimate. In our problem, we compare two profit sharing schemes in hedge funds and show that the first-loss scheme (with a suitable incentive rate) is better than the traditional one across a wide range of model parameters. Therefore, our approach bypasses the difficulty of estimating model parameters. Furthermore, given the model parameters, we are also able to find the optimal first-loss scheme, i.e., the optimal incentive rate, from investors' and managers' perspectives, respectively; see Section 6.

The remainder of the paper is organized as follows. We propose our model in Section 2 and solve the corresponding optimization problems explicitly in Section 3. In Section 4, we compare the traditional and first-loss schemes theoretically. Section 5 is devoted to calibrating the parameters in CPT to hedge fund data. In Section 6, we compare the two schemes numerically and propose a first-loss scheme with 30% incentive rate. In Section 7, we compare the fund values when the manager has CPT and EU preferences, respectively. Finally, Section 8 concludes. All proofs are in the appendix.

## 2. BASIC SETTING

### 2.1. Hedge Fund Investment

We assume that the fund manager invests in two assets, a risk-free asset and a risky asset, whose price dynamics are given, respectively, by

$$S_{0,t} = e^{rt},\ t \geq 0, \quad dS_{1,t} = \mu S_{1,t} dt + \sigma S_{1,t} dW_t,\ t \geq 0.$$

Here, $r \geq 0$ is the risk-free rate, $\mu > r$ is the appreciation rate, $\sigma > 0$ is the volatility, and $W_t, t \geq 0$ is a one-dimensional standard Brownian motion. This geometric Brownian motion model is standard in the literature. If the manager invests $\pi_t$ dollars in the risky asset at time $t$, then the asset value of the fund, $X_t, t \geq 0$, evolves according to

$$dX_t = rX_t dt + \pi_t \left[(\mu - r)dt + \sigma dW_t\right]. \tag{2.1}$$

A lower boundary $B_t, 0 \leq t \leq T$ is imposed on the investment strategy, i.e., $X_t \geq B_t := be^{-r(T-t)} X_0, 0 \leq t \leq T$, for some $b \in [0, 1)$. This lower boundary, which is known as the *liquidation boundary* of the fund, exists in practice and has also been included in the models proposed by Goetzmann, Ingersoll, and Ross (2003) and Hodder and Jackwerth (2007).

Note that the risk-free and risky assets constitute a complete market. In the following, we denote $\kappa := \sigma^{-1}(\mu - r)$ as the *market price of risk* of the risky asset and define the *state price density* process as $\xi_t := \exp\left[-\left(r + \frac{\kappa^2}{2}\right) t - \kappa W_t\right],\ 0 \leq t \leq T$.

### 2.2. Two Compensation Schemes

In the traditional scheme, let $w$ be the *managerial ownership ratio*, i.e., the proportion of the fund that belongs to the manager, and $\alpha$ be the *incentive rate*. Then, at the terminal date $T$, if the fund makes a profit, the manager charges a performance fee that is $\alpha$ proportion of the external investors' profit.[3] In other words, the performance fee is $\alpha(1 - w)\left(X_T - X_0\right)^+$. Therefore, under the traditional scheme, the manager's net profit-or-loss at time $T$ is

$$\Theta(X_T) := \begin{cases} (w + \alpha(1 - w))\left(X_T - X_0\right), & X_T \geq X_0, \\ w(X_T - X_0), & X_T < X_0. \end{cases} \tag{2.2}$$

[3] We do not consider explicitly the management fee that exists in a typical hedge fund in the United States, but this is not a restriction. Indeed, one can consider the management fee to be part of the managerial ownership. A typical U.S. hedge fund charges a management fee that ranges from 1% to 2% of the investors' capital. This management fee is used to cover operating expenses and is not invested. Thus, effectively, assets under management consist of 98% or 99% of the investors' capital plus the manager's own capital. To take the management fee into account, the initial asset value $X_0$ here should be understood as the value of the effective assets under management. For instance, if the management fee is 2% and the investors' capital and manager's capital are 100 million and 10 million, respectively, then $X_0 = 100 \times 98\% + 10 = 108$ million. In addition, the managerial ownership ratio $w$ is $10/108 = 9.26\%$. It is possible, though not common, that the management fee can be invested. In this case, it is not distinct from the manager's own capital invested in the fund, so we can regard the managerial ownership ratio $w$ as the total of the manager's own capital and management fee. For instance, if the manager's own capital is 8% and the management fee is 2%, the managerial ownership ratio $w$ should be $8\% + 2\% = 10\%$.

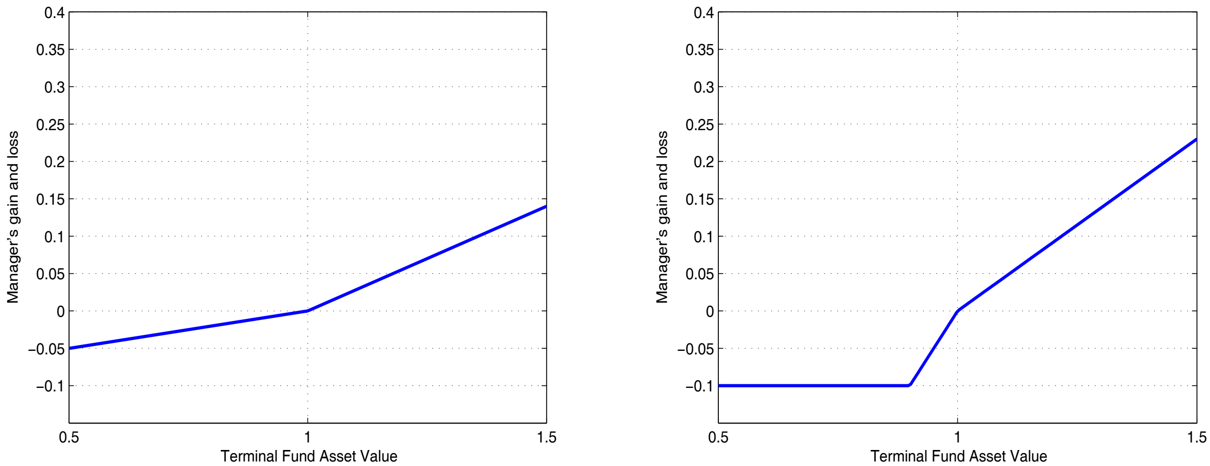


Figure 2.1: Manager's gain and loss in the traditional scheme (left panel) and in the first-loss scheme (right panel) as a function of fund asset value at terminal time. The managerial ownership ratio in both schemes is 10%. The incentive rates in the traditional and first-loss schemes are 20% and 40%, respectively. The benchmark for the performance fee, i.e., the initial asset value of the fund, is 1.

In the first-loss scheme, the fund manager's own capital invested in the fund will be used to offset any losses before any external investors take a loss. Thus, with managerial ownership ratio (for the first-loss capital) $\widetilde{w}$ and incentive rate $\widetilde{\alpha}$, under the first-loss scheme, the manager's net profit-or-loss at time $T$ is

$$\widetilde{\Theta}(X_T) := \begin{cases} (\widetilde{w} + \widetilde{\alpha}(1-\widetilde{w}))(X_T - X_0), & \text{if } X_T \geq X_0, \\ X_T - X_0, & \text{if } (1-\widetilde{w})X_0 < X_T < X_0, \\ -\widetilde{w}X_0 & \text{otherwise}. \end{cases} \tag{2.3}$$

Figure 2.1 illustrates the manager's gains and losses in the traditional scheme (left panel) and in the first-loss scheme (right panel), respectively. Note that the manager's gain is linear in the fund asset value whichever scheme is used. The manager's loss is linear in the fund asset value in the traditional scheme but is convex in the first-loss scheme. Overall, the manager's profit-or-loss is convex in the fund asset value in the traditional scheme. The manager's profit-or-loss in the first-loss scheme, however, is not convex in the fund asset value: the manager's profit-or-loss is concave in the fund asset value around the benchmark (i.e., the initial asset value). This concavity marks the difference of the first-loss scheme not only from the traditional scheme but also from other incentive contracts in the literature, such as those that involve an option payment.

## 2.3. The Manager's Preferences

In contrast to EUT, which is axiomatized from several normative principles that rational individuals are supposed to obey, behavioral economics attempts to describe or account for human behavior in decision-making. One of the most notable theories in behavioral economics is CP-T, which was proposed in Kahneman and Tversky (1979) and in Tversky and Kahneman (1992).

In this theory, individuals base decisions on comparisons to certain *reference points* (also known as benchmarks) rather than evaluating absolute values directly, a practice that has long been confirmed by empirical evidence and that is widely held to be a result of human cognitive processes (Kahneman, 2003). When applying CPT to portfolio choice, it is assumed that investors first decide, intentionally or unintentionally, their *evaluation period* and reference point before evaluating random gains and losses (relative to the reference point at the end of the evaluation period) according to some criterion. The choice of the evaluation period and the reference point is associated with the phenomena known as the *framing effect* (Tversky and Kahneman, 1981) and *mental accounting* (Thaler, 1999).

CPT suggests that (i) individuals tend to be risk averse with respect to random gains of moderate probability and risk seeking with respect to random losses of moderate probability and (ii) individuals tend to be *loss averse*, i.e., to be more sensitive to a loss than to a gain of the same amount. The literature supporting these two assertions is vast, with examples including Tversky and Kahneman (1992), Kahneman, Knetsch, and Thaler (1990), Abdellaoui, Bleichrodt, and Paraschiv (2007), and Abdellaoui, Bleichrodt, and Kammoun (2013), and provides extensive support for our choice of CPT to model the managers' preferences.

Formally, we assume that the manager in our hedge fund model evaluates random gain/loss $Y$, where a gain is recorded as $Y \geq 0$ and a loss is recorded as $Y \leq 0$, by $\mathbb{E}[u(Y)]$ for some *S-shaped* utility function $u(\cdot)$. We use the following functional form of $u(\cdot)$, which is also used in Tversky and Kahneman (1992):

$$u(x) = x^p, x \geq 0, \quad u(x) = -\lambda(-x)^p, x \leq 0, \tag{2.4}$$

where $0 < p < 1$ is called the *diminishing sensitivity parameter*, which measures the degree of risk aversion and risk seeking with respect to random gains and losses, respectively, of moderate probability. The parameter $\lambda > 1$ is called *loss aversion degree*, and it measures the extent to which individuals are loss averse.[4]

In our hedge fund management problem, in addition to her investment in the fund, namely $wX_0$ and $\widetilde{w}X_0$ in the traditional and first-loss schemes, respectively, the fund manager can also have her personal wealth outside the fund, and we denote it as $aX_0$ for some $a > 0$. We assume that the fund manager uses the performance fee payment date $T$ as the evaluation period. In addition, when evaluating her total wealth at $T$, she uses her initial investment in the fund plus her personal wealth outside the fund as the reference point, so the gain and loss experienced by the manager are $\Theta(X_T)$ and $\widetilde{\Theta}(X_T)$ in the traditional and first-loss schemes, respectively.[5] Then, the manager evaluates her gain/loss under CPT, in which two critical parameters—diminishing sensitivity parameter $p$ and loss aversion degree $\lambda$—are involved. As a result, under the traditional scheme the fund manager

[4]Various definitions of loss aversion degree have been proposed in the literature, including the definition $-u(-x)/u(x), \forall x > 0$ by Kahneman and Tversky (1979), the definition $-u(-1)/u(1)$ by Tversky and Kahneman (1992), the definition $u'(-x)/u'(x), \forall x > 0$ by Wakker and Tversky (1993), the definition $\lim_{x\downarrow 0} -u(-x)/u(x)$ by Köbberling and Wakker (2005), and the definition $\lim_{x\uparrow+\infty} -u(-x)/u(x)$ by He and Zhou (2011). When the utility function takes the form (2.4), all of these definitions coincide with the parameter $\lambda$.

[5]In most applications of CPT, reference points are set exogenously to be benchmarks (such as purchase prices and risk-free returns) that investors would naturally use to distinguish gains and losses; see e.g., Barberis and Huang (2008) and Barberis and Xiong (2009, 2012). In our model, the initial investment in the fund plus the manager's personal wealth outside the fund is a natural benchmark because whether the manager takes a performance fee or suffers a loss for her investment in the fund is benchmarked to the fund's initial asset value.

solves the following optimization problem:

$$(2.5)\qquad \begin{aligned} \underset{\pi}{\text{Max}} \quad & \mathbb{E}\left[u(\Theta(X_T))\right], \\ \text{Subject to} \quad & dX_t = rX_t dt + \pi_t\left[(\mu - r)dt + \sigma dW_t\right], \\ & X_t \geq b e^{-r(T-t)} X_0,\ 0 \leq t \leq T. \end{aligned}$$

Under the first-loss scheme, the fund manager solves another optimization problem:

$$(2.6)\qquad \begin{aligned} \underset{\pi}{\text{Max}} \quad & \mathbb{E}\left[u(\widetilde{\Theta}(X_T))\right], \\ \text{Subject to} \quad & dX_t = rX_t dt + \pi_t\left[(\mu - r)dt + \sigma dW_t\right], \\ & X_t \geq b e^{-r(T-t)} X_0,\ 0 \leq t \leq T. \end{aligned}$$

### 2.4. Some Comments on the Model Setting

Note that our model features both managerial ownership and a liquidation boundary. Fung and Hsieh (1999, p. 316) argue that the managerial ownership parameter is indispensable, noting that "the significant amount of personal wealth that hedge fund managers place at risk alongside investors' inhibits excessive risk taking" (indeed, a recent empirical study by Agarwal, Daniel, and Naik (2009) showed that the managerial ownership averaged 7.1% of total fund values). Similar to Hodder and Jackwerth (2007), we consider a liquidation boundary to avoid problems associated with the case in which the asset value goes to zero (as in Carpenter (2000)).

The geometric Brownian motion setting is also used in Carpenter (2000), Hodder and Jackwerth (2007), and Kouwenberg and Ziemba (2007) to study risk taking in hedge funds under the traditional scheme. Bichuch and Sturm (2014) assume a general semi-martingale for the risky asset price dynamics in the study of risk taking in hedge funds.[6] Except for Kouwenberg and Ziemba (2007), all the other papers set up models in the EUT framework. Moreover, all of these works focus on investment risk taking and optimal trading strategies but do not study the fund managers' or investors' utilities. Compared to these papers, the contribution of the current paper is twofold: First, in addition to investment risk taking and optimal trading strategies, we study profit sharing by deriving analytical formulae for both the fund managers' and investors' utilities. Second, the present paper is the first work of which we are aware to study both the first-loss scheme (which has a non-convex payoff) and the traditional scheme, whereas the literature to date has focused exclusively on the latter.

