# Dynamic Portfolio Choice and Consumption Plan under Inflation with Nominal and Indexed Bonds

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## Abstract

We solve for an intertemporal portfolio-consumption choice problem under inflation. We assume that the nominal interest rate is observable while the expected inflation rate is not. The inclusion of the indexed bond in the investor's portfolio provides the investor an opportunity to perfectly hedge against the inflation risk. While the hedging demand of the nominal bonds would be crowded out proportional to the demand of the indexed bonds. The estimation risk of the estimated inflation rate would also introduce an additional hedging demand. We also show that the direction in which the interest rate and the inflation rate affect the optimal consumption-wealth ratio would rely on the elasticity of intertemporal substitution of the investor. When the elasticity of intertemporal substitution is smaller than one, the consumption-wealth ratio is increasing in the nominal interest rate and decreasing in the inflation rate; the income effect dominates. When the elasticity of intertemporal substitution is greater than one, the consumption-wealth ratio is affected in an opposite way; the substitution effect dominates. However, the consumption-wealth ratio is not decided by the real interest rate, i.e., the difference of the nominal interest rate and the inflation rate. It also depends on the absolute levels of the nominal interest rate and the inflation rate. The nominal and real consumption growth rates are derived. The nominal consumption growth is decided by the sum of the real consumption growth rate and inflation rate.

**Keywords: Inflation Risk, Indexed Bond, Dynamic Portfolio Choice, Elasticity of Intertemporal Substitution, Consumption**

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# 1. Introduction

How the inflation risk affects the investor's portfolio choice and consumption plan has long been a hot issue in both financial and economic fields. Since the change of nominal price level would affect the future purchasing power of the investor, take the inflation risk into account is important when the non-myopic investor makes her inter-temporal decision. In fact, all uncertain changes in the investment opportunity set would affect the investor's intertemporal behavior and the recognition of the inter-temporal changes in the investment opportunity makes the investor's financial behavior quite different to what has been suggested in the static mean-variance analysis because the uncertain changes in the investment opportunity set would introduce an additional inter-temporal hedging demand of risky assets. Efforts have been made in various aspects. For example, Kim and Omberg (1996) and Wachter (2001) show that under stochastic mean-reverting risk premium of stocks the investor with a longer investment horizon should hold more stocks, while Brennan and Xia (2000) solve a bond/stock mix portfolio problem under stochastic interest rate and show that the zero-coupon bond is the corresponding security to hedge against the interest rate risk. Their finding fits with the popular institutional recommendation that conservative investors should hold more bonds in their portfolio since long-term bond could provide certain payoff at the end of the investor's investment horizon.

Intertemporal portfolio choice problem under inflation risk has been surveyed by Campbell and Viceira (2001), Brennan and Xia (2002) and Munk, Sørensen and Vinther (2004). Campbell and Viceira (2001) establish a discrete-time model with a two-factor term structure of nominal interest rate. The time variation of nominal interest rate is driven by the processes of real interest rate and inflation rate. They solve the problem with a log-linear approximation and show that in a world with inflation risk, a long-term nominal bond is no longer a safe asset for a risk-averse investor. When the bonds available in the investor's investment opportunity set are inflation-indexed, an infinitely risk-averse investor with zero elasticity of intertemporal substitution would hold a portfolio composed of only indexed bonds to form a portfolio equivalent to the indexed perpetuity in order to finance a riskless real consumption stream. When bonds available are only nominal zeros, the investor would short long-term nominal bonds to reduce her exposure to inflation risk. Brennan and Xia (2002) provide an exact solution to a continuous-time problem similar to Campbell and Viceira (2001). Brennan and Xia (2002) show that without the explicit inclusion of indexed bonds, the infinitely risk-averse would hold a portfolio of two nominal bonds with different maturities which perfectly mimics a hypothetical indexed bond. Both works of Campbell and Viceira (2001) and Brennan and Xia (2002) tell us that a long-term risk-averse investor prefers the indexed bond or a perfect substitution of indexed bond in order to hedge against the inflation risk. In fact, the indexed-bond market is growing fast since last decade. The U.S. Treasury has been issuing the Treasury Inflation Protected Securities (TIPS) since January 1997. There is over $515 billion of TIPS outstanding in 2008, over two times the amount in 2004 and the daily trading volume has grown from $2 billion to $9 billion during the years of 2002-2008.

In this paper, we would like to solve an intertemporal portfolio choice problem with interim consumption for an infinitely lived investor under uncertain inflation. We try to find out the optimal consumption plan and the optimal portfolio rule for stocks, nominal bonds as well as inflation-indexed bonds. In contrast with Campbell and Viceira (2001) and Brennan and Xia (2002), we assume that only the nominal interest rate rather than the real interest rate is directly observable to the investor. In Campbell and Viceira (2001) and Brennan and Xia (2002), they assume that the real interest rate and the expected inflation rate are described as two Ornstein-Uhlenbeck process respectively. The nominal interest rate is hence a two-factor process driven by the two state variables of real interest rate and expected inflation rate. However, since the investors could trade the financial assets and consumption goods only in nominal term, it is plausible to argue the observability of the state variables. In this paper, we assume that the directly observable variable is the nominal interest rate while the expected inflation rate is still unobservable. In reality, the investor should infer the expected inflation rate according to the realized changes of the nominal price level in the past. We employ the nonlinear filtering technique introduced in Lipster and Shiryayev (1977) to model this estimation problem. Our setting is much like the setting in Munk, Sørensen and Vinther (2004). Rather than a one-factor real term structure, they also adopt a one-factor term structure of nominal interest rate with inflation and, as an immediate result, come up to a two-factor real interest rate model. However, they ignore the fact that the expected inflation rate is still unobservable and the problem is defined on the terminal wealth not on the interim consumptions. Besides, they also assume that there are only nominal bonds to be traded so that the inflation risk is never perfectly hedged. Our model, in contrast, would solve the problem for portfolio choice as well as the interim consumption decision under inflation. We provide a more realistic assumption that the nominal interest rate is the observable state variable while the inflation rate is indeed unobservable and the investor must infer the value from the past. We would show that with the inclusion of the indexed bond in the investor's investment opportunity, the demand of the indexed bonds would crowd out a proportional position to the holdings of nominal bonds. The estimation risk of the estimated inflation rate would also give rise to an additional hedging demand of the bond portfolio.