Let us also comment on the setting of CPT preferences in the present paper. First, we assume the parametric form of the S-shaped utility function as in (2.4) because it is supported by some

[6] Note that neither these papers nor ours consider high-water marks. Goetzmann, Ingersoll, and Ross (2003) study high-water marks by modeling the asset value of a hedge fund directly (without portfolio optimization). They include a liquidation boundary but do not consider managerial ownership. Panageas and Westerfield (2009) study the portfolio selection problem of a risk-neutral hedge fund manager who is compensated by high-water mark contracts and maximizes the expected cumulative performance fee in an infinite horizon. They show that high-water mark contracts can prohibit the manager from taking infinite risk. Guasoni and Obłój (2016) extend the results in Panageas and Westerfield (2009) by assuming that the manager is risk averse and has a power utility function. Alongside the aforementioned papers considering the high-water mark at a continuous-time basis, Mitchell, Muthuraman, and Titman (2013) use a discrete-time setting, which is the market practice, and show that the fund risk is increasing with respect to the incentive rate.

experimental studies—e.g., Tversky and Kahneman (1992), in which $p$ is estimated to be 0.88—and is employed in many applications of CPT, e.g., Barberis and Huang (2008), Barberis and Xiong (2009), and Barberis (2012). With this parametric form, we must enforce $p$ to be positive because $u(0) = 0$.[7]

Second, we did not consider probability weighting, another important ingredient in CPT; see Tversky and Kahneman (1992). With probability weighting, the fund manager's optimization problem is difficult to solve; moreover, time inconsistency arises as a result of probability weighting, see Barberis (2012). Therefore, as in some applications of CPT in the literature, such as Barberis and Xiong (2009), we chose not to consider probability weighting in the present paper. The impact of probability weighting on profit sharing in hedge funds will be studied in the future.

## 3. OPTIMAL FUND VALUE AND OPTIMAL TRADING STRATEGY

Before presenting the optimal strategies taken by the manager, we define

$$\gamma := \frac{(1-b)w}{w + \alpha(1-w)} \tag{3.1}$$

as the *loss-gain ratio* in the traditional scheme and

$$\widetilde{\gamma} := \frac{\min(\widetilde{w}, 1-b)}{\widetilde{w} + \widetilde{\alpha}(1-\widetilde{w})} \tag{3.2}$$

as the *loss-gain ratio* in the first-loss scheme. Indeed, in the traditional scheme, if the fund ends up at a gain, the manager collects $w + \alpha(1-w)$ proportion of the total gain. If the fund suffers a loss, the maximum loss of the manager is $w(1-b)$ proportion of the fund's initial asset value. Therefore, $\gamma$ measures the ratio of loss impediment and gain incentive for the manager. Similarly, in the first-loss scheme, the manager receives $\widetilde{w} + \widetilde{\alpha}(1-\widetilde{w})$ proportion of the fund's gain and loses $\min(\widetilde{w}, 1-b)$ proportion of the fund's initial asset value, i.e., covers the fund loss in the worst case, which is $1-b$ proportion of the fund's initial asset value, until her first-loss capital, which is $\widetilde{w}$ proportion of the fund's initial asset value, is exhausted. Thus, $\widetilde{\gamma}$ is the loss-gain ratio of the manager's incentive in the first-loss scheme.

**Theorem 3.1.** *In the first-loss scheme, the optimal asset value of the fund and the optimal percentage allocation to the risky asset are*

$$X_t^* = e^{-r(T-t)} X_0 \left[ b + (1-b)\Phi\left(d_{1,t}\right) + \widetilde{c}^* \frac{\Phi'(d_{1,t})}{\Phi'(d_{2,t})} \Phi\left(d_{2,t}\right) \right], \ 0 \le t \le T \tag{3.3}$$

*and*

$$\frac{\pi_t^*}{X_t^*} = \left\{ \frac{1}{1-p} + \frac{X_0}{e^{r(T-t)} X_t^*} \left[ \frac{\widetilde{c}^* + 1 - b}{\kappa\sqrt{T-t}} \Phi'(d_{1,t}) - \frac{1}{1-p}\left(b + (1-b)\Phi(d_{1,t})\right) \right] \right\} \frac{\kappa}{\sigma}, \ 0 \le t \le T, \tag{3.4}$$

[7] Alternative forms of the utility function are available in the literature. For example, one can consider utility functions with different exponents for gains and losses, i.e., $u(x) = x^p \mathbf{1}_{\{x \ge 0\}} - \lambda(-x)^q \mathbf{1}_{\{x<0\}}$ for some $p, q \in (0, 1]$, and piece-wise transformed negative power functions, i.e., $u(x) = (1 - (1+x)^p)\mathbf{1}_{\{x \ge 0\}} - \lambda(1 - (1-x)^q)\mathbf{1}_{\{x<0\}}$ for some $p, q < 0$. In both cases, we have the same conclusion as in the case of the setting in the present paper that the fund either has no loss or is liquidated. However, in neither case can we conclude that the optimal strategy depends only on the loss-gain ratio that measures the ratio of loss impediment and gain incentive for the manager.

*respectively, where* $\Phi(\cdot)$ *is the cumulative distribution function of the standard normal distribution and*

$$d_{1,t} = \frac{\ln \widetilde{\nu}^* - \ln \xi_t + \left(r - \frac{1}{2}\kappa^2\right)(T-t)}{\kappa\sqrt{T-t}}, \quad d_{2,t} = d_{1,t} + \frac{\kappa\sqrt{T-t}}{1-p}, \quad 0 \le t \le T.$$

*In particular, the terminal asset value is*

$$X_T^* := \left[ b\mathbf{1}_{\{\xi_T > \widetilde{\nu}^*\}} + \left(1 + \widetilde{c}^* \left(\frac{\xi_T}{\widetilde{\nu}^*}\right)^{\frac{1}{p-1}}\right) \mathbf{1}_{\{\xi_T \le \widetilde{\nu}^*\}} \right] X_0. \tag{3.5}$$

*Here,* $\widetilde{c}^* > 0$ *is the unique solution to*

$$(1-p)(\widetilde{c}^*)^p - (1-b)p(\widetilde{c}^*)^{p-1} + \lambda\widetilde{\gamma}^p = 0 \tag{3.6}$$

*and* $\widetilde{\nu}^*$ *is the unique positive number determined by* $\mathbb{E}\left[\xi_T X_T^*\right] = X_0$.

*In the traditional scheme, the optimal asset value of the fund,* $X_t^*, 0 \le t \le T$, *and the optimal percentage allocation to the risky asset,* $\pi_t^*/X_t^*, 0 \le t \le T$, *are obtained by replacing* $\widetilde{c}^*$ *and* $\widetilde{\nu}^*$ *in* (3.3) *and* (3.4) *with* $c^*$ *and* $\nu^*$, *respectively, where* $c^* > 0$ *is the unique solution to*

$$(1-p)(c^*)^p - (1-b)p(c^*)^{p-1} + \lambda\gamma^p = 0 \tag{3.7}$$

*and* $\nu^*$ *is the unique positive number determined by* $\mathbb{E}\left[\xi_T X_T^*\right] = X_0$.

It is interesting to observe that in both schemes the managerial ownership ratio and incentive rate combine into a single number—the loss-gain ratio ($\gamma$ in the traditional scheme and $\widetilde{\gamma}$ in the first-loss scheme)—to determine the optimal solutions. More strikingly, if the loss-gain ratios are the same in the two schemes, i.e., $\gamma = \widetilde{\gamma}$, the resulting optimal trading strategies and asset values are the same as well. However, for more general incentive schemes rather than the traditional and first-loss schemes, or when the asset price model is no longer the geometric Brownian motion, there is no such simple ratio fully characterizing the scheme. In general, the optimal terminal asset value will no longer be as simple as in (3.5), in which the only possible loss amount is $(1-b)X_0$.

Figure 3.1 illustrates the trading strategies that the fund manager implements, i.e., $\pi_t^*/X_t^*$ as a function of the asset value $X_t^*$, in the traditional scheme with 10% managerial ownership ratio and 20% incentive rate and in the first-loss scheme with 10% first-loss capital and 20%, 30%, and 40% incentive rates, respectively, when the time to the performance fee payment date is 0.1, 0.5, 1, 3, 5, and 10 years, respectively (corresponding to panels from left to right and from top to bottom, respectively). We can see that when it is near the performance fee payment date, the optimal percentage allocation to the risky asset has a *peak-valley* pattern: the percentage allocation is low when the fund value is either close to the liquidation barrier (as the manager tries to avoid the liquidation boundary) or slightly above the initial value (due to the risk aversion regarding gains and loss aversion). When it is far from the performance fee payment date, the percentage allocation is increasing with respect to the asset value. Moreover, the difference in the percentage allocation in different schemes becomes less significant as the time to the performance fee payment date becomes longer.

We also observe from Figure 3.1 that in the first-loss scheme the percentage allocation to the risky asset is increasing with respect to the incentive rate. Moreover, with the parameter values

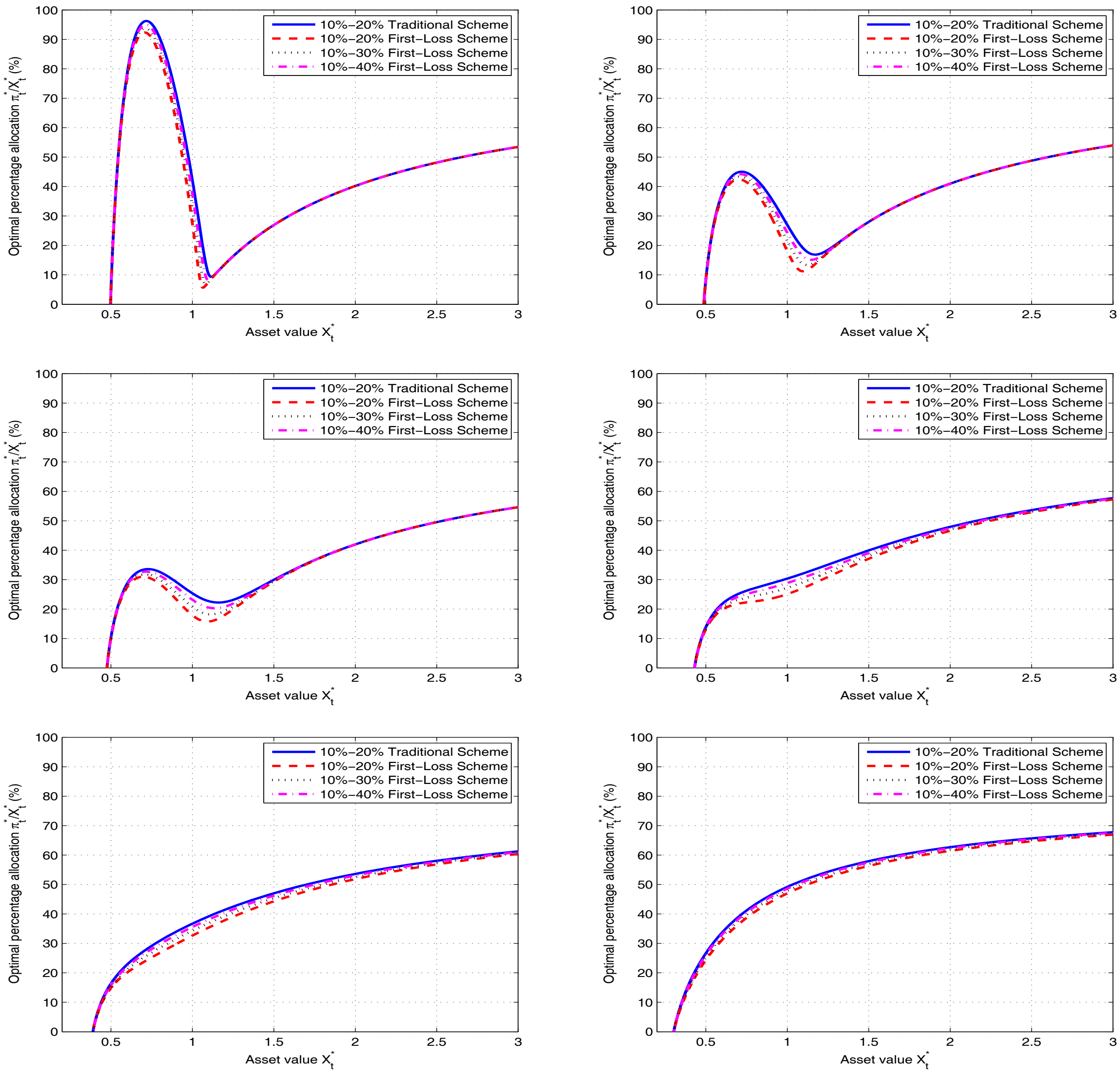


Figure 3.1: Optimal percentage allocation to the risky asset, $\pi_t^*/X_t^*$, with respect to $X_t^*$ in the traditional scheme with 10% managerial ownership ratio and 20% incentive rate and in the first-loss scheme with 10% first-loss capital and 20%, 30%, and 40% incentive rates, respectively. The six panels from left to right and from top to bottom correspond to the cases $T - t = 0.1$, 0.5, 1, 3, 5, and 10, respectively. The parameter values are chosen as: $X_0 = 1$, $p = 0.5$, $\lambda = 2.25$, $b = 0.5$, $r = 5\%$, $\kappa = 40\%$, and $\sigma = 1$. The Merton line, i.e., $\sigma^{-1}\kappa/(1-p)$, is 80%.

used in Figure 3.1, the percentage allocation is higher in the traditional scheme than in the first-loss scheme. These observations will be confirmed by the theoretical result in Theorem 4.1 in the following section.

**Remark 3.2.** *We can prove that* $\lim_{X_t^*\downarrow be^{-r(T-t)}X_0}\pi_t^*/X_t^*=0$ *and* $\lim_{X_t^*\uparrow+\infty}\pi_t^*/X_t^*=\sigma^{-1}\kappa/(1-p)$. *Indeed, from the proof of Theorem 3.1,* $X_t^*$ *is a function of* $t$ *and* $\xi_t$, *and is strictly decreasing in* $\xi_t$. *In addition, it is straightforward to show that* $\lim_{\xi_t\uparrow+\infty}d_{i,t}=-\infty$, $\lim_{\xi_t\downarrow 0}d_{i,t}=+\infty$, $i=1,2$, $\lim_{\xi_t\uparrow+\infty}\Phi'(d_{1,t})/\Phi'(d_{2,t})=0$, *and* $\lim_{\xi_t\downarrow 0}\Phi'(d_{1,t})/\Phi'(d_{2,t})=+\infty$. *Thus, from* (3.3), *we have* $\lim_{\xi_t\uparrow+\infty}X_t^*=e^{-r(T-t)}bX_0$ *and* $\lim_{\xi_t\downarrow 0}X_t^*=+\infty$. *If we regard* $\xi_t$ *as a function of* $t$ *and* $X_t^*$, *then* $\xi_t$ *is strictly decreasing in* $X_t^*$ *and* $\lim_{X_t^*\downarrow e^{-r(T-t)}bX_0}\xi_t=+\infty$ *and* $\lim_{X_t^*\uparrow+\infty}\xi_t^*=0$. *Passing* $X_t^*$ *in* (3.4) *to* $e^{-r(T-t)}bX_0$ *and* $+\infty$, *respectively, we immediately obtain the limits we want to prove. Note that the limit* $\sigma^{-1}\kappa/(1-p)$ *is exactly the Merton line, i.e., the fixed constant percentage allocation for a manager who has a power utility function and invests on her own. In Figure 3.1, the Merton line is 80%. We can observe from the figure that the convergence of the optimal portfolio to the Merton line as the wealth level goes to infinity is slow.*

## 4. ANALYTICAL COMPARISON OF THE TRADITIONAL AND FIRST-LOSS SCHEMES

### 4.1. The Risk of the Fund

Before we discuss the profit sharing between hedge fund investors and managers, it is important to study fund risk. In particular, under a particular profit sharing scheme, it is more suitable to talk about a utility increase for either investors or managers if the fund risk is kept the same or becomes lower; otherwise, it is difficult to justify the benefit of profit sharing if the managers' and investors' utilities and the fund risk increase at the same time.

How do we quantify the risk of a fund? The literature on risk measures is quite rich, and the discussion and use of risk measures are also very common in practice. For instance, the Basel Committee uses value-at-risk (VaR) as a measure of market risk to regulate banks. Academically, Artzner, Delbaen, Eber, and Heath (1999) proposed coherent risk measures, including conditional value-at-risk (CVaR) as a special case. Each of these two types of risk measures has its own advantages and disadvantages; overall, the former is suitable for external use and the latter for internal use (Cont, Deguest, and He, 2013, Kou, Peng, and Heyde, 2013).

Fortunately, in our model, the choice of risk measures makes no difference. Indeed, from the optimal terminal wealth (3.5) obtained in Theorem 3.1, the loss of the fund follows a simple distribution: it is a fixed amount $(1-b)X_0$ if $\xi_T>\nu^*$ and zero otherwise. Therefore, if a risk measure is loss-based (depending solely on losses), monotone (larger losses in all scenarios leading to higher risk), and law-invariant (depending only on loss distributions), then, fixing the magnitude of loss $(1-b)X_0$, the risk measure must be simply determined by the probability of the loss event $\{\xi_T>\nu^*\}$. Therefore, we can regard the *loss probability* $\mathcal{R}:=\mathbb{P}(\xi_T>\nu^*)$ as the measure of fund risk in the traditional scheme. Similarly, in the first-loss scheme, we regard the loss probability $\widetilde{\mathcal{R}}:=\mathbb{P}(\xi_T>\widetilde{\nu}^*)$ as fund risk.

Carpenter (2000) considers the risk of a fund to be the asset volatility of the fund. From the wealth equation (2.1) we can see that the asset volatility of the fund in our model is equal to $\sigma\pi_t^*/X_t^*, 0\le t\le T$, where $\pi^*$ and $X^*$ are the optimal portfolio and wealth processes, respectively.

It happens that our definition of risk is consistent with the definition by asset volatility, which will be shown in the following theorem.

**Theorem 4.1.** *In both schemes, both the loss probability and the asset volatility of the fund are strictly decreasing in the loss-gain ratio, i.e.,* $\gamma$ *and* $\widetilde{\gamma}$*, for the manager. Consequently, the fund risk (measured either by loss probability or asset volatility) in the first-loss scheme is the same as (strictly higher than or strictly lower than) in the traditional scheme if and only if* $\widetilde{\gamma}$ *is equal to (strictly less than or strictly greater than)* $\gamma$.

Theorem 4.1 provides us with a very simple way of comparing hedge fund risk in the two different schemes: comparing the loss-gain ratios.[8] Such a comparison is independent of the investment opportunities of the funds (corresponding to parameters $r$ and $\kappa$) and of the preferences of the managers (corresponding to parameters $p$ and $\lambda$), which are usually difficult to estimate accurately.

For example, suppose $b = 50\%$, which means that the fund can only lose 50% of its initial assets in the worst case. In a typical first-loss scheme newly introduced in the United States, the managerial ownership ratio is 10% and the incentive rate is 40%. Let us compare this to a traditional scheme with the same managerial ownership ratio but with a 20% incentive rate, a typical number for U.S. funds. It is easy to calculate that $\gamma = 17.86\%$ and $\widetilde{\gamma} = 21.74\%$. As a result, *the risk in the first-loss scheme is lower*.

Finally, we study the effect of the liquidation boundary on the fund risk.

**Proposition 4.2.** *The loss probability of the fund in both the traditional and first-loss schemes is decreasing with respect to* $b \in [0, 1)$*. The monotonicity becomes strict in the traditional scheme if and only if* $r > 0$*, and becomes strict in the first-loss scheme if and only if* $r > 0$ *or* $b \in [0, 1 - \widetilde{w}]$*. Moreover, when* $b$ *approaches 1, the limit of the loss probability is strictly positive if* $r = 0$ *and is zero if* $r > 0$*. Finally, the asset volatility in both schemes is strictly decreasing with respect to* $b$.

With a lower liquidation boundary $b$, the potential loss of the fund, i.e., $(1 - b)X_0$, becomes larger. Proposition 4.2 shows that the loss probability becomes higher in both the traditional and first-loss schemes and, consequently, the fund risk becomes higher. Proposition 4.2 also shows that a lower liquidation boundary leads to higher asset volatility, i.e., more investment in the risky asset.

In view of Theorem 4.1 and of the definition of the loss-gain ratios (3.1)–(3.2), we can see that, in the absence of the liquidation boundary (i.e., $b = 0$), the first-loss scheme yields a lower loss probability than the traditional scheme if and only if $\widetilde{w}/(\widetilde{w} + \widetilde{\alpha}(1 - \widetilde{w})) \geq w/(w + \alpha(1 - w))$. This is not surprising because without the liquidation boundary, the manager in either scheme can lose all her stake in the fund in the worst case and thus behaves the same in the two schemes given the same managerial ownership ratio and incentive rate.

[8] A consequence of Theorem 4.1 is that a higher incentive rate increases while a higher managerial ownership ratio reduces hedge fund risk. This reminds us that in the empirical study of hedge fund risk, the managerial ownership ratio, which is usually neglected, should be taken into account. For instance, Ackermann, McEnally, and Ravenscraft (1999) study how different characteristics of a hedge fund affect the performance of the fund that is represented by the Sharpe ratio of the fund's return. The managerial ownership was not included in these characteristics. Similarly, Kouwenberg and Ziemba (2007) also neglect managerial ownership in their empirical study of hedge fund risk.

On the other hand, we observe from Proposition 4.2 that in both schemes the loss probability of the fund converges to zero when the liquidation boundary $b$ approaches 1 if and only if the risk-free rate $r > 0$. Intuitively, even when $b$ approaches 1, the manager wants to take some risk for a potential gain. When the risk-free rate $r > 0$, simply holding the risk-free asset can yield a gain because the manager's benchmark is the initial fund value. This gain can be used as a cushion to offset losses from limited risk taking and, consequently, the loss probability of the fund is nearly zero. When the risk-free rate $r = 0$, however, such a cushion does not exist, so the loss probability is not zero even when $b$ approaches 1.

## 4.2. The Utility of the Manager

Next, we compare the traditional and first-loss schemes from the manager's perspective. To this end, we compare the *utility per capital*, i.e., the utility of the manager if her initial investment is one dollar, under these two schemes.[9] In the traditional scheme, if the manager invests one dollar in the fund, the initial asset value of the fund is $X_0 = 1/w$. In the first-loss scheme, it is $X_0 = 1/\widetilde{w}$.

As will be clear in Section 5.1, typical values of $b$ in the market satisfy $b \leq 1 - \widetilde{w}$; i.e., in the worst case, the manager is unable to cover the fund loss completely in the first-loss scheme. Thus, for simplicity, we assume this condition in the following analysis. However, let us comment that in the case in which $b > 1 - \widetilde{w}$ we can still compute the utilities per capital of the manager and of the investor in closed form, and the analytical result regarding the comparison of the manager's utility per capital in the traditional and first-loss schemes in Theorem 4.3-(iii) and -(iv) still holds.

**Theorem 4.3.** *Assume $b \leq 1 - \widetilde{w}$. The utility per capital of the manager is strictly increasing in the incentive rate and is strictly decreasing in the liquidation boundary $b$ in both schemes. Furthermore,*

*(i) The utility per capital of the manager in the traditional scheme is*

$$\begin{aligned}\mathcal{M} := (1-b)^p \Bigg[ \left(\frac{c^*}{\gamma}\right)^p (\nu^*)^{\frac{p}{1-p}} e^{\left(\frac{p}{1-p}r + \frac{p}{2(1-p)^2}\kappa^2\right)T} \Phi\left(\frac{\ln \nu^* + \left(r + \frac{1+p}{2(1-p)}\kappa^2\right)T}{\kappa\sqrt{T}}\right) \\ -\lambda \Phi\left(-\frac{\ln \nu^* + (r + \frac{1}{2}\kappa^2)T}{\kappa\sqrt{T}}\right)\Bigg],\end{aligned} \tag{4.1}$$

*where $\gamma$ is the loss-gain ratio and $c^*$ and $\nu^*$ are given as in Theorem 3.1. Moreover, this utility is strictly decreasing in $\gamma$.*

[9] Equivalently, we can also compute and compare the "certainty equivalent wealth" of the manager in these two schemes. Note that we assume the same preferences for the manager in the traditional and first-loss schemes when comparing them, so it makes no difference whether we use utility per capital or certainty equivalent wealth to carry out the comparison. Indeed, because the utility function $u$ is strictly increasing, the utility per unit capital in the traditional scheme, denoted as $\mathcal{M}$, is larger than or equal to that in the first-loss scheme, denoted as $\widetilde{\mathcal{M}}$, if and only if $u^{-1}(\mathcal{M}) \geq u^{-1}(\widetilde{\mathcal{M}})$, i.e., the certainty equivalent wealth of the manager in the traditional scheme is larger than or equal to that in the first-loss scheme.

*(ii) The utility per capital of the manager in the first-loss scheme is*

$$\widetilde{\mathcal{M}} := \left(\frac{\widetilde{c}^*}{\widetilde{\gamma}}\right)^p (\widetilde{\nu}^*)^{\frac{p}{1-p}} e^{\left(\frac{p}{1-p}r + \frac{p}{2(1-p)^2}\kappa^2\right)T} \Phi\left(\frac{\ln\widetilde{\nu}^* + \left(r + \frac{1+p}{2(1-p)}\kappa^2\right)T}{\kappa\sqrt{T}}\right) - \lambda\Phi\left(-\frac{\ln\widetilde{\nu}^* + (r+\frac{1}{2}\kappa^2)T}{\kappa\sqrt{T}}\right), \tag{4.2}$$

*where $\widetilde{\gamma}$ is the loss-gain ratio and $\widetilde{c}^*$ and $\widetilde{\nu}^*$ are given as in Theorem 3.1. Moreover, this utility is strictly decreasing in $\widetilde{\gamma}$.*

*(iii) If the loss-gain ratio is the same in the two schemes, then $\widetilde{\mathcal{M}} = \mathcal{M}$ in the case of $b = 0$ and $\widetilde{\mathcal{M}} > \mathcal{M}$ in the case of $b > 0$.*

*(iv) If the same managerial ownership ratio and incentive rate are employed in the two schemes, then $\widetilde{\mathcal{M}} = \mathcal{M}$ in the case of $b = 0$ and $\widetilde{\mathcal{M}} < \mathcal{M}$ in the case of $b > 0$.*

Theorem 4.3 shows that no matter which scheme is used, the utility per capital of the manager is strictly decreasing in the loss-gain ratio. Consequently, it is strictly increasing in the incentive rate and strictly decreasing in the managerial ownership ratio. On the other hand, the utility per capital of the manager is strictly decreasing in the liquidation boundary: the lower the boundary is, the more strategies the manager can take, and consequently the larger the manager's utility is.

When $b > 0$, Theorem 4.3-(iii) and-(iv), together with Theorem 4.1, have interesting implications. First, if keeping the managerial ownership ratio and incentive rate the same, the utility of the manager is reduced when switching from the traditional to the first-loss scheme. Second, if keeping the risk the same in the two schemes by manipulating some contractual components in the two schemes, e.g., the incentive rate, then the utility of the manager can be strictly increased. This observation implies that it *is possible to design a first-loss scheme* so that this scheme is better than the existing traditional scheme from the perspectives of risk taking and the manager's utility at the same time!

## 4.3. The Utility of the Investor

We again argue that because the initial asset value of the fund is a salient component in the incentive schemes that determine the manager's compensation, any investor in the fund would also choose this value to be her benchmark to distinguish gains and losses. Therefore, we use CPT as well to model the investors' preferences. For simplicity, we assume the utility function to be $v(x) = x^q, x \geq 0, v(x) = -\eta(-x)^q, x \leq 0$, where $q \in (0,1)$ and $\eta > 1$.

**Theorem 4.4.** *Assume $b \leq 1 - \widetilde{w}$.*

*(i) The utility per capital of the investor in the traditional scheme is*

$$\mathcal{I} := (1-\alpha)^q (c^*)^q (\nu^*)^{\frac{q}{1-p}} e^{\left(\frac{q}{1-p}r + \frac{q(1-p+q)}{2(1-p)^2}\kappa^2\right)T} \Phi\left(\frac{\ln\nu^* + \left(r + \frac{1-p+2q}{2(1-p)}\kappa^2\right)T}{\kappa\sqrt{T}}\right) - \eta(1-b)^q \Phi\left(-\frac{\ln\nu^* + \left(r+\frac{1}{2}\kappa^2\right)T}{\kappa\sqrt{T}}\right) \tag{4.3}$$

*where $c^*$ and $\nu^*$ are given as in Theorem 3.1.*

*(ii) The utility per capital of the investor in the first-loss scheme is*

$$
\begin{aligned}
\widetilde{\mathcal{I}} = {} & (1-\widetilde{\alpha})^q (\widetilde{c}^*)^q (\widetilde{\nu}^*)^{\frac{q}{1-p}} e^{\left(\frac{q}{1-p}r + \frac{q(1-p+q)}{2(1-p)^2}\kappa^2\right)T} \Phi\left(\frac{\ln \widetilde{\nu}^* + \left(r + \frac{1-p+2q}{2(1-p)}\kappa^2\right)T}{\kappa\sqrt{T}}\right) \\
& - \eta \left(\frac{1-b-\widetilde{w}}{1-\widetilde{w}}\right)^q \Phi\left(-\frac{\ln \widetilde{\nu}^* + \left(r + \frac{1}{2}\kappa^2\right)T}{\kappa\sqrt{T}}\right)
\end{aligned}
\tag{4.4}
$$

*where $\widetilde{c}^*$ and $\widetilde{\nu}^*$ are given as in Theorem 3.1.*

Unfortunately, unlike in Theorem 4.3-(iii) and -(iv), we are unable to compare the utility of the investor in these two schemes analytically. Thus, we resort to numerical computation in Section 6.

## 5. CALIBRATING CPT TO HEDGE FUND DATA

### 5.1. Hedge Fund Contractual Parameters and Asset Return Parameters

The first set of parameters consists of the managerial ownership ratio and the incentive rate. We set the incentive rate in the traditional scheme $\alpha$ at 20%, which is a typical value in U.S. hedge funds. There is no convention or hard rule regarding the managerial ownership ratio under the traditional scheme in the United States. Using a hedge fund data set from 1994 to 2002, however, Agarwal, Daniel, and Naik (2009) estimated the average managerial ownership ratio at 7.1%. If the management fee, which is typically 1–2%, can be invested, then the management fee can be regarded as managerial ownership, and the total "effective" managerial ownership ratio is actually close to 10%. Thus, we set $w = 10\%$.

Next, we specify the interest rate $r$ and the market price of risk of the risky asset $\kappa$. Historical data suggest that the nominal interest rate is around 5%.[10] Therefore, we set $r = 5\%$ in our following numerical study. The market price of risk $\kappa$ is less clear because hedge funds do not publish their investment strategies. We believe that the Sharpe ratio of hedge fund returns is a good proxy for $\kappa$. The most reported Sharpe ratio in hedge funds is around 40%; see, for instance, Liang (1999, 2001). To be conservative, we let $\kappa$ take the following three values: 30%, 40%, and $50\%$.

The evaluation period is set at one year, i.e., $T = 1$, because performance fees are paid annually in many hedge funds. The reference point is simply the initial value of the fund $X_0$.

Finally, we specify the value of the parameter $b$ that determines the liquidation boundary. Hodder and Jackwerth (2007) set $b$ at 50%. This value is also used by Goetzmann, Ingersoll, and Ross (2003). Therefore, we take the following three different values for $b$: 40%, 50%, and 60%.