For the representation of the investor's intertemporal utility, we follow the work of Chacko and Viceira (2005) to use the stochastic differential utility (SDU) proposed by Duffie and Epstein (1992). The stochastic differential utility is a generalization of the conventional time-additive power utility. Unlike the power utility, SDU allows the level of the investor's risk aversion and the attitude toward intertemporal substitution of consumption to be represented by two distinct parameters. It is more suitable to employ the SDU to disentangle these two factors when solving a problem incorporating with the intertemporal consumption decision. We would show that the portfolio choice depends on the investor's risk aversion while the consumption plan is affected by the elasticity of intertemporal substitution. We also show that the elasticity of intertemporal substitution decides the direction in which the interest rate and the inflation rate affect the investor's consumption.

# 2. The Economy

### 2.1 The Dynamics of Price Level and Expected Inflation

Let $P_t$ denote the nominal price level per unit of the consumption good at time $t$ and we assume that it follows a diffusion process:

$$\frac{dP_t}{P_t} = \tilde{\pi}_t dt + \sigma_p dZ_1 \tag{1}$$

where $\tilde{\pi}_t$ is expected inflation rate at time $t$ and $dZ_1$ is the increment of a standard Brownian motion representing the shock to the instantaneous unexpected inflation. In line with Brennan and Xia (2002), we assume that the expected inflation rate $\tilde{\pi}_t$ is also stochastic and follows an Ornstein-Uhlenbeck process:

$$d\tilde{\pi}_t = \ell(\bar{\pi} - \tilde{\pi}_t)dt + \sigma_1 dZ_1 + \sigma_2 dZ_2 \tag{2}$$

We argue that $\tilde{\pi}_t$ is unobservable to the investor and is not necessarily perfectly correlated to the instantaneous price level. In Equation (2), $dZ_2$ represents the remaining part of innovation that is uncorrelated to the innovation of the price level. According to Liptser and Shiryayev (1977), if at time $t = 0$ the distribution of $\tilde{\pi}_0$ is conditionally Gaussian, i.e. $\text{Prob}\left(\tilde{\pi}_0 \le a \middle| P_0\right) \sim N\left(\pi_0, v(0)\right)$, the conditional distribution of $\tilde{\pi}_t$ would also be Gaussian $N\left(\pi_t, v(t)\right)$. The conditional mean $\pi_t = \mathrm{E}\left(\tilde{\pi}_t \middle| \mathrm{F}_t^P\right)$ is the optimal estimator of $\tilde{\pi}_t$ where $\mathrm{F}_t^P$ is the $\sigma$-field generated by $\{P_s : s \le t\}$ and the conditional variance $v(t)$ could be viewed as the estimation error. The investor would then substitute the estimator $\pi_t$ for the unobservable $\tilde{\pi}_t$ in Equation (1):

$$\frac{dP_t}{P_t} = \pi_t dt + \sigma_p dZ_p \tag{3}$$

where

$$d\pi_t = \ell\left(\overline{\pi} - \pi_t\right) dt + \left(\sigma_1 + \frac{v(t)}{\sigma_p}\right) dZ_p \tag{4}$$

$$\frac{dv(t)}{dt} = -2\ell v(t) + \sigma_1^2 + \sigma_2^2 - \left(\sigma_1 + \frac{v(t)}{\sigma_p}\right) \tag{5}$$

The estimator of the expected inflation rate is perfectly correlated to the instantaneous change of the price level. As shown in Liptser and Shiryayev (1977), the common innovation to these two variables:

$$dZ_p = \frac{1}{\sigma_p}\left(\frac{dP_t}{P_t} - \pi_t dt\right) \tag{6}$$

is observable and $Z_p$ would be a standard Brownian motion. In fact, $dZ_p$ is decided by the unexpected excess inflation relative to its current estimated value $\pi_t$. As to the conditional variance $v(t)$, it could be explicitly solved from the Riccati equation shown in Equation (5) and it could be shown that in the steady state the value of $v(t)$ would approach a constant

$$v = \lim_{t\to\infty} v(t) = \sigma_p^2 \left[ \sqrt{\left(\ell + \frac{\sigma_1}{\sigma_p}\right)^2 + \sigma_2^2} - \left(\ell + \frac{\sigma_1}{\sigma_p}\right) \right] \tag{7}$$