[10]The average nominal one-year interest rate for the period 1871 to 2011 is 4.72%; see e.g., the data set available at `http://www.econ.yale.edu/~shiller/data.htm`. The reason we use nominal instead of real interest rate is that individuals are usually subject to *money illusion*, a tendency to think in terms of nominal rather than real dollars (Shafir, Diamond, and Tversky, 1997). Given the low interest rate in recent years, we also performed the numerical analysis with 2% interest rate and found that none of the results changed significantly. For example, the calibrated diminishing sensitivity $p$ is within the range from 0.42 to 0.66. In most cases, the first-loss scheme with a 30% incentive rate is better than the traditional scheme for both managers and investors, but with a 40% incentive rate, the first-loss scheme becomes worse for investors.

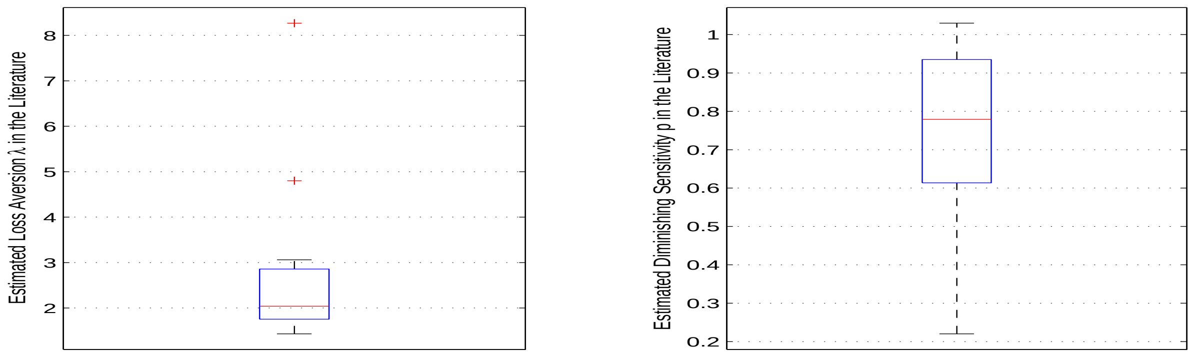


Figure 5.1: Boxplots of estimates of loss aversion degree $\lambda$ (left panel) and diminishing sensitivity parameter $p$ (right panel) in the experimental literature.

### 5.2. Calibrating CPT Parameters $\lambda$ and $p$

There is a vast literature on conducting laboratory experiments to calibrate CPT. In particular, the diminishing sensitivity parameter $p$ and loss aversion degree $\lambda$ have been estimated extensively. Figure 5.1 summarizes these laboratory estimates. The left panel of Figure 5.1 is a boxplot of the laboratory estimates of $\lambda$.[11] Note that, except for two outliers, these estimates lie in the range $[1.25, 3.25]$ and the median is slightly larger than 2. The right panel of Figure 5.1 plots the laboratory estimates of $p$, which range from 0.22 to 1.03.[12]

In all of these experimental studies, the value of loss aversion parameter $\lambda$ is always larger than 1, which is consistent with CPT. However, it is more difficult to estimate the parameter $p$ because many of the experimental estimates of $p$ are very close to or even larger than 1, making the individual risk neutral or even risk seeking with respect to random gains, thus leading to problematic infinite risk-taking in many cases.

Given the significant differences between experimental settings and financial markets, especially the different magnitudes of monetary payoffs involved, it becomes important to use financial data to calibrate CPT. Unfortunately, literature on estimating CPT parameters using financial data is scarce.[13] A main difficulty lies in the complexity of financial investment activities, an issue that

[11] The numbers are reported in Fishburn and Kochenberger (1979), Tversky and Kahneman (1992), Bleichrodt, Pinto, and Wakker (2001), Schmidt and Traub (2002), Pennings and Smidts (2003), Abdellaoui, Bleichrodt, and Paraschiv (2007), and Booij and van de Kuilen (2009). All these numbers are taken from Tables 1 and 5 in Abdellaoui, Bleichrodt, and Paraschiv (2007). For the numbers in Abdellaoui, Bleichrodt, and Paraschiv (2007, Table 5), we only select the mean of the estimates in the first, third, and fifth rows because other rows refer to definitions of loss aversion that are not represented by $\lambda$. Abdellaoui, Bleichrodt, and Kammoun (2013) measure the loss aversion degree of a selected group of financial professionals. However, their experiments did not involve monetary payoffs and the experiment questions were formulated in terms of the company's money rather than the professionals' own money, so we chose not to include their results here.

[12] These numbers are taken from Tversky and Kahneman (1992), Camerer and Ho (1994), Wu and Gonzalez (1996), Bleichrodt and Pinto (2000), Booij and van de Kuilen (2009), and other works as summarized in Booij, van Praag, and van de Kuilen (2010, Table 1). Some of those works estimated the diminishing sensitivity parameter $p$ in the gain and loss regions separately. In this case, we use in the boxplot the average of the estimates of $p$ in these two regions.

[13] The only papers we are aware of in this regard are Kliger and Levy (2009) and Gurevich, Kliger, and Levy (2009).

can be mitigated significantly in laboratories through experiment design. For example, in the calibration of CPT to financial data, the misspecification of the evaluation period and reference point could lead to totally different results, while in laboratories such a danger does not exist because the evaluation period and reference point are controlled by experiment design.

In the present paper, we use our hedge fund profit sharing model and hedge fund data to calibrate CPT. One advantage of this approach is that the managers use the performance fee payment date as the evaluation period and the benchmark for the performance fee as the reference point to distinguish gains and losses. The main disadvantage is that there is little data available, as hedge funds report their fund performance at most quarterly (certainly not daily) and as the holdings in their portfolios change from time to time and remain to a large extent secret.

Facing this data challenge, we shall use the historical hedge fund attrition rates to perform the calibration. A substantial percentage of hedge funds disappear from the hedge fund databases each year, mainly due to liquidation. This historically observed attrition rate can be regarded as the hedge fund liquidation probability. In our model, the liquidation probability in the traditional scheme is $\mathcal{R} = \mathbb{P}(\xi_T > \nu^*)$ where $\nu^*$ is defined as in Theorem 3.1. More precisely,

$$\mathcal{R} = \mathbb{P}(\xi_T > \nu^*) = \mathbb{P}\left(e^{-(r+\kappa^2/2)T-\kappa W_T} > \nu^*\right) = \Phi\left(-\frac{\log \nu^* + (r+\kappa^2/2)T}{\kappa\sqrt{T}}\right)$$

and $\nu^*$ is determined by $\mathbb{E}\left[\xi_T X_T^*\right] = X_0$, i.e., by

$$e^{-rT}\left[b + (1-b)\Phi(d_{1,0}) + c^*\frac{\Phi'(d_{1,0})}{\Phi'(d_{2,0})}\Phi(d_{2,0})\right] = 1,$$

where $c^*$ is the solution to (3.7) and $d_{1,0} = \frac{\ln \nu^* + (r-\kappa^2/2)T}{\kappa\sqrt{T}}$, $d_{2,0} = d_{1,0} + \kappa\sqrt{T}/(1-p)$. Note that with other parameters fixed, the liquidation probability $\mathcal{R}$ is a function of $\kappa$, $b$, $\lambda$, and $p$.

We only have one number—attrition rate—so it is impossible to calibrate two parameters—namely, $\lambda$ and $p$. As we have argued, the experimental estimates summarized in Figure 5.1 suggest that it is more difficult to estimate $p$ than $\lambda$. Moreover, except for two outliers, the estimates of $\lambda$ lie in the range $[1.25, 3.25]$. Therefore, we choose to fix $\lambda$ at five different values: 1.25, 1.75, 2.25, 2.75, and 3.25 and calibrate $p$ to the attrition rate of U.S. hedge funds. More precisely, for each fixed $\lambda$, the liquidation probability $\mathcal{R}$ is a function of $\kappa$, $b$, and $p$, and we denote it as $\mathcal{R} = h(p, \kappa, b)$. This connection between $p$ and $\mathcal{R}$ allows us to calibrate $p$ from the historical values of $\mathcal{R}$.

The historical attrition rate of U.S. hedge funds is well within the range 10%-20%; see e.g., Brown, Goetzmann, and Ibbotson (1999), Brown, Goetzmann, and Park (2001), Fung and Hsieh (2000), Liang (2001), and Malkiel and Saha (2005). Therefore, we take $\mathcal{R}$, the probability of liquidation in the traditional scheme, to be 10%, 12.5%, 15%, 17.5%, and 20%. To summarize, $\kappa$ and $b$ take three values and $\mathcal{R}$ takes five values, leading to $3 \times 3 \times 5 = 45$ triples of $(\mathcal{R}_i, \kappa_j, b_k)$. For each triple, we calibrate $p$ from $h(p, \kappa_j, b_k) = \mathcal{R}_i$.

Summary statistics of the calibration results are reported in Table 5.1. We can see that the calibrated $p$ is in the range $[0.33, 0.81]$. Table 5.2 summarizes the difference between the CPT calibration results in the literature and in the present paper. In particular, we find that the range of

In these two papers, the authors assume the reference point of the representative agent therein to be status quo and the evaluation period to be one month. Another related paper is Polkovnichenko and Zhao (2013), where rank-dependent expected utility (RDEU) theory (Quiggin, 1982), a theory related to CPT, is examined.

Table 5.1: Summary statistics of the calibrated $p$.

| | minimum | 25% quantile | medium | 75% quantile | maximum |
|---|---|---|---|---|---|
| $\lambda = 1.25$ | 0.33 | 0.39 | 0.43 | 0.48 | 0.53 |
| $\lambda = 1.75$ | 0.39 | 0.46 | 0.52 | 0.56 | 0.61 |
| $\lambda = 2.25$ | 0.46 | 0.54 | 0.58 | 0.62 | 0.69 |
| $\lambda = 2.75$ | 0.52 | 0.61 | 0.65 | 0.69 | 0.76 |
| $\lambda = 3.25$ | 0.58 | 0.67 | 0.71 | 0.75 | 0.81 |

Table 5.2: Comparison between our paper and the CPT calibration literature. The first row provides the range of estimates of $p$ in the works summarized in the right panel of Figure 5.1 and in footnote 12. The second and third rows summarize the results in Kliger and Levy (2009) and Gurevich, Kliger, and Levy (2009), where the estimates and the corresponding standard errors (in parentheses) are provided. The last row provides our results.

| | Data | CPT Calibration | | |
|---|---|---|---|---|
| | | Evaluation Period | Reference Point | Diminishing Sensitivity ($p$) |
| Works Summarized in the Right Panel of Figure 5.1 | Experimental Data | Experimental Setting | Designed to be Status Quo | Statistically Estimated in Range [0.22,1.03] |
| Kliger and Levy (2009) | S&P 500 Index Options | Assumed to be One Month | Assumed to be Status Quo | Statistically Estimated at 0.947 (0.009) |
| Gurevich et al. (2009) | U.S. Equity and S&P 500 Index Options | Assumed to be One Month | Assumed to be Status Quo | Statistically Estimated at 0.98 (0.005) |
| This paper | U.S. Hedge Fund Attrition Rate | Performance Fee Payment Date | Performance Fee Benchmark | Calibrated to Attrition Rate in Range [0.33,0.81] |

$p$ obtained for hedge funds in the present paper is quite different from that found in the previous laboratory studies or in empirical studies using other financial data, and it is significantly less than 1.

Finally, regarding the CPT parameters of fund investors, it is notable that most hedge funds require a minimum investment, e.g., $1 million. As hedge fund investors thus also deal with large stakes, they can have similar preferences to those of hedge fund managers. Therefore, we assume the ranges of CPT parameters for hedge fund investors to be the same as those for fund managers; i.e., we assume $\eta \in [1.25, 3.25]$ and $q \in [0.3, 0.8]$.

## 6. APPLICATION: A 10%-30% FIRST-LOSS SCHEME

In this section, we investigate whether the first-loss scheme with a 10% managerial ownership ratio and a carefully chosen incentive rate is better than the traditional scheme with a 10% managerial ownership ratio and a 20% incentive rate, as well as the question of the optimal incentive rate in the first-loss scheme. We choose the same 10% managerial ownership ratio in both schemes because this number is approximately a typical value in both schemes. In addition, 20% is also a typical incentive rate in the traditional scheme.

With any parameter values considered in the following, the loss-gain ratio in the first-loss scheme is higher than in the traditional scheme, thus leading to lower fund risk in the former. Therefore, in the following we only consider the utility changes of the manager and investor when the traditional scheme is switched to the first-loss scheme. According to Theorem 4.3, the utility of the manager is increasing with respect to the incentive rate in the first-loss scheme. In the following numerical studies, the utility of the investor turns out to be decreasing with respect to the incentive rate in the first-loss scheme. We thus define $\widetilde{\alpha}_M^*$ to be the highest incentive rate (in the range [20%,40%]) in the first-loss scheme so that the investor's utility is not reduced when the traditional scheme is replaced with the first-loss scheme. In other words, $\widetilde{\alpha}_M^*$ is the highest incentive rate that the manager can choose in the first-loss scheme without losing investors to the traditional scheme. On the other hand, we define $\widetilde{\alpha}_I^*$ to be the lowest incentive rate (in the range [20%,40%]) in the first-loss scheme so that the manager's utility is not reduced when the traditional scheme is switched to the first-loss scheme; i.e., $\widetilde{\alpha}_I^*$ is the best incentive rate the investor can expect from the manager. Therefore, $\widetilde{\alpha}_M^*$ and $\widetilde{\alpha}_I^*$ can be regarded as the "optimal" incentive rates from the manager's and investor's perspectives, respectively.

In Figure 6.1, we plot $\widetilde{\alpha}_M^*$ and $\widetilde{\alpha}_I^*$. As in Section 5, we set $r = 5\%$. In addition, as discussed in Section 5, the reasonable ranges of $\kappa$, $b$, $\mathcal{R}$, $\lambda$, $\eta$, and $q$ are $[0.3, 0.5]$, $[0.4, 0.6]$, $[10\%, 20\%]$, $[1.25, 3.25]$, $[1.25, 3.25]$, and $[0.3, 0.8]$, so we choose the following benchmark values of these parameters: $\kappa = 0.4$, $b = 0.5$, $\mathcal{R} = 15\%$, $\lambda = 2.25$, $q = 0.6$, and $\eta = 2.25$.

In Figure 6.1, we fix any five of the six parameters $\kappa$, $b$, $\mathcal{R}$, $\lambda$, $\eta$, and $q$, let the remaining parameter vary in its reasonable range, and plot $\widetilde{\alpha}_M^*$ (dash-dotted blue line) and $\widetilde{\alpha}_I^*$ (dashed red line) with respect to this parameter. Note that the manager's and investor's utilities are increasing and decreasing, respectively, with respect to the incentive rate in the first-loss scheme. Thus, the region above the dashed line stands for the incentive rates in the first-loss scheme for which this scheme is preferred over the traditional one by the manager, and the region below the dash-dotted line stands for the incentive rates in the first-loss scheme for which this scheme is preferred over the traditional one by the investor. As a result, the interval $[\widetilde{\alpha}_I^*, \widetilde{\alpha}_M^*]$, which is depicted in Figure 6.1 as the grey region above the dashed line and below the dash-dotted line, stands for the range of incentive rates in the first-loss scheme that make both the manager and investor better off when this scheme replaces the traditional scheme. We can observe that this interval is large, indicating the possibility of improving the satisfaction of both the manager and investor simultaneously. Moreover, with almost all the parameter values under consideration, 30% is within this interval. Therefore, we conclude that a switch from the traditional to a first-loss scheme with a 30% incentive rate renders both the manager and investor better off.

When the incentive rate in the first-loss scheme is set at 40%, however, which is a typical number in the market, the traditional scheme is preferable from the investor's perspective. Indeed, we observe from Figure 6.1 that, with most parameter values, 40% is above the dash-dotted line that represents the highest incentive rate the manager can choose without making investors worse off. Therefore, under typical practice (i.e., with a 40% incentive rate), investors become worse off when switching from the traditional to the first-loss scheme.

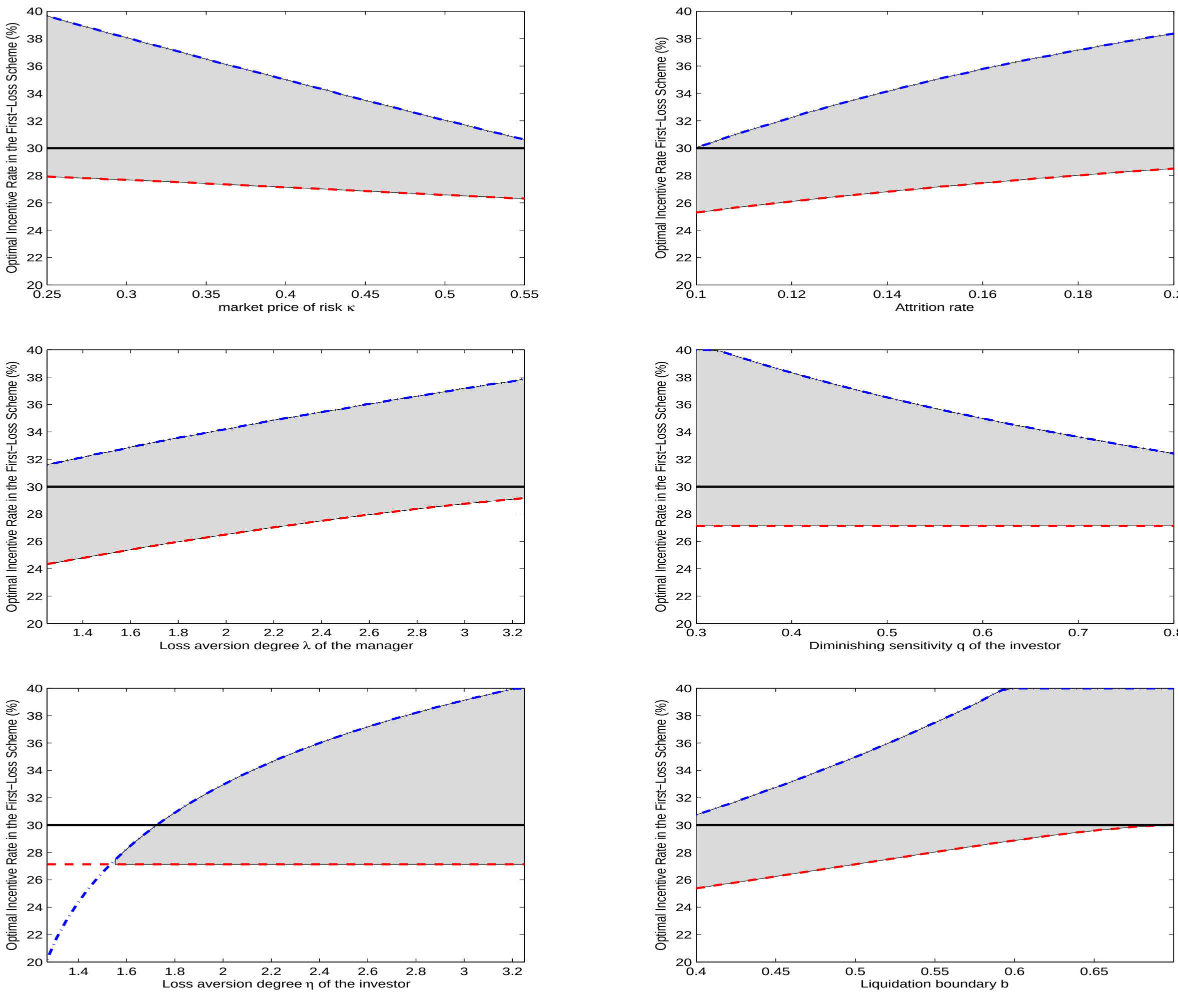


Figure 6.1: Optimal incentive rate $\widetilde{\alpha}_I^*$ from the investor's perspective (dashed red line) and the optimal incentive rate $\widetilde{\alpha}_M^*$ from the manager's perspective (dash-dotted blue line) in the first-loss scheme. The grey region above $\widetilde{\alpha}_I^*$ and below $\widetilde{\alpha}_M^*$ represents the range of the incentive rate in the first-loss scheme with which both the manager and the investor are better off when the first-loss scheme replaces the traditional scheme. The solid black line stands for the 30% incentive rate. In most cases, 30% lies inside the grey region and 40% lies above the grey region, showing that switching to the first-loss scheme makes both the manager and investor better off when the incentive rate in this scheme is 30% and makes the investor worse off when the incentive rate is 40%. The managerial ownership ratios in both schemes are set at 10% and the incentive rate in the traditional scheme is fixed at 20%. The following benchmark values are chosen: $r = 5\%$, $\kappa = 0.4$, $b = 0.5$, $\mathcal{R} = 15\%$, $\lambda = 2.25$, $q = 0.6$, and $\eta = 2.25$.

Next, we perform robust checking by allowing the model parameters to vary simultaneously. More precisely, as in Section 5, we fix $r$ at $5\%$ and choose $\kappa$ to be one of $0.3$, 0.4, and 0.5, $b$ to be one of 0.4, 0.5, and 0.6, and $\mathcal{R}$ to be one of 10%, 12.5%, 15%, 17.5%, and 20%. We set $q$ to be one of 0.3, 0.4, 0.5, 0.6, 0.7, and 0.8, which covers the estimated range of the manager's diminishing sensitivity. We choose both $\lambda$ and $\eta$ to be one of the following four values: 1.75, 2.25, 2.75, and 3.25, which cover the range of loss aversion degrees widely suggested in the literature (as 1.75 is around the 25th percentile in Figure 5.1). For each fixed pair of $(\lambda, \eta)$, we have $3 \times 3 \times 5 \times 6 = 270$ 4-tuples of parameters for $(\kappa, b, \mathcal{R}, q)$. Table 6.1 shows the percentage of the parameter 4-tuples (out of 270 4-tuples) that make both the manager and investor better off simultaneously when the traditional scheme is replaced with the first-loss scheme with incentive rate $\widetilde{\alpha}$, where $\widetilde{\alpha} = 25\%$, $30\%$, $35\%$, or $40\%$. Table 6.1 shows that it is possible to have both the investor and the manager better off at the same time for most parameter 4-tuples. Moreover, it is clear from the table that 30% outperforms 25%, 35%, and 40% as the incentive rate in the first-loss scheme because it leads to the largest number of parameter 4-tuples under which both the investor and the manager can be better off.

## 7. PROFIT SHARING WITH EU MANAGERS

The existing literature on the study of risk taking in hedge funds mainly assumes EU managers. A classical paper in this regard is Carpenter (2000). Although we argue that CPT is a better preference model for hedge fund managers, we compute in the following the optimal strategies of an EU manager in order to compare our results to those found in the literature.

Following Carpenter (2000), we assume the manager to maximize the expected utility of her total wealth, which includes the investment in the fund and her personal wealth. The utility function is assumed to be a power function $x^p/p$ for some $p < 1$.[14] The personal wealth is assumed to be $aX_0$ for some $a > 0$. Then, the manager's total wealth is $aX_0 + wX_0 + \Theta(X_T)$ in the traditional scheme and $aX_0 + \widetilde{w}X_0 + \widetilde{\Theta}(X_T)$ in the first-loss scheme. As a result, the manager chooses trading strategies to maximize $\mathbb{E}[(aX_0 + wX_0 + \Theta(X_T))^p/p]$ and $\mathbb{E}[(aX_0 + \widetilde{w}X_0 + \widetilde{\Theta}(X_T))^p/p]$ in the traditional and first-loss schemes, respectively.

Denote $\zeta := (a + w)/(w + \alpha(1 - w))$ and $\widetilde{\zeta} := (a + \widetilde{w})/(\widetilde{w} + \widetilde{\alpha}(1 - \widetilde{w}))$. Denote

$$\begin{aligned} U(x) &:= [(w + \alpha(1 - w))^p(\zeta + x - 1)^p/p]\, \mathbf{1}_{\{x \geq 1\}} + [(a + wx)^p/p]\, \mathbf{1}_{\{x<1\}}, \\ \widetilde{U}(x) &:= \left[(\widetilde{w} + \widetilde{\alpha}(1 - \widetilde{w}))^p(\widetilde{\zeta} + x - 1)^p/p\right] \mathbf{1}_{\{x \geq 1\}} + [(a + \max(\widetilde{w} + x - 1, 0))^p/p]\, \mathbf{1}_{\{x<1\}}. \end{aligned}$$

Then, the manager maximizes $\mathbb{E}[U(X_T/X_0)]$ and $\mathbb{E}[\widetilde{U}(X_T/X_0)]$ in the traditional and first-loss schemes, respectively. Therefore, $U$ and $\widetilde{U}$ are the effective utility functions of the manager in these two schemes, respectively.

**Theorem 7.1.** *Assume $b \leq 1 - \widetilde{w}$.*

*(i) There exists a unique line passing through $(b, \widetilde{U}(b))$ and tangent to $\widetilde{U}(\cdot)$. Denote $\hat{x}$ as the point of tangency and $\widetilde{k}$ as the slope of the tangent line. Then $\hat{x} > b$, and $\hat{x} = 1$ ($\hat{x} > 1$ and $\hat{x} < 1$,*

[14] When $p = 0$, the utility function is defined as $\log(x)$.

Table 6.1: Percentage of the 270 parameter 4-tuples $(\kappa, b, \mathcal{R}, q)$ with which both the manager and investor are better off when the traditional scheme is replaced with the first-loss scheme. The managerial ownership ratio is 10% for both the traditional and first-loss schemes. The incentive rate is 20% in the traditional scheme and is chosen to be one of 25%, 30%, 35%, and 40% in the first-loss scheme. Both $\lambda$ and $\eta$ take four values: 1.75, 2.25, 2.75, and 3.25, which cover the range of loss aversion degrees widely suggested in the literature (as 1.75 is around the 25th percentile in Figure 5.1).

| | | $\eta = 1.75$ | $\eta = 2.25$ | $\eta = 2.75$ | $\eta = 3.25$ |
|---|---|---|---|---|---|
| first-loss scheme with a 25% incentive rate | $\lambda = 1.75$ | 24% | 34% | 36% | 36% |
| | $\lambda = 2.25$ | 9% | 14% | 16% | 16% |
| | $\lambda = 2.75$ | 2% | 4% | 4% | 4% |
| | $\lambda = 3.25$ | 0% | 0% | 0% | 0% |
| | | $\eta = 1.75$ | $\eta = 2.25$ | $\eta = 2.75$ | $\eta = 3.25$ |
| first-loss scheme with a 30% incentive rate | $\lambda = 1.75$ | 48% | 76% | 90% | 97% |
| | $\lambda = 2.25$ | 49% | 76% | 87% | 92% |
| | $\lambda = 2.75$ | 44% | 66% | 77% | 79% |
| | $\lambda = 3.25$ | 40% | 62% | 71% | 73% |
| | | $\eta = 1.75$ | $\eta = 2.25$ | $\eta = 2.75$ | $\eta = 3.25$ |
| first-loss scheme with a 35% incentive rate | $\lambda = 1.75$ | 23% | 41% | 57% | 70% |
| | $\lambda = 2.25$ | 28% | 51% | 67% | 78% |
| | $\lambda = 2.75$ | 34% | 58% | 75% | 85% |
| | $\lambda = 3.25$ | 42% | 65% | 80% | 89% |
| | | $\eta = 1.75$ | $\eta = 2.25$ | $\eta = 2.75$ | $\eta = 3.25$ |
| first-loss scheme with a 40% incentive rate | $\lambda = 1.75$ | 8% | 19% | 32% | 42% |
| | $\lambda = 2.25$ | 13% | 27% | 40% | 51% |
| | $\lambda = 2.75$ | 17% | 33% | 47% | 56% |
| | $\lambda = 3.25$ | 22% | 38% | 51% | 58% |

*respectively) if and only if $\widetilde{k} \in [\widetilde{U}'(1+), \widetilde{U}'(1-)]$ ($\widetilde{k} < \widetilde{U}'(1+)$ and $\widetilde{k} > \widetilde{U}'(1-)$, respectively). Furthermore, the terminal fund value in the first-loss scheme is*

$$
\begin{aligned}
X_T^* =& \Big\{ \Big[(\widetilde{\zeta} + \max(\hat{x}, 1) - 1) \left( (\xi_T/(\widetilde{\tau}_2\widetilde{\nu}^*))^{1/(p-1)} - 1 \right) + \max(\hat{x}, 1) \Big] \mathbf{1}_{\{\xi_T < \widetilde{\tau}_2\widetilde{\nu}^*\}} + \mathbf{1}_{\{\widetilde{\tau}_2\widetilde{\nu}^* \le \xi_T \le \widetilde{\tau}_1\widetilde{\nu}^*\}} \\
&+ \left[(a + \widetilde{w}) \left( (\xi_T/(\widetilde{\tau}_1\widetilde{\nu}^*))^{1/(p-1)} - 1 \right) + 1 \right] \mathbf{1}_{\{\widetilde{\tau}_1\widetilde{\nu}^* < \xi_T < \widetilde{\nu}^*\}} + b\mathbf{1}_{\{\xi_T \ge \widetilde{\nu}^*\}} \Big\} X_0,
\end{aligned}
$$

*where $\widetilde{\tau}_1 := \min(\widetilde{U}'(1-)/\widetilde{k}, 1)$, $\widetilde{\tau}_2 := \min(\widetilde{U}'(1+)/\widetilde{k}, 1)$, and $\widetilde{\nu}^* > 0$ is determined by $\mathbb{E}[\xi_T X_T^*] = X_0$.*

*(ii) There exist unique $\hat{x}_1 \in [b, 1)$ and $\hat{x}_2 > 1$ such that $U(x)\mathbf{1}_{\{x \notin (\hat{x}_1, \hat{x}_2)\}} + (U(\hat{x}_1) + k(x - \hat{x}_1))\mathbf{1}_{\{x \in (\hat{x}_1, \hat{x}_2)\}}$ is the concave envelop of $U(\cdot)$ on $[b, +\infty)$, where $k := (U(\hat{x}_2) - U(\hat{x}_1))/(\hat{x}_2 - \hat{x}_1)$. Moreover, $\hat{x}_1 = b$ ($\hat{x}_1 > b$) if and only if $k \ge U'(b)$ ($k < U'(b)$). Furthermore, the terminal fund value in the traditional scheme is*

$$
\begin{aligned}
X_T^* = \Big\{ & \left[(\zeta + \hat{x}_2 - 1) \left( (\xi_T/\nu^*)^{1/(p-1)} - 1 \right) + \hat{x}_2 \right] \mathbf{1}_{\{\xi_T < \nu^*\}} \\
&+ \left[ \left( \frac{a}{w} + \hat{x}_1 \right) \left( (\xi_T/\nu^*)^{1/(p-1)} - 1 \right) + \hat{x}_1 \right] \mathbf{1}_{\{\nu^* < \xi_T < \tau\nu^*\}} + b\mathbf{1}_{\{\xi_T \ge \tau\nu^*\}} \Big\} X_0,
\end{aligned}
$$