By Equation (7), the estimation error $v$ would not vanish unless $\sigma_2 = 0$, i.e. the expected inflation is in fact a constant or is perfectly correlated to the instantaneous change of $P_t$. With an infinitely-lived investor, we ignore the transient time variation of $v(t)$ and rewrite Equation (4) as

$$d\pi_t = \ell\left(\overline{\pi} - \pi_t\right)dt + \sigma_\pi dZ_p \tag{8}$$

where

$$\sigma_\pi \equiv \sigma_1 + \frac{v}{\sigma_p} = \sigma_p \left[ \sqrt{\left(\ell + \frac{\sigma_1}{\sigma_p}\right)^2 + \sigma_2^2} - \ell \right] \tag{9}$$

## 2.2 The Bond Market

We assume that the investor could directly observe the nominal interest rate which could be described as an Ornstein-Uhlenbeck process of Vasicek (1977) type:

$$dR_t = \kappa\left(\overline{R} - R_t\right)dt + \sigma_R dZ_R \tag{10}$$

Let $N\left(R_t, T-t\right)$ denote the price of the nominal zero-coupon bond which pays one money unit when it matures at time $T$. $N\left(R_t, T-t\right)$ would satisfy the following partial differential equation with the boundary condition $N(R,0)=1$:

$$N_R \kappa\left(\overline{R} - R_t\right) + \frac{\partial N}{\partial t} + \frac{1}{2} N_{RR}\sigma_R^2 = R_t N - \lambda_R N_R \sigma_R \tag{11}$$

where the constant $\lambda_R$ represents the risk premium of the interest rate risk, $N_R$ is the first-order partial derivative of $N(\cdot)$ with respect to $R$ and $N_{RR}$ is the second-order partial derivative. The solution of Equation (11) gives the dynamic of the return of the nominal zero-coupon bond:

$$\frac{dN\left(R_t, T-t\right)}{N\left(R_t, T-t\right)} = \left(R_t + \lambda_R b_1(T-t)\sigma_R\right)dt - b_1(T-t)\sigma_R dZ_R \tag{12}$$

where $b_1(T-t) = \kappa^{-1}[1-\exp\left(-\kappa(T-t)\right)]$ is a function of the time to maturity $T-t$. Since the infinitely-lived investor has to roll over her investment of bonds whenever the bonds in the portfolio expire, we could assume that investor always sells the expiring bonds and buys the newly-issued bonds continuously to keep the maturity of the bonds in her portfolio a constant. Thus, we would take the value of $b_1\left(T-t\right)$ as a constant $b$ for a given $\tau^*$ such that $T-t=\tau^*$ and $b=b(\tau^*)$. For this sake, we simplify the expression of Equation (12):

$$\frac{dN_t}{N_t} = \left(R_t + \eta_N\right)dt - \sigma_N dZ_R \tag{13}$$

where $\eta_N \equiv \lambda_R \sigma_N$ and $\sigma_N \equiv b\sigma_R$.

The return of the inflation-indexed zero-coupon bond is defined as the price of one unit of consumption good when it matures at time $T$.The indexed bond would simultaneously bear the risks of interest rate, inflation rate and the nominal price. The price of the indexed bond, $I(R_t, \pi_t, P_t, T-t)$, could be decided by the following partial differential equation with the boundary condition $I(R_t, \pi_t, P_t, 0) = P_T$:

$$\begin{aligned} & I_R \kappa\left(\overline{R}-R\right) + I_P P\pi + I_\pi \ell\left(\overline{\pi}-\pi\right) + \frac{\partial I}{\partial t} + \frac{1}{2} I_{RR}\sigma_R^2 + \frac{1}{2} I_{PP}\sigma_p^2 + \frac{1}{2} I_{\pi\pi}\sigma_\pi^2 \\ & + I_{RP}\sigma_{RP} P + I_{R\pi}\sigma_{R\pi} + I_{P\pi} P\sigma_p\sigma_\pi = R_t I - \lambda_R I_R \sigma_R + \lambda_p\left(I_P P\sigma_p + I_\pi \sigma_\pi\right) \end{aligned} \tag{14}$$

where $\sigma_{XY} \equiv \mathrm{E}\left(dXdY\right)$ denotes the covariance of the state variables $X$ and $Y$.

$\lambda_p$ represents the measure of the risk premium with respect to the innovation $dZ_p$ defined as Equation (6). Solving Equation (14), we then derive the return of the indexed bond as the following process:

$$\begin{aligned}\frac{dI\left(R_t,\pi_t,P_t,T-t\right)}{I\left(R_t,\pi_t,P_t,T-t\right)} &= \left(R_t+\lambda_R b_1(T-t)\sigma_R+\lambda_p(\sigma_p+b_2(T-t)\sigma_\pi)\right)dt \\ &\quad -b_1(T-t)\sigma_R dZ_R+\left(\sigma_p+b_2(T-t)\sigma_\pi\right)dZ_p\end{aligned} \tag{15}$$

where $b_1(T-t)$ has been shown earlier and $b_2(T-t)=\ell^{-1}\left[1-\exp(-\ell(T-t))\right]$. We also assume that, for an infinitely-lived investor, he would adopt the trading strategy of substituting the newly-issued bonds for the expiring bonds continuously to keep the time to maturity of the indexed bond to be a constant $\tau^{**}$ such that $b_1\equiv b_1(\tau^{**})$ and $b_2\equiv b_2(\tau^{**})$ would be two constants. If $\tau^{**}$ equals $\tau^*$,the time to maturity of the nominal bond, the values of $b$ and $b_1$ would be equal. For brevity, we rewrite Equation (15) to be

$$\frac{dI_t}{I_t}=\left(R_t+\eta_I\right)dt-\sigma_{I1}dZ_R+\sigma_{I2}dZ_p \tag{16}$$

where $\eta_I\equiv\lambda_R\sigma_{I1}+\lambda_p\sigma_{I2}$, $\sigma_{I1}\equiv b_1\sigma_R$ and $\sigma_{I2}\equiv\sigma_p+b_2\sigma_\pi$.