*where $\tau := \max(U'(b)/k, 1)$ and $\nu^* > 0$ is determined by $\mathbb{E}[\xi_T X_T^*] = X_0$.*

We conclude the following from Theorem 7.1:

(1) With CPT managers, the fund does not suffer any loss unless it is liquidated; with EU managers, however, the fund can suffer a loss other than being liquidated. To see this, note that, from Theorem 7.1, in the first-loss scheme with EU managers, the optimal terminal asset value $X_T^* \ge \max(\hat{x}, 1)X_0 \ge X_0$ on $\{\xi_T < \widetilde{\tau}_2\widetilde{\nu}^*\}$, $X_T^* = X_0$ on $\{\widetilde{\tau}_2\widetilde{\nu}^* \le \xi_T \le \widetilde{\tau}_1\widetilde{\nu}^*\}$, and $X_T^* = bX_0$ on $\{\xi_T \ge \widetilde{\nu}^*\}$. When $\widetilde{\tau}_1 < 1$, i.e., when $\widetilde{U}'(1-) < k$, we have $\hat{x} < 1$ and

$$
\begin{aligned}
X_T^* &= [(a + \widetilde{w}) \left( (\xi_T/(\widetilde{\tau}_1\widetilde{\nu}^*))^{1/(p-1)} - 1 \right) + 1]X_0 \\
&= [(a + \widetilde{w} + \hat{x} - 1) \left( (\xi_T/\widetilde{\nu}^*)^{1/(p-1)} - 1 \right) + \hat{x}]X_0 \\
&\in (\hat{x}X_0, X_0)
\end{aligned}
$$

on $\{\widetilde{\tau}_1\widetilde{\nu}^* < \xi_T < \widetilde{\nu}^*\}$. Therefore, in this case, the fund can lose any amount between 0 and $(1 - \hat{x})X_0$. Similarly, in the traditional scheme with EU managers, the optimal terminal asset value $X_T^* \ge \hat{x}_2 X_0 > X_0$ on $\{\xi_T < \nu^*\}$, and $X_T^* = bX_0$ on $\{\xi_T \ge \tau\nu^*\}$. When $\tau > 1$, i.e., when $k < U'(b)$, we have $\hat{x}_1 \in (b, 1)$ and

$$
\begin{aligned}
X_T^* &= \left[ \left( \frac{a}{w} + \hat{x}_1 \right) \left( (\xi_T/\nu^*)^{1/(p-1)} - 1 \right) + \hat{x}_1 \right] X_0 \\
&= \left[ \left( \frac{a}{w} + b \right) \left( (\xi_T/(\tau\nu^*))^{1/(p-1)} - 1 \right) + b \right] X_0 \\
&\in (bX_0, \hat{x}_1 X_0)
\end{aligned}
$$

on $\{\nu^* < \xi_T < \tau\nu^*\}$. Therefore, in this case, the fund can lose any amount between $(1 - \hat{x}_1)X_0$ and $(1 - b)X_0$.

As observed in Figure 2.1, when the fund is in the loss region, the managerial contract is linear in the traditional scheme and convex in the first-loss scheme with respect to the fund asset value.

Intuitively, because CPT managers are risk seeking with respect to losses, when these managers are at a loss, they may consider taking further risk, which may either lead to the liquidation of the fund or bring the fund asset value into the gain territory. On the other hand, when EU managers are at a loss, they may avoid taking further risk if they are strongly risk averse, and, as a result, the fund asset value may stay at a certain loss level without touching the liquidation boundary.

(2) With CPT managers, in both schemes the managerial ownership ratio and incentive rate combine into the loss-gain ratio to determine the optimal fund asset value; with EU managers, however, the loss-gain ratio is insufficient to determine the fund asset value. For instance, in the first-loss scheme with EU managers, Theorem 7.1 shows that the fund asset value depends on $\widetilde{\zeta}$, $\hat{x}$, $\widetilde{\tau}_1$, and $\widetilde{\tau}_2$; the values of these quantities can be different for different pairs of the managerial ownership ratio and incentive rate, even if these pairs imply the same loss-gain ratio.

(3) With CPT managers, the traditional and first-loss schemes cannot be differentiated if the loss-gain ratio is the same in these two schemes; with EU managers, however, the fund asset values can be qualitatively different in these two schemes. To show this, consider the case of EU managers and recall Theorem 7.1. Suppose $\widetilde{\tau}_1 < 1$ in the first-loss scheme and $\tau > 1$ in the traditional scheme. We have concluded that, except for the case of liquidation, the fund can only suffer small losses (with magnitude between 0 and $(1-\hat{x})X_0$) in the first-loss scheme and can only suffer large losses (with magnitude between $(1-\hat{x}_1)X_0$ and $(1-b)X_0$) in the traditional scheme.

The reason the traditional and first-loss schemes can be qualitatively differentiated with EU managers can be explained intuitively as follows: Imagine that the fund has already suffered a large loss and is about to be liquidated, and an EU manager decides whether to take further risk. Note that taking further risk may deepen the fund loss, which may eventually force the fund to liquidate, or bring the fund value to a small-loss position or even to a gain position. In the traditional scheme, if the fund loss deepens, so does the manager's loss, and thus the manager chooses not to take further risk due to risk aversion. In the first-loss scheme, the manager does not lose extra money if the fund loss deepens because the first-loss capital is already exhausted. As a result, she chooses to take further risk. On the other hand, imagine that the fund just suffered a small loss. Again, the EU manager decides whether to take further risk, which will either deepen the fund loss or bring the fund asset value to a gain position. In the traditional scheme, the manager may do so because her loss per one dollar fund loss is strictly less than her gain per one dollar fund gain. In the first-loss scheme, however, she chooses not to do so because her loss per one dollar fund loss is strictly larger than her gain per one dollar fund gain (because she covers the fund loss completely).

(4) With CPT managers, we can compare the traditional and first-loss schemes in terms of fund risk and the manager's utility analytically; with EU managers, however, such analytical comparison cannot be obtained. Indeed, with an EU manager, we can also explicitly compute the portfolios that replicate the optimal terminal asset values of the fund under both schemes. However, the optimal utility of the manager and the investor cannot be computed in closed form. In addition, the fund risk in these two schemes cannot be easily compared because the fund loss is qualitatively different in these two schemes: except for the case of liquidation, the fund can only suffer small and large losses under the first-loss and traditional schemes, respectively.

# 8. CONCLUSION

We have proposed an analytical framework to compare two managerial incentive schemes for hedge funds: traditional and first-loss. In the traditional scheme, which is currently used in most U.S. funds, the manager takes part of the investors' profit as a performance fee but does not cover investors' losses. In the first-loss scheme, which is common in China and currently emerging in the United States, the manager uses her investment in the fund to cover investors' losses first. Assuming CPT preferences for the manager, we have obtained closed-form formulae for the trading strategies, fund risk, and manager's and investors' utilities. In particular, we found that the optimal trading strategy is fully determined by the loss-gain ratio of the manager, regardless of the scheme being used. Consequently, the fund risk is also fully determined by the loss-gain ratio: to compare the fund risk in the traditional and first-loss schemes, we only need to compare the loss-gain ratio in these two schemes. The analytical comparison of the manager's utility in the traditional and first-loss schemes shows that one can reduce the fund risk and improve the manager's utility at the same time by replacing the traditional with the first-loss scheme.

We have used the historical attrition rates of hedge funds to calibrate utility parameters in the model. The calibrated parameter values turn out to be smaller than the estimates that have been arrived at using experimental data in the literature. Assuming reasonable parameter values, we have found that one can reduce the fund risk and improve the utilities of fund managers and investors at the same time by substituting the traditional scheme (with 10% internal capital and 20% performance fee) with the first-loss scheme (with 10% first-loss capital and 30% performance fee). When the performance fee in the first-loss scheme is 40%, a current market practice, however, such substitution makes the investors worse off.

We have also obtained the trading strategies when the manager has EU preferences. In contrast to the case of CPT managers, we found that the loss-gain ratio cannot fully determine the trading strategies for EU managers. In addition, with CPT managers, the fund does not suffer any loss unless it is liquidated. With EU managers, however, the fund can suffer small losses but not large losses without being liquidated under the first-loss scheme, but the opposite is true under the traditional scheme.

# APPENDIX A: PROOFS

*Proof of Theorem 3.1* We give the proof only for the case of the first-loss scheme, that of the traditional scheme being similar. The proof consists of three steps. First, apply the standard martingale approach (Karatzas and Shreve, 1998) to convert the dynamic asset allocation problem (2.5) to an optimization problem in which the optimal terminal asset value is to be identified. Secondly, apply the concavification technique employed in Carpenter (2000) and Kouwenberg and Ziemba (2007) to find the optimal terminal asset value. Finally, find the trading strategy that leads to the optimal terminal asset value.

According to the standard approach to portfolio selection problems (Karatzas and Shreve, 1998), the optimization problem (2.6) is equivalent to

$$\begin{aligned} &\underset{X_T}{\text{Max}} && \mathbb{E}\left[u(\widetilde{\Theta}(X_T))\right], \\ &\text{Subject to} && X_T \in \mathcal{F}_T,\ \mathbb{E}\left[\xi_T X_T\right] \le X_0,\ X_T \ge bX_0, \end{aligned} \tag{A.1}$$

where $X_T$ stands for the terminal wealth of a certain portfolio. Next, we solve (A.1) by the standard Lagrange dual method in Karatzas and Shreve (1998) together with the concavification technique in Carpenter (2000) and Kouwenberg and Ziemba (2007).

Let $x^*$ be the tangent point of the line starting at $(bX_0, u(\widetilde{\Theta}(bX_0)))$ to the curve $u(\widetilde{\Theta}(x)), x \ge bX_0$. Because $u(\widetilde{\Theta}(\cdot))$ is convex in $[bX_0, X_0]$ and $\lim_{x\downarrow X_0} u'(\widetilde{\Theta}(x))\widetilde{\Theta}'(x) = +\infty$, the tangent point $x^*$ must be strictly larger than $X_0$ and satisfy the following equation: $u'(\widetilde{\Theta}(x^*))\widetilde{\Theta}'(x^*)(x^* - bX_0) = u(\widetilde{\Theta}(x^*)) - u(\widetilde{\Theta}(bX_0))$. By the change-of-variable $x^* = (1+\widetilde{c}^*)X_0$, it is easy to see that the above equation is equivalent to (3.6). Moreover, one can check that (3.6) admits a unique solution. Thus, the tangent point is $x^* = (1+\widetilde{c}^*)X_0$ and the slope of the tangent line is $k := p(\widetilde{w} + \widetilde{\alpha}(1-\widetilde{w}))^p(\widetilde{c}^* X_0)^{p-1}$. Then, it is easy to verify that

$$\underset{x\ge bX_0}{\text{argmax}}\left[u(\widetilde{\Theta}(x)) - yx\right] = \begin{cases} X_0 + \left(\frac{y}{p(\widetilde{w}+\widetilde{\alpha}(1-\widetilde{w}))^p}\right)^{\frac{1}{p-1}}, & 0 < y < k, \\ \{bX_0, (1+\widetilde{c}^*)X_0\}, & y = k, \\ bX_0, & y > k. \end{cases} \tag{A.2}$$

Consider the following problem:

$$\begin{aligned} &\underset{X_T}{\text{Max}} && \mathbb{E}\left[u(\widetilde{\Theta}(X_T)) - \tfrac{k}{\nu}\xi_T X_T\right] + \tfrac{k}{\nu}X_0 \\ &\text{Subject to} && X_T \in \mathcal{F}_T,\ X_T \ge bX_0, \end{aligned} \tag{A.3}$$

where $\nu > 0$ is a Lagrange multiplier. It can be easily verified from (A.2) that

$$X_T^*(\nu) := \left[b\mathbf{1}_{\{\xi_T > \nu\}} + \left(1 + \widetilde{c}^*\left(\frac{\xi_T}{\nu}\right)^{\frac{1}{p-1}}\right)\mathbf{1}_{\{\xi_T \le \nu\}}\right]X_0 \tag{A.4}$$

is an optimal solution to (A.3). Moreover, it is the unique optimal solution because the event $\{\xi_T = \nu\}$ has zero probability.

Denote by $V(\nu)$ the optimal value of (A.3). Weak duality immediately leads to $\inf_{\nu>0} V(\nu) \ge V$, where $V$ is the optimal value of (A.1). On the other hand, because $\xi_T$ has no atom, $\mathcal{X}(\nu) := \mathbb{E}\left[\xi_T X_T^*(\nu)\right]$ is continuous and strictly increasing in $\nu$. Moreover, $\lim_{\nu\to 0}\mathcal{X}(\nu) = bX_0$ and $\lim_{\nu\to\infty}\mathcal{X}(\nu) = +\infty$. Thus, there exists a unique $\widetilde{\nu}^*$ such that $\mathbb{E}\left[\xi_T X_T^*(\widetilde{\nu}^*)\right] = X_0$. As a result, $X_T^*(\widetilde{\nu}^*)$ is feasible to (A.1). Therefore, we have

$$\inf_{\nu>0} V(\nu) \ge V \ge \mathbb{E}\left[u(\widetilde{\Theta}(X_T^*(\widetilde{\nu}^*)))\right] = \mathbb{E}\left[u(\widetilde{\Theta}(X_T^*(\widetilde{\nu}^*))) - \frac{k}{\widetilde{\nu}^*}\xi_T X_T^*(\widetilde{\nu}^*)\right] + \frac{k}{\widetilde{\nu}^*}X_0 = V(\widetilde{\nu}^*) \ge \inf_{\nu>0} V(\nu).$$

As a result, $X_T^*(\widetilde{\nu}^*)$ is an optimal solution to (A.1). Moreover, it is easy to see from the above dual argument that any optimal solution to (A.1) must be optimal to (A.3) with the multiplier $\widetilde{\nu}^*$. Because the optimal solution to (A.3) is unique, so is that of (A.1).

Finally, we find the optimal portfolio. Using the standard theory in portfolio selection literature (see, for instance, Karatzas and Shreve, 1998), we obtain $X_t^* = \xi_t^{-1}\mathbb{E}\left[\xi_T X_T^* \mid \mathcal{F}_t\right], 0 \le t \le T$. Recalling that $X_T^* = X_T^*(\widetilde{\nu}^*)$, as in (A.4), and the definition of $\xi_t, t \ge 0$, we have

$$X_t^* = \xi_t^{-1}\mathbb{E}\left[\xi_T\left(b\mathbf{1}_{\{\xi_T>\widetilde{\nu}^*\}} + \left(1+\widetilde{c}^*\left(\frac{\xi_T}{\widetilde{\nu}^*}\right)^{\frac{1}{p-1}}\right)\mathbf{1}_{\{\xi_T\le\widetilde{\nu}^*\}}\right)X_0 \mid \mathcal{F}_t\right]$$
$$= \mathbb{E}\left[Z_{t,T}\left(b\mathbf{1}_{\{\xi_t Z_{t,T}>\widetilde{\nu}^*\}} + \left(1+\widetilde{c}^*\left(\frac{\xi_t Z_{t,T}}{\widetilde{\nu}^*}\right)^{\frac{1}{p-1}}\right)\mathbf{1}_{\{\xi_t Z_{t,T}\le\widetilde{\nu}^*\}}\right)X_0 \mid \mathcal{F}_t\right]$$

where $Z_{t,T} := \exp\left[-\left(r+\frac{\kappa^2}{2}\right)(T-t) - \kappa(W_T - W_t)\right]$. Because $Z_{t,T}$ is independent of $\mathcal{F}_t$, we have $X_t^* = f(t,\xi_t)$ where $f(t,\xi) := \mathbb{E}\left[Z_{t,T}\left(b\mathbf{1}_{\{\xi Z_{t,T}>\widetilde{\nu}^*\}} + \left(1+\widetilde{c}^*\left(\frac{\xi Z_{t,T}}{\widetilde{\nu}^*}\right)^{\frac{1}{p-1}}\right)\mathbf{1}_{\{\xi Z_{t,T}\le\widetilde{\nu}^*\}}\right)X_0\right]$. It is obvious to see that $f$ is strictly decreasing in $\xi$, and some tedious calculation leads to (3.3). To find the optimal portfolio, we apply Itô's lemma to $X_t^* = f(t,\xi_t)$ and compare it with (2.1). Then, we have $\pi_t^* = \sigma^{-1}\left(-\kappa\xi_t\frac{\partial f}{\partial\xi}(t,\xi_t)\right)$. After some tedious but straightforward calculation, we derive (3.4). □

*Proof of Theorem 4.1* We prove it for the case of the first-loss scheme. The case of the traditional scheme can be treated similarly.

We first show that the risk of the fund, defined as the loss probability $\widetilde{\mathcal{R}}$, is strictly decreasing in $\widetilde{\gamma}$. From Theorem 3.1, fixing $b$ and the utility parameters $p$ and $\lambda$, $\widetilde{\nu}^*$ is determined by $\widetilde{c}^*$, the latter depending on $\widetilde{\gamma}$. We first show that $\widetilde{\nu}^*$ is strictly decreasing in $\widetilde{c}^*$. Recall $X_T^*(\nu)$ in the proof of Theorem 3.1. It is easy to see that $X_T^*(\nu)$ is strictly increasing in $\widetilde{c}^*$ and $\nu$ almost surely. As a result, $\widetilde{\nu}^*$, which is determined by $\mathbb{E}\left[\xi_T X_T^*(\widetilde{\nu}^*)\right] = X_0$, is strictly decreasing in $\widetilde{c}^*$. On the other hand, it is easy to see that $\widetilde{c}^*$ is strictly decreasing in $\widetilde{\gamma}$. As a consequence, $\widetilde{\nu}^*$ is strictly increasing in $\widetilde{\gamma}$ and thus $\widetilde{\mathcal{R}}$ is strictly decreasing in $\widetilde{\gamma}$.

We then show that the asset volatility is strictly decreasing in $\widetilde{\gamma}$ as well. To this purpose, let us recall the optimal percentage allocation to the risky asset in Theorem 3.1. We only need to show that $\varphi(t,X_t) := \frac{\pi_t^*}{X_t^*}$ is strictly decreasing in $\widetilde{\gamma}$. Without loss of generality, we can show it only for $t=0$. To simplify notation, let $d_i := d_{i,0}, i = 1,2$. Taking derivatives with respect to (w.r.t.) $\widetilde{c}^*$ on both sides of (3.4), we have

$$\frac{\partial}{\partial\widetilde{c}^*}\varphi = e^{-rT}\sigma^{-1}\kappa\frac{1}{\kappa\sqrt{T}}\Phi'(d_1)\left\{1 - \left[\widetilde{c}^* d_1 + (1-b)d_2\right]\frac{\partial d_1}{\partial\widetilde{c}^*}\right\}.$$

On the other hand, from the optimal wealth (3.3), $d_1$ is determined by $\widetilde{c}^*$ according to

$$1 = e^{-rT}\left[b + (1-b)\Phi(d_1) + \widetilde{c}^*\frac{\Phi'(d_1)}{\Phi'(d_2)}\Phi(d_2)\right].$$

Taking derivatives w.r.t. $\widetilde{c}^*$ on both sides, we have

$$\left[(1-b) + \widetilde{c}^* + \widetilde{c}^*(d_2-d_1)\frac{\Phi(d_2)}{\Phi'(d_2)}\right]\frac{\partial d_1}{\partial\widetilde{c}^*} + \frac{\Phi(d_2)}{\Phi'(d_2)} = 0.$$

Therefore,

$$\begin{aligned}
&\left[(1-b)+\widetilde{c}^*+\widetilde{c}^*(d_2-d_1)\frac{\Phi(d_2)}{\Phi'(d_2)}\right]\cdot\left\{1-\left[\widetilde{c}^*d_1+(1-b)d_2\right]\frac{\partial d_1}{\partial\widetilde{c}^*}\right\}\\
=&(1-b)+\widetilde{c}^*+\widetilde{c}^*(d_2-d_1)\frac{\Phi(d_2)}{\Phi'(d_2)}+\left[\widetilde{c}^*d_1+(1-b)d_2\right]\frac{\Phi(d_2)}{\Phi'(d_2)}\\
=&(1-b+\widetilde{c}^*)\left(1+d_2\frac{\Phi(d_2)}{\Phi'(d_2)}\right)>0,
\end{aligned}$$

where the inequality is the case because $1+x\Phi(x)/\Phi'(x)>0$ for all $x\in\mathbb{R}$. Thus, $\frac{\partial}{\partial\widetilde{c}^*}\varphi>0$. Finally, because we have shown that $\widetilde{c}^*$ is strictly decreasing in $\widetilde{\gamma}$, the optimal percentage allocation $\varphi$ is strictly decreasing in $\widetilde{\gamma}$. $\square$

*Proof of Proposition 4.2* We only consider the case of the first-loss scheme. The case of the traditional scheme can be treated similarly.

We first prove the monotonicity of the loss probability $\widetilde{\mathcal{R}}$ w.r.t. $b$. Denote $\widetilde{z}^*:=\widetilde{c}^*/(1-b)$. Then, $\widetilde{z}^*$ is solved by

$$(1-p)(\widetilde{z}^*)^p-p(\widetilde{z}^*)^{p-1}+\lambda\left[\min(\widetilde{w},1-b)/\big((\widetilde{w}+\widetilde{\alpha}(1-\widetilde{w}))(1-b)\big)\right]^p=0.$$

One can see that $\widetilde{z}^*$ is strictly decreasing in $b\in[0,1-\widetilde{w}]$ and is constant w.r.t. $b\in[1-\widetilde{w},1)$. On the other hand,

$$X_T^*=X_0+(1-b)\left[-\mathbf{1}_{\{\xi_T>\widetilde{\nu}^*\}}+\widetilde{z}^*\left(\xi_T/\widetilde{\nu}^*\right)^{\frac{1}{p-1}}\mathbf{1}_{\{\xi_T\le\widetilde{\nu}^*\}}\right]X_0$$

and $\widetilde{\nu}^*$ is determined by $\mathbb{E}[\xi_TX_T^*]=X_0$, i.e., by

$$\mathbb{E}\left[\xi_T\left(-\mathbf{1}_{\{\xi_T>\widetilde{\nu}^*\}}+\widetilde{z}^*\left(\xi_T/\widetilde{\nu}^*\right)^{\frac{1}{p-1}}\mathbf{1}_{\{\xi_T\le\widetilde{\nu}^*\}}\right)\right]=\frac{1-\mathbb{E}[\xi_T]}{1-b}=\frac{1-e^{-rT}}{1-b}. \tag{A.5}$$

One can see that the random variable inside the expectation on the left-hand side of (A.5) is strictly increasing in $\widetilde{\nu}^*$ and $\widetilde{z}^*$. The right-hand side is increasing w.r.t. $b$ and is strictly increasing w.r.t. $b$ when the risk-free rate $r>0$. Recalling the monotonicity of $\widetilde{z}^*$ w.r.t. $b$, we conclude that $\widetilde{\nu}^*$ is increasing w.r.t. $b\in[0,1)$ and the monotonicity becomes strict if and only if $r>0$ or $b\in[0,1-\widetilde{w}]$. As a result, $\widetilde{\mathcal{R}}$ is decreasing w.r.t. $b\in[0,1)$ and the monotonicity becomes strict if and only if $r>0$ or $b\in[0,1-\widetilde{w}]$.

Next, we study the loss probability when $b$ goes to 1. Note that $\widetilde{z}^*$ is constant for $b\ge1-\widetilde{w}$. Thus, we conclude from (A.5) that when $b$ goes to 1, $\widetilde{\nu}^*$ remains constant if $r=0$ and tends to infinity if $r>0$. Therefore, when $b$ goes to 1, the limit of the loss probability $\widetilde{\mathcal{R}}$ is positive when $r=0$ and is zero when $r>0$.

Finally, we show that the asset volatility is strictly decreasing w.r.t. $b$. As in the proof of Theorem 4.1, we only need to show that $\pi_0^*/X_0^*$ is strictly decreasing w.r.t. $b$. To simplify notation, let $d_i:=d_{i,0}$, $i=1,2$. Taking derivatives w.r.t. $b$ on both sides of (3.4) with $t=0$, we have

$$\begin{aligned}
e^{rT}\sigma\sqrt{T}\frac{\partial(\pi_0^*/X_0^*)}{\partial b}=&-\left[(\widetilde{z}^*+1)\Phi'(d_1)+(d_2-d_1)(1-\Phi(d_1))\right]\\
&+(1-b)\Phi'(d_1)\left[\frac{\partial\widetilde{z}^*}{\partial b}-(\widetilde{z}^*d_1+d_2)\frac{\partial d_1}{\partial b}\right].
\end{aligned}$$

On the other hand, taking derivatives w.r.t. $b$ on both sides of (3.3) with $t = 0$, we obtain

$$\begin{aligned}
&-(1-b)\Phi'(d_1)\frac{\partial d_1}{\partial b}\left(1+\widetilde{z}^* + \widetilde{z}^*(d_2-d_1)\frac{\Phi(d_2)}{\Phi'(d_2)}\right)\\
=&(1-b)\frac{\Phi'(d_1)}{\Phi'(d_2)}\Phi(d_2)\frac{\partial \widetilde{z}^*}{\partial b} + 1 - \Phi(d_1) - \widetilde{z}^*\frac{\Phi'(d_1)}{\Phi'(d_2)}\Phi(d_2).
\end{aligned}$$

Consequently, tedious but straightforward calculation yields

$$\begin{aligned}
&\left(1+\widetilde{z}^* + \widetilde{z}^*(d_2-d_1)\frac{\Phi(d_2)}{\Phi'(d_2)}\right)e^{rT}\sigma\sqrt{T}\frac{\partial(\pi_0/X_0)}{\partial b}\\
=&-(d_2-d_1)\left[\widetilde{z}^*(1-\Phi(d_1)) + \widetilde{z}^*\frac{\Phi'(d_1)}{\Phi'(d_2)}\Phi(d_2) + \widetilde{z}^*(d_2-d_1)(1-\Phi(d_1))\frac{\Phi(d_2)}{\Phi'(d_2)}\right]\\
&-(1+\widetilde{z}^*)\Phi'(-d_1)\left[1+\widetilde{z}^* + (-d_1)\frac{\Phi(-d_1)}{\Phi'(-d_1)} + \widetilde{z}^* d_2\frac{\Phi(d_2)}{\Phi'(d_2)}\right]\\
&+(1-b)\Phi'(d_1)(1+\widetilde{z}^*)\frac{\partial \widetilde{z}^*}{\partial b}\left[1+d_2\frac{\Phi(d_2)}{\Phi'(d_2)}\right]\\
<&0,
\end{aligned}$$

where the inequality is the case because $d_2 > d_1$, $\widetilde{z}^* > 0$, $1 + x\Phi(x)/\Phi'(x) > 0$ for all $x \in \mathbb{R}$, and $\partial\widetilde{z}^*/\partial b \le 0$. Therefore, the asset volatility is strictly decreasing w.r.t. $b$. $\square$

*Proof of Theorem 4.3* Note that the manager's profit is strictly increasing in the incentive rate and the fund makes a profit in good market scenarios, so the manager's utility must be strictly increasing in the incentive rate. On the other hand, for any $b_1 < b_2$, denote the corresponding unique fund asset values as $X^*_{1,T}$ and $X^*_{2,T}$, respectively. It is obvious that $X^*_{2,T}$, which is optimal with liquidation boundary $b_2$, is also feasible with liquidation boundary $b_1$, so the manager's utility must be strictly decreasing in the liquidation boundary.

*(i)* Let $X^*_T$ be the optimal terminal asset value of (2.5). Recalling $X_0 = 1/w$, we have $\Theta(X^*_T) = \frac{1-b}{\gamma}\left(\frac{X^*_T}{X_0} - 1\right)$ if $\frac{X^*_T}{X_0} \ge 1$ and $\Theta(X^*_T) = -(1-b)$ otherwise. Tedious but straightforward calculation yields that the utility per capital, $\mathbb{E}\left[u(\Theta_U(X^*_T))\right]$, is given by (4.1). Next, we show the monotonicity in $\gamma$. By the change-of-variable $\bar{X}_T := wX_T$, and recalling $X_0 = 1/w$, we observe that the utility per capital, $\mathbb{E}\left[u(\Theta_U(X^*_T))\right]$, is the optimal value of

$$\begin{aligned}
\underset{\bar{X}_T}{\text{Max}} \quad & \mathbb{E}\left\{u\left[\tfrac{1-b}{\gamma}\left(\bar{X}_T - 1\right)\mathbf{1}_{\{\bar{X}_T \ge 1\}} - (1-\bar{X}_T)\mathbf{1}_{\{\bar{X}_T < 1\}}\right]\right\},\\
\text{Subject to} \quad & \bar{X}_T \in \mathcal{F}_T, \quad \mathbb{E}\left[\xi_T\bar{X}_T\right] = 1, \quad \bar{X}_T \ge b.
\end{aligned}$$

Because the constraints do not depend on $\gamma$ and the objective function is strictly decreasing in $\gamma$, we conclude that the optimal value $\mathbb{E}\left[u(\Theta_U(X^*_T))\right]$ is strictly decreasing in $\gamma$.

*(ii)* Using the similar argument in (i), we can show that the utility per capital of the manager in the first-loss scheme is given by (4.2). Moreover, fixing $b$, the utility per capital depends only on $\widetilde{\gamma}$. On the other hand, by the change-of-variable $\bar{X}_T := \widetilde{w}X_T$ and recalling $X_0 = \frac{1}{\widetilde{w}}$, we observe

that the utility per capital is the optimal value of

$$\begin{aligned} \underset{\bar{X}_T}{\text{Max}} \quad & \mathbb{E}\left\{u\left[\frac{\widetilde{w}+\widetilde{\alpha}(1-\widetilde{w})}{\widetilde{w}}\left(\bar{X}_T-1\right)\mathbf{1}_{\{\bar{X}_T\geq 1\}}-\frac{1}{\widetilde{w}}(1-\bar{X}_T)\mathbf{1}_{\{1-\widetilde{w}\leq \bar{X}_T<1\}}-\mathbf{1}_{\{\bar{X}_T<1-\widetilde{w}\}}\right]\right\}, \\ \text{Subject to} \quad & \bar{X}_T\in\mathcal{F}_T, \quad \mathbb{E}\left[\xi_T\bar{X}_T\right]=1, \quad \bar{X}_T\geq b. \end{aligned}$$

The objective function is strictly increasing in $\widetilde{\alpha}$ when fixing $\widetilde{w}$, and thus so is the utility per capital. Because the utility per capital is a function of $\widetilde{\gamma}$ and $\widetilde{\gamma}$ is strictly decreasing in $\widetilde{\alpha}$, we can conclude that it is strictly decreasing in $\widetilde{\gamma}$.