We define $I_t/P_t$ as the real bond. According to Equation (3), (15) and Itô's lemma, the return of the real bond is

$$\begin{aligned}\frac{d(I_t/P_t)}{I_t/P_t} &= \left(R_t-\pi_t+\lambda_p\sigma_p+\lambda_R b_1\sigma_R+\lambda_P b_2\sigma_\pi+b_1\sigma_{RP}-b_2\sigma_p\sigma_\pi\right)dt \\ &\quad -b_1\sigma_R dZ_R+b_2\sigma_\pi dZ_P\end{aligned} \tag{17}$$

The instantaneous real risk-free interest rate, $r_t$, is obtained by taking the limit of the return of the real bond in Equation (17) when $T-t\to 0$:

$$r_t = R_t - \pi_t + \lambda_p \sigma_p \tag{18}$$

The real interest rate is the difference of the nominal interest rate and the inflation rate plus the risk premium of the nominal price level for consumption goods. The Fish Equation is not hold unless the risk premium of the price risk is zero.

### 2.3 The Optimization Problem

We assume that the investor's preference is represented by the stochastic differential utility proposed by Duffie and Epstein (1992):

$$J_t = \mathrm{E}_t\left[\int_t^\infty f\left(C_s, J_s\right) ds\right] \tag{19}$$

and

$$f\left(C, J\right) = \beta\left(1 - \frac{1}{\varphi}\right)^{-1}\left(1-\gamma\right) J\left[\left(\frac{C}{\left((1-\gamma)J\right)^{1/(1-\gamma)}}\right)^{1-(1/\varphi)} - 1\right] \tag{20}$$

$f\left(C, J\right)$ is called the normalized aggregator of the investor's current consumption and utility. $\beta$ is the time preference, $\gamma$ is the measure of the relative risk aversion for the investor and $\varphi$ stands for the elasticity of intertemporal substitution of consumption. The benefit of this utility representation is that it separates the elasticity of intertemporal substitution from the relative risk aversion. For the widely adopted time-additive power utility function, the reciprocal of the risk aversion represents the elasticity of intertemporal substitution as well. Accordingly, an investor who is more risk-averse is more unwilling to substitute consumption intertemporally while it is not always the case. Obviously, the stochastic differential utility is a more generalized setting and the standard time-additive power utility could be viewed as a special case when $\varphi = 1/\gamma$ in Equation (20).

In the financial market, there are three kinds of risky assets to be traded. One is the stock with nominal price $S_t$ following:

$$\frac{dS_t}{S_t} = \left(R_t + \eta_s\right)dt + \sigma_s dZ_s \tag{21}$$

and the other two are the zero coupon nominal and indexed bonds as shown in Equation (13) and (16) respectively. In Equation (21), $\eta_s$ is the excess return of stock and $dZ_s$ is the unexpected disturbance of stock return. The investor's problem is to choose her optimal consumption and the portfolio weights on the three kinds of risky assets to maximize the utility represented in Equation (19) subjected to the intertemporal budget constraint:

$$dW_t = \left[\left(x^{\mathrm{T}}\eta + R_t\right)W_t - C_t P_t\right]dt + x^{\mathrm{T}}\Gamma\,\mathrm{dZ} \tag{22}$$

where $C_t$ is the instantaneous consumption at time $t$ in terms of real units. $x^{\mathrm{T}}$ is the transpose of the vector of the portfolio weights $x$

$$x = \begin{pmatrix} x_S \\ x_N \\ x_I \end{pmatrix} \tag{23}$$

$\eta$ is the vector of excess return

$$\eta = \begin{pmatrix} \eta_S \\ \eta_N \\ \eta_I \end{pmatrix} \tag{24}$$

and

$$\Gamma = \begin{pmatrix} \sigma_s & 0 & 0 \\ 0 & -\sigma_N & 0 \\ 0 & -\sigma_{I1} & \sigma_{I2} \end{pmatrix} \tag{25}$$

$$\mathrm{dZ} = \begin{pmatrix} dZ_s \\ dZ_R \\ dZ_p \end{pmatrix} \tag{26}$$

The optimal policies for the investor must satisfy the following Bellman equation:

$$\begin{aligned} &\max_{x,C}\Big[ f\left(C,J\right) + J_W W\left(x^{\mathrm{T}}\eta + R\right) - J_W CP + J_R \kappa\left(\bar{R} - R\right) + J_P P\pi + J_\pi \ell\left(\bar{\pi} - \pi\right) \\ &\quad + \frac{1}{2}\left(J_{WW} W^2 x^{\mathrm{T}} \Sigma x + J_{RR}\sigma_R^2 + J_{PP} P^2 \sigma_p^2 + J_{\pi\pi}\sigma_\pi^2\right) + J_{WR} W x^{\mathrm{T}} \Gamma \rho e_2 \sigma_R + J_{WP} W P x^{\mathrm{T}} \Gamma \rho e_3 \sigma_p \\ &\quad + J_{W\pi} W x^{\mathrm{T}} \Gamma \rho e_3 \sigma_\pi + J_{RP} P \sigma_{RP} + J_{R\pi}\sigma_{R\pi} + J_{P\pi} P \sigma_p \sigma_\pi \Big] = 0 \end{aligned} \tag{27}$$