*(iii)* From the proof of Theorem 3.1, if $\widetilde{\gamma}=\gamma$, then $\widetilde{c}^*=c^*$ and $\widetilde{\nu}^*=\nu^*$. As a consequence, $\widetilde{\mathcal{M}}>\mathcal{M}$ when $b>0$ and $\widetilde{\mathcal{M}}=\mathcal{M}$ when $b=0$.

*(iv)* If $\widetilde{w}=w$ and $\widetilde{\alpha}=\alpha$, we have $u(\widetilde{\Theta}(x))=u(\Theta(x))$ when $x\geq X_0$ or $x=0$ and $u(\widetilde{\Theta}(x))<u(\Theta(x))$ otherwise. When $b=0$, the optimal terminal asset values of (2.5) and (2.6) are the same and take either $0$ or a value larger than $X_0$. As a consequence, the manager's utility is the same. When $b>0$, the optimal terminal asset values can take values in $(0,X_0)$, showing that $\widetilde{\mathcal{M}}<\mathcal{M}$. □

*Proof of Theorem 4.4* We only consider the case of the first-loss scheme. The case of the traditional scheme can be treated similarly. Because we assume the investor's initial investment is one dollar, the initial asset value is $X_0=1/(1-\widetilde{w})$. As a consequence, the gain/loss of the investor at the terminal time is $(1-\widetilde{\alpha})\widetilde{c}^*\left(\frac{\xi_T}{\widetilde{\nu}^*}\right)^{\frac{1}{p-1}}$ when $\xi_T\leq\widetilde{\nu}^*$ and is $-(1-b-\widetilde{w})/(1-\widetilde{w})$ otherwise. Then, straightforward calculation leads to (4.3). □

*Proof of Theorem 7.1* We only prove for the case of the first-loss scheme, as the case of the traditional scheme is similar.

Note that $\widetilde{U}(x)$ is strictly concave for $x\geq 1-\widetilde{w}$ and $\widetilde{U}(x)\equiv\widetilde{U}(1-\widetilde{w})$ for $x\in[b,1-\widetilde{w}]$. Therefore, the line that starts from $(b,\widetilde{U}(b))$ and is tangent to $\widetilde{U}$ uniquely exists. Moreover, the point of tangency $\hat{x}>b$, and $\hat{x}=1$ ($\hat{x}>1$ and $\hat{x}<1$, respectively) if and only if the slope of the tangent line $\widetilde{k}\in[\widetilde{U}'(1+),\widetilde{U}'(1-)]$ ($\widetilde{k}<\widetilde{U}'(1+)$ and $\widetilde{k}>\widetilde{U}'(1-)$, respectively).

Recall that in the first-loss scheme the manager maximizes $\mathbb{E}[\widetilde{U}(X_T/X_0)]$ over $X_T$ with the constraints $X_T/X_0\geq b$ and $\mathbb{E}[\xi_T X_T/X_0]\leq 1$. Following the proof of Theorem 3.1, we apply the Lagrange dual method to solve this problem. For each $\nu>0$, we solve $\mathbb{E}[\widetilde{U}(X_T/X_0)-(\widetilde{k}/\nu)\xi_T X_T/X_0]$ subject to the constraint $X_T/X_0\geq b$.

By discussing the cases of $\widetilde{k}>U'(1-)$, $\widetilde{k}<U'(1+)$, and $\widetilde{k}\in[U'(1+),U'(1-)]$, respectively, and using straightforward calculation, we can show

$$\begin{aligned} & \underset{x\geq b}{\text{argmax}}[U(x)-(\widetilde{k}/\nu)\xi_T x] \\ = & \begin{cases} (\widetilde{\zeta}+\max(\hat{x},1)-1)\left((\xi_T/(\widetilde{\tau}_2\nu))^{1/(p-1)}-1\right)+\max(\hat{x},1), & \xi_T<\widetilde{\tau}_2\nu, \\ 1, & \widetilde{\tau}_2\nu\leq\xi_T\leq\widetilde{\tau}_1\nu, \\ (a+\widetilde{w})\left((\xi_T/(\widetilde{\tau}_1\nu))^{1/(p-1)}-1\right)+1, & \widetilde{\tau}_1\nu<\xi_T<\widetilde{\nu}, \\ b, & \xi_T\geq\nu. \end{cases} \end{aligned}$$

Therefore, for each $\nu > 0$,

$$X_T^*(\nu) := \Big\{ \Big[(\widetilde{\zeta} + \max(\hat{x}, 1) - 1) \Big( (\xi_T / (\widetilde{\tau}_2 \nu))^{1/(p-1)} - 1 \Big) + \max(\hat{x}, 1) \Big] \mathbf{1}_{\{\xi_T < \widetilde{\tau}_2 \nu\}} + \mathbf{1}_{\{\widetilde{\tau}_2 \nu \le \xi_T \le \widetilde{\tau}_1 \nu\}} + \big[ (a + \widetilde{w}) \big( (\xi_T / (\widetilde{\tau}_1 \nu))^{1/(p-1)} - 1 \big) + 1 \big] \mathbf{1}_{\{\widetilde{\tau}_1 \nu < \xi_T < \nu\}} + b \mathbf{1}_{\{\xi_T \ge \nu\}} \Big\} X_0$$

maximizes $\mathbb{E}[\widetilde{U}(X_T/X_0) - (\widetilde{k}/\nu)\xi_T X_T/X_0]$. Following the same proof as of Theorem 3.1, $X_T^*(\widetilde{\nu}^*)$ is the optimal terminal asset value of the fund in the first-loss scheme, where $\widetilde{\nu}^* > 0$ is the one such that $\mathbb{E}[\xi_T X_T^*(\widetilde{\nu}^*)] = X_0$. $\square$

## References


ABDELLAOUI, M., H. BLEICHRODT, and H. KAMMOUN (2013): Do Financial Professionals Behave According to Prospect Theory? An Experimental Study, *Theory and Decision*, 74(3), 411–429.

ABDELLAOUI, M., H. BLEICHRODT, and C. PARASCHIV (2007): Loss Aversion Under Prospect Theory: A Parameter-Free Measurement, *Management Science*, 53(10), 1659–1674.

ACKERMANN, C., R. MCENALLY, and D. RAVENSCRAFT (1999): The Performance of Hedge Funds: Risk, Return, and Incentives, *Journal of Finance*, 54(3), 833–874.

AGARWAL, V., N. D. DANIEL, and N. Y. NAIK (2009): Role of Managerial Incentives and Discretion in Hedge Fund Performance, *Journal of Finance*, 64(5), 2221–2256.

ARTZNER, P., F. DELBAEN, J.-M. EBER, and D. HEATH (1999): Coherent Measures of Risk, *Mathematical Finance*, 9(3), 203–228.

BARBERIS, N. (2012): A Model of Casino Gambling, *Management Science*, 58(1), 35–51.

BARBERIS, N., and M. HUANG (2008): Stocks as Lotteries: The Implications of Probability Weighting for Security Prices, *American Economic Review*, 98(5), 2066–2100.

BARBERIS, N., and R. THALER (2003): 'A Survey of Behavioral Finance,' in *Financial Markets and Asset Pricing*, eds. M. H. G.M. Constantinides and R. Stulz, vol. 1 of *Handbook of the Economics of Finance*, 1053–1128. Elsevier.

BARBERIS, N., and W. XIONG (2009): What Drives the Disposition Effect? An Analysis of a Long-Standing Preference-Based Explanation, *Journal of Finance*, 64(2), 751–784.

BARBERIS, N., and W. XIONG (2012): Realization Utility, *Journal of Financial Economics*, 104(2), 251–271.

BICHUCH, M., and S. STURM (2014): Portfolio Optimization under Convex Incentive Schemes, *Finance and Stochastics*, 18(4), 873–915.

BLEICHRODT, H., and J. L. PINTO (2000): A Parameter-Free Elicitation of the Probability Weighting Function in Medical Decision Analysis, *Management Science*, 46(11), 1485–1496.

BLEICHRODT, H., J. L. PINTO, and P. P. WAKKER (2001): Making Descriptive Use of Prospect Theory to Improve the Prescriptive Use of Expected Utility, *Management Science*, 47(11), 1498–1514.

BOOIJ, A., B. VAN PRAAG, and G. VAN DE KUILEN (2010): A Parametric Analysis of Prospect Theory's Functionals for the General Population, *Theory and Decision*, 68, 115–148.

BOOIJ, A. S., and G. VAN DE KUILEN (2009): A Parameter-Free Analysis of the Utility of Money for the General Population under prospect theory, *Journal of Economic Psychology*, 30(4), 651–666.

BROWN, S. J., W. N. GOETZMANN, and R. G. IBBOTSON (1999): Offshore Hedge Funds: Survival and Performance, 1989 - 95, *Journal of Business*, 72(1), 91–117.

BROWN, S. J., W. N. GOETZMANN, and J. PARK (2001): Careers and Survival: Competition and Risk in the Hedge Fund and CTA Industry, *Journal of Finance*, 56(5), 1869–1886.

CAMERER, C. F., and T.-H. HO (1994): Violations of the Betweenness Axiom and Nonlinearity in Probability, *Journal of Risk and Uncertainty*, 8(2), 167–196.

CARPENTER, J. N. (2000): Does Option Compensation Increase Managerial Risk Appetite?, *Journal of Finance*, 55(5), 2311–2331.

CHANG, H., J. CVITANIĆ, and X. Y. ZHOU (2015): Optimal Contracting with Moral Hazard and Behavioral Preferences, *Journal of Mathematical Analysis and Applications*, 428(2), 959–981.

CONT, R., R. DEGUEST, and X. D. HE (2013): Loss-Based Risk Measures, *Statistics and Risk Modeling*, 30(2), 133–167.

FISHBURN, P. C., and G. A. KOCHENBERGER (1979): Two-Piece von Neumann-Morgenstern Utility Functions, *Decision Sciences*, 10(4), 503–518.

FUNG, W., and D. A. HSIEH (1999): A Primer on Hedge Funds, *Journal of Empirical Finance*, 6(3), 309–331.

FUNG, W., and D. A. HSIEH (2000): Performance Characteristics of Hedge Funds and Commodity Funds: Natural vs. Spurious Biases, *Journal of Financial and Quantitative Analysis*, 35(03), 291–307.

GOETZMANN, W. N., J. E. INGERSOLL, and S. A. ROSS (2003): High-Water Marks and Hedge Fund Management Contracts, *Journal of Finance*, 58(4), 168–1718.

GUASONI, P., and J. OBŁÓJ (2016): The Incentives of Hedge Fund Fees and High-Water Marks, *Mathematical Finance*, 26(2), 269–295.

GUREVICH, G., D. KLIGER, and O. LEVY (2009): Decision-Making under Uncertainty—A Field Study of Cumulative Prospect Theory, *Journal of Banking and Finance*, 33(7), 1221–1229.

HE, X. D., and X. Y. ZHOU (2011): Portfolio Choice under Cumulative Prospect Theory: An Analytical Treatment, *Management Science*, 57(2), 315–331.

HODDER, J. E., and J. C. JACKWERTH (2007): Incentive Contracts and Hedge Fund Management, *Journal of Financial and Quantitative Analysis*, 42(4), 811–826.

KAHNEMAN, D. (2003): Maps of Bounded Rationality: Psychology for Behavioral Economics, *American Economic Review*, 93(5), 1449–1475.

KAHNEMAN, D., J. L. KNETSCH, and R. H. THALER (1990): Experimental Tests of the Endowment Effect and the Coase Theorem, *Journal of Political Economy*, 98(6), 1325–1348.

KAHNEMAN, D., and A. TVERSKY (1979): Prospect Theory: An Analysis of Decision Under Risk, *Econometrica*, 47(2), 263–291.

KARATZAS, I., and S. E. SHREVE (1998): *Methods of Mathematical Finance*. Springer, New York.

KLIGER, D., and O. LEVY (2009): Theories of Choice under Risk: Insights from Financial Markets, *Journal of Economic Behavior and Organization*, 71(2), 330–346.

KÖBBERLING, V., and P. P. WAKKER (2005): An Index of Loss Aversion, *Journal of Economic Theory*, 122(1), 119–131.

KOU, S. G., X. PENG, and C. C. HEYDE (2013): External Risk Measures and Basel Accords, *Mathematics of Operations Research*, 38(3), 393–417.

KOUWENBERG, R., and W. T. ZIEMBA (2007): Incentives and Risk Taking in Hedge Funds, *Journal of Banking and Finance*, 31(11), 3291–3310.

LHABITANT, F.-S. (2011): *Handbook of Hedge Funds*, vol. 579. John Wiley & Sons, West Sussex.

LIANG, B. (1999): On the Performance of Hedge Funds, *Financial Analysts Journal*, 55(4), 72–85.

LIANG, B. (2001): Hedge Fund Performance: 1990-1999, *Financial Analysts Journal*, 57(1), 11–18.

MALKIEL, B. G., and A. SAHA (2005): Hedge Funds: Risk and Return, *Financial Analysts Journal*, 61(6), 80–88.

MCCRARY, S. A. (2004): *Hedge Fund Course*, vol. 304. John Wiley & Sons, New Jersey.

MITCHELL, D., K. MUTHURAMAN, and S. TITMAN (2013): Money Management with Performance Fees, Working Paper.

PANAGEAS, S., and M. M. WESTERFIELD (2009): High-Water Marks: High Risk Appetites? Convex Compensation, Long Horizons, and Portfolio Choice, *Journal of Finance*, 64(1), 1–36.

PENNINGS, J. M., and A. SMIDTS (2003): The Shape of Utility Functions and Organizational Behavior, *Management Science*, 49(9), 1251–1263.

POLKOVNICHENKO, V., and F. ZHAO (2013): Probability Weighting Functions Implied in Options Prices, *Journal of Financial Economics*, 107(3), 580–609.

QUIGGIN, J. (1982): A Theory of Anticipated Utility, *Journal of Economic Behavior and Organization*, 3, 323–343.

SCHMIDT, U., and S. TRAUB (2002): An Experimental Test of Loss Aversion, *Journal of Risk and Uncertainty*, 25(3), 233–249.

SHAFIR, E., P. DIAMOND, and A. TVERSKY (1997): Money Illusion, *Quarterly Journal of Economics*, 112(2), 341–374.

THALER, R. H. (1999): Mental Accounting Matters, *Journal of Behavioral Decision Making*, 12(3), 183–206.

TVERSKY, A., and D. KAHNEMAN (1981): The Framing of Decisions and the Psychology of Choice, *Science*, 211, 453–458.

TVERSKY, A., and D. KAHNEMAN (1992): Advances in Prospect Theory: Cumulative Representation of Uncertainty, *Journal of Risk and Uncertainty*, 5(4), 297–323.

WAKKER, P., and A. TVERSKY (1993): An Axiomatization of Cumulative Prospect Theory, *Journal of Risk and Uncertainty*, 7(2), 147–175.

WU, G., and R. GONZALEZ (1996): Curvature of the Probability Weighting Function, *Management Science*, 42(12), 1676–1690.