In Equation (25), $e_2$ is $\begin{pmatrix}0 & 1 & 0\end{pmatrix}^{\mathrm{T}}$ and $e_3$ is $\begin{pmatrix}0 & 0 & 1\end{pmatrix}^{\mathrm{T}}$. Besides, $\Sigma \equiv \Gamma \rho \Gamma^{\mathrm{T}}$ represents the variance-covariance matrix of the nominal risky asset returns and $\Gamma \rho e_2 \sigma_R$ is the vector of covariance between the risky asset returns and the nominal interest rate (and so forth for the similar terms) where $\rho$ ( $3 \times 3$ ) is the matrix of correlation coefficients:

$$\rho \equiv \begin{pmatrix} 1 & \rho_{SR} & \rho_{SP} \\ \rho_{SR} & 1 & \rho_{RP} \\ \rho_{SP} & \rho_{RP} & 1 \end{pmatrix} \tag{28}$$

$\rho_{SR}$ is defined by $E(dZ_s dZ_R) = \rho_{SR} dt$ and others are defined in a similar fashion.

The first-order conditions for the Bellman equation are:

$$C^* = \beta^{\varphi}\left(J_W P\right)^{-\varphi}\left[\left(1-\gamma\right)J\right]^{(1-\gamma\varphi)/1-\gamma} \tag{29}$$

$$x^* = \frac{-J_W}{J_{WW} W}\Sigma^{-1}\eta + \frac{-J_{WR}}{J_{WW} W}\Sigma^{-1}\Gamma\rho e_2 \sigma_R + \frac{-J_{WP} P}{J_{WW} W}\Sigma^{-1}\Gamma\rho e_3 \sigma_p + \frac{-J_{W\pi}}{J_{WW} W}\Sigma^{-1}\Gamma\rho e_3 \sigma_\pi \tag{30}$$

Equation (29) gives the optimal real consumption plan once the value function is given and Equation (30) shows that the optimal portfolio weights are composed of four parts. The first term in the right-hand side is the demand due to the excess returns of the risky assets and is often called the speculative or myopic demand. The rest parts represent the hedging demands against the risks of interest rate, nominal price level of consumption goods and the expected inflation rate sequentially.

## 3. Results

### 3.1 The Approximate Solution with Log-Linearization

The partial differential equation after we substitute Equation (29) and (30) back into (27) is complicated to solve. However, by conjecture, it could be verified that the solution of $J$ would have the following form:

$$J\left(W_t, R_t, P_t, \pi_t\right) = \left[H\left(R_t, \pi_t\right)\right]^{(\gamma-1)/(1-\varphi)} \frac{\left(W_t / P_t\right)^{1-\gamma}}{1-\gamma} \tag{31}$$

$H\left(R, \pi\right)$ is the solution of the following partial differential equation:

$$\begin{aligned}
&\beta^{\varphi}\frac{1}{H} - \beta\varphi + (\varphi-1)\left(x^{\mathrm{T}}\eta + R\right) + \frac{H_R}{H}\kappa\left(\bar{R} - R\right) - (\varphi-1)\pi + \frac{H_\pi}{H}\ell\left(\bar{\pi} - \pi\right) \\
&+\frac{1}{2}\left[\frac{\gamma+\varphi-2}{1-\varphi}\left(\frac{H_R}{H}\right)^2 + \frac{H_{RR}}{H}\right]\sigma_R^2 + \frac{1}{2}\left[\frac{\gamma+\varphi-2}{1-\varphi}\left(\frac{H_\pi}{H}\right)^2 + \frac{H_{\pi\pi}}{H}\right]\sigma_\pi^2 \\
&-\frac{1}{2}\left(\gamma-2\right)\left(\varphi-1\right)\sigma_P^2 - \frac{\gamma(\varphi-1)}{2}x^{\mathrm{T}}\Sigma x - (\gamma-1)\frac{H_R}{H}x^{\mathrm{T}}\Gamma\rho e_2\sigma_R \\
&-(\gamma-1)\frac{H_\pi}{H}x^{\mathrm{T}}\Gamma\rho e_3\sigma_\pi + (\gamma-1)(\varphi-1)x^{\mathrm{T}}\Gamma\rho e_3\sigma_P + (\gamma-1)\frac{H_R}{H}\sigma_{RP} \\
&+\left[\frac{\gamma+\varphi-2}{1-\varphi}\left(\frac{H_R}{H}\right)\left(\frac{H_\pi}{H}\right) + \frac{H_{R\pi}}{H}\right]\sigma_{R\pi} + (\gamma-1)\frac{H_\pi}{H}\sigma_p\sigma_\pi = 0
\end{aligned} \tag{32}$$

As mentioned in Chacko and Viceira (2005), the nonlinear partial differential equation presented above would have no exact analytical solution in general. In line with Chacko and Viceira (2005), we employ the method of log-linear approximation to find an approximated analytical solution to investigate more insights of the solution to our problem. In the first place, we substitute Equation (31) into Equation (29) and find that the envelope condition would be expressed as

$$\frac{P_t C_t}{W_t} = \beta^{\varphi} \frac{1}{H(R_t, \pi_t)} \tag{33}$$

Denoting $\left(c_t - w_t\right) \equiv \log(P_t C_t / W_t)$ and using the first-order Taylor expansion of $\exp(c_t - w_t)$ around its unconditional mean $\mathrm{E}(c_t - w_t) \equiv \overline{c - w}$, the envelope condition could be rewritten as

$$\begin{aligned} \beta^{\varphi} \frac{1}{H} &= \exp\left(c_t - w_t\right) \approx \exp(\overline{c - w}) + \exp(\overline{c - w})[c_t - w_t - (\overline{c - w})] \\ &= h_0 - h_1 \log H \end{aligned} \tag{34}$$

where $h_1 \equiv \exp(\overline{c - w})$ and $h_0 \equiv h_1(1 + \varphi \log \beta - \log h_1)$. Substituting the approximated result in Equation (34) for $\beta^{\varphi} H^{-1}$ in Equation (33), it is easy to see that the solution of $H$ would take the form of $H\left(R_t, \pi_t\right) = \exp(a_0 - a_1 R_t + a_2 \pi_t)$. The undetermined coefficients would then be solved as following:

$$a_1 = \frac{1 - \varphi}{h_1 + \kappa} \tag{35}$$

$$a_2 = \frac{1 - \varphi}{h_1 + \ell} \tag{36}$$

$$\begin{aligned} a_0 = \frac{1}{h_1} \Bigg[ & h_0 - \beta\varphi + (\varphi - 1) x^{\mathrm{T}} \eta + \frac{1}{2} \frac{\gamma - 1}{1 - \varphi} \left( a_1^2 \sigma_R^2 + a_2^2 \sigma_\pi^2 - 2 a_1 a_2 \sigma_{R\pi} \right) \\ & + (\gamma - 1) x^T \Gamma \rho (a_1 \sigma_R e_2 - a_2 \sigma_\pi e_3 - (1 - \varphi) \sigma_p e_3) - a_1 \kappa \bar{R} + a_2 \ell \bar{\pi} \\ & - (\gamma - 1) a_1 \sigma_{RP} + (\gamma - 1) a_2 \sigma_p \sigma_\pi + \frac{1}{2} (\gamma - 2)(1 - \varphi) \sigma_p^2 + \frac{\gamma (1 - \varphi)}{2} x^{\mathrm{T}} \Sigma x \Bigg] \end{aligned} \tag{37}$$

### 3.2 The Optimal Policies

Up to now, we have derived the approximate solution for the value function of the investor. It would then be an immediate result to show the optimal policy for the investor:

**Proposition 1** *The approximate analytical solution for the investor's value function is*

$$J\left(W_t, R_t, P_t, \pi_t\right) = \exp\left[\frac{\gamma - 1}{1-\varphi}\left(a_0 - a_1 R_t + a_2 \pi_t\right)\right]\frac{\left(W_t / P_t\right)^{1-\gamma}}{1-\gamma} \tag{38}$$

*and the optimal consumption and portfolio policies implied by the value function are*

$$\frac{P_t C_t}{W_t} = \beta^{\varphi}\exp(-a_0 + a_1 R_t - a_2 \pi_t) \tag{39}$$

*and*

$$\begin{aligned} x = {} & \frac{1}{\gamma}\Sigma^{-1}\eta + \left(1 - \frac{1}{\gamma}\right)\frac{-a_1}{1-\varphi}\Sigma^{-1}\Gamma\rho e_2 \sigma_R + \left(1 - \frac{1}{\gamma}\right)\frac{a_2}{1-\varphi}\Sigma^{-1}\Gamma\rho e_3 \sigma_\pi \\ & + \left(1 - \frac{1}{\gamma}\right)\Sigma^{-1}\Gamma\rho e_3 \sigma_p \end{aligned} \tag{40}$$

**Proof.** Substituting the approximate solution of $H(R_t, \pi_t)$ derived in last section into Equation (31) we immediately get the solution of the value function. The optimal policies stem from the value function and Equation (29), (30).■

In Proposition 1, we find that the consumption-wealth ratio $P_t C_t / W_t$ is an exponentially affine function of the interest rate $R_t$ and the (estimated) inflation rate $\pi_t$. The exact relationship between the consumption-wealth ratio and the two state variables is decided by the value of $\varphi$, the elasticity of intertemporal substitution. According to Equation (35) and (36), when $\varphi = 1$, the values of $a_1$ and $a_2$ are both identical to zero and hence the consumption-wealth ratio turn out to be a constant over time. A constant consumption-wealth ratio makes $c_t - w_t$ exactly identical to its unconditional mean. This implies that the solution of $\varphi = 1$ is an exact solution. When $\varphi < 1$, $a_1$ and $a_2$ are positive. The consumption-wealth ratio would rise as $R_t$ rises or $\pi_t$ falls. However, when $\varphi > 1$, the consumption-wealth ratio would fall as $R_t$ rises or $\pi_t$ falls. As $R_t$ increases or $\pi_t$ decreases, the investor's income or the purchasing power would be higher and the investor could consume more. This is the positive income effect. However, an increase in $R_t$ or a decrease in $\pi_t$ would induce an incentive to cut the current consumption since consumption in the future becomes less expensive under this circumstance; this is the negative substitution effect of current consumption. The relative importance of intertemporal substitution and income effects would affect the investor's attitude toward her consumption plan. We could conclude that when $\varphi < 1$, the income effect dominate such that the investor's current consumption rises relative to wealth. However, when $\varphi > 1$, the substitution effect dominates and the investor cuts her current consumption relative to wealth. Contrary to the earlier related works which solve the problem with a explicit real interest rate process showing that the consumption-wealth ratio is a function of the real interest rate, our result in Equation (39) implies that the consumption-wealth ratio is not determined by the real interest rate which is often approximately referred as the difference of $R_t - \pi_t$. The consumption-wealth ratio derived in our model is decided by $a_1 R_t - a_2 \pi_t$, which is not a multiple of $R_t - \pi_t$ unless $a_1 = a_2$ or equivalently $\kappa = \ell$. In the case of $\varphi < 1$ and $\kappa < \ell$, given the level of the real interest rate $R_t - \pi_t$ unchanged, the consumption-wealth ratio would be higher with a higher absolute level of the nominal interest rate $R_t$ and the inflation rate $\pi_t$. In our model $\kappa$ and $\ell$ represent the degree of mean-reverting of the nominal interest rate and the inflation rate respectively. When $\kappa < \ell$, any deviation to the average level of the nominal interest rate would have a stronger persistency than that of the inflation rate. The investor would think that the nominal interest rate keeps in the high level longer than the inflation rate does and, as a result, a higher real interest rate follows. On the other hand, when $\varphi < 1$ and $\kappa > \ell$, increasing the absolute level of $R_t$ and $\pi_t$ while the real interest rate $R_t - \pi_t$ unchanged would result in an decrease in the consumption-wealth ratio since in this case the high inflation rate persists longer than the nominal interest rate, which means that there is a lower real interest rate in the following future. This finding implies that the consumption-wealth ratio is not purely decided by the real interest rate. The absolute level of the nominal variables and the relative persistency of the disturbance to the interest rate and inflation rate would also affect the investor's optimal consumption plan.

As to the optimal portfolio weights on the risky assets, we investigate Equation (40) in more details. By Equation (25), (28), (35) and (36), we substitute the full expressions of $a_1$, $a_2$, $\rho$ and $\Gamma$ for the corresponding items in Equation (40), we find that

$$\begin{aligned} x = \begin{pmatrix} x_S \\ x_N \\ x_I \end{pmatrix} &= \underbrace{\frac{1}{\gamma}\Sigma^{-1}\begin{pmatrix} \eta_S \\ \eta_N \\ \eta_I \end{pmatrix}}_{x_1} + \underbrace{\left(1-\frac{1}{\gamma}\right)\frac{1}{(h_1+\kappa)}\begin{pmatrix} 0 \\ 1/b \\ 0 \end{pmatrix}}_{x_2} \\ &\quad + \underbrace{\left(1-\frac{1}{\gamma}\right)\frac{\sigma_\pi}{(h_1+\ell)\sigma_{I2}}\begin{pmatrix} 0 \\ -b_1/b \\ 1 \end{pmatrix}}_{x_3} + \underbrace{\left(1-\frac{1}{\gamma}\right)\frac{\sigma_p}{\sigma_{I2}}\begin{pmatrix} 0 \\ -b_1/b \\ 1 \end{pmatrix}}_{x_4} \end{aligned} \qquad (41)$$

The term $x_1$ is the speculative (or myopic) demand, $x_2$ is the hedging demand against the unexpected innovation of interest rate and $x_3$ , $x_4$ are to hedge against the innovation of the inflation rate and the instantaneous nominal price level. Equation (41) shows that the investor's portfolio choice is represented by a weighted average of the speculative demand and the hedging demands. It is obviously that the portfolio policy depends only on the risk aversion but does not depend on the elasticity of intertemporal substitution explicitly. The elasticity of intertemporal substitution only affects the optimal portfolio implicitly by the unconditional mean of log consumption-wealth ratio through the coefficient $h_1$. For an infinitely risk-averse investor ($\gamma \to \infty$), the speculative demand would vanish and the optimal portfolio for the investor is composed of the mix of nominal and indexed zero coupon bonds. This meets the common advice that the more conservative investor should put more weights on bonds; documented as the asset allocation puzzle in Canner, Mankiw and Weil (1999). The absolute magnitude of hedging demand is deceasing with $\kappa$ and $\ell$, the degree of mean-reverting process of interest rate $R_t$ and inflation rate $\pi_t$ respectively. For large $\kappa$ and $\ell$, any undesirable disturbances that damage the future investment opportunity would not persist for long and the incentive to hold assets for the hedging purpose would mitigate. Observing $x_2$, we find that the interest rate risk could be perfectly hedged by holding a long position of the nominal zero coupon bond since the return of the nominal bond is perfectly negatively correlated to the instantaneous nominal interest rate. Regarding the demand of $x_3$ and $x_4$, which are in need to hedge the inflation and price risk, there is a long position of the inflation-indexed bonds while a short position of the nominal bonds. As mentioned earlier, the nominal bonds account for the demand to hedge against the interest rate risk. However, in a world with inflation, the nominal bond which pays certain monetary payoffs in the future is no longer a safe asset in real term since the real purchasing power is uncertain. The holding of nominal bonds under inflation would expose the long term investor to the inflation risk. According to Equation (16), the return of the inflation-indexed bond is positive related to the inflation. This implies that the real purchasing power would be compensated by the return of the indexed bond when the inflation rises. The indexed bonds thus provide an opportunity to hedge against the inflation risk. On the other hand, the return of the indexed bond is also negatively related to the instantaneous nominal interest rate in part and this means that the indexed bond also provide a channel to hedge against the interest rate risk. As a result, the risk-averse investor shorts parts of her holdings of the nominal zero-coupon bond and turn to the indexed bond which is a relatively safe asset under inflation. The short position of nominal bond in $x_3$ and $x_4$ is proportional to the long position of indexed bond. When $b = b_1$, i.e. with identical time to maturity for both the nominal and indexed bond, the investor shorts the nominal bond by the amounts identical to which she invests in the indexed bond. The need of the indexed bonds would crowd out the need of nominal bonds. This also implies that the demand of indexed bonds could be financed by selling the corresponding amount of nominal bonds. The sign of the net position of nominal bonds would depend on the values of the sensitivity of $b$, $b_1$, $\kappa$ and $\ell$.

Regarding to the estimation risk $v$, it affects the optimal portfolio through $\sigma_\pi$ in $x_3$ since $\sigma_\pi = \sigma_1 + v / \sigma_p$. This implies that the hedging demand against the risk of the expected inflation could be divided into two parts. One part is to hedge against the uncertainty that is perfectly explained by the nominal price level and the other part is to hedge against the estimation risk with respect to the unobserved innovations in the nominal world. As long as the expected inflation is not indeed perfectly correlated to the nominal price, the estimation risk and therefore this corresponding hedging demand exist.

### 3.3 Dynamics of Nominal and Real Consumptions

In this section, we derive the nominal and real consumption dynamics respectively. By Equation (22), (33), (34) and the solution of $H(R_t, \pi_t)$, the intertemporal budget constraint could be rewritten as

$$\frac{dW_t}{W_t} = \left[ x^{\mathrm{T}}\eta + R_t - h_0 + h_1(a_0 - a_1 R_t + a_2 \pi_t) \right] dt + x^{\mathrm{T}} \Gamma \, \mathrm{d}Z \tag{42}$$

where we use the approximate consumption-wealth ratio to substitute for its exact expression. According to Equation (39), (42) and Itô's lemma we obtain the following proposition:

**Proposition 2**

(i) *The dynamic of the nominal consumption* $P_tC_t$ *could be expressed as following:*

$$\frac{d(P_tC_t)}{P_tC_t} = \mu_{NC}(R_t, \pi_t)dt + \sigma_{NC}^{\mathrm{T}} \,\mathrm{dZ} \tag{43}$$

*where*

$$\mu_{NC}(R_t, \pi_t) = \varphi(R_t - \beta + x^{\mathrm{T}}\eta) + (1-\varphi)\pi_t + \phi_0 \tag{44}$$

*and*

$$\sigma_{NC} = \Gamma^{\mathrm{T}}x + a_1\sigma_R e_2 - a_2\sigma_\pi e_3 \tag{45}$$

$\beta$ *is the investor's subjective time preference and* $\phi_0$ *is a collection of the variance-covariance terms in our model.*

(ii) *The dynamic of real consumption* $C_t$ *is:*

$$\frac{dC_t}{C_t} = \mu_C(R_t, \pi_t)dt + \sigma_C^{\mathrm{T}} \,\mathrm{dZ} \tag{46}$$

*where*

$$\mu_C(R_t, \pi_t) = \varphi(R_t - \pi_t - \beta + x^{\mathrm{T}}\eta) + \phi_1 \tag{47}$$

$\phi_1$ *represents a collection of the variance-covariance terms and*

$$\sigma_C = \Gamma^{\mathrm{T}}x + a_1\sigma_R e_2 - a_2\sigma_\pi e_3 - \sigma_p e_3 \tag{48}$$

Proposition 2 shows that the elasticity of intertemporal substitution decides the sensitivity of the expected nominal and real consumption growth with respect to the nominal interest rate, estimated inflation and the excess return of the portfolio. The coefficient of risk aversion has no explicit effect on consumption growth. In contrast, the coefficient of risk aversion decides the portfolio rule while the elasticity of substitution has no effect on portfolio choice. This is why we argue that we should separate the elasticity of substitution from the measure of risk aversion when solving a problem involving the portfolio and consumption choice simultaneously. To compare the expected growth rate of the real consumption with the nominal consumption, we firstly examine Equation (44) and (47). It is clear that the expected real consumption growth is decided by the difference of $R_t - \pi_t$, which could be viewed as the proxy for the real interest rate and the expected real consumption growth is positively related to the real interest rate. However, the nominal consumption growth is not a function of the real interest rate $R_t - \pi_t$. In fact, for a given real interest level, say $R_t - \pi_t \equiv \overline{r}$, the nominal and the real consumption growth could be re-expressed as

$$\mu_C = \varphi(\overline{r} - \beta + x^{\mathrm{T}}\eta) + \phi_1 \tag{49}$$

and

$$\begin{aligned} \mu_{NC} &= \varphi\left(\overline{r} - \beta + x^{\mathrm{T}}\eta\right) + \pi_t + \phi_0 \\ &= \mu_C + \pi_t + \phi_0 - \phi_1 \end{aligned} \tag{50}$$

The growth rate of nominal consumption is positively related the sum of the growth of real consumption and the estimated inflation since the nominal amount of consumption would change according to the change of real units of consumption and the change of the nominal price to purchase the consumption goods as well.

# 4. Conclusions

We have derived the optimal intertemporal portfolio-consumption choice of the investor with the stochastic differential utility under inflation. The optimal portfolio rule depends on the coefficient of risk aversion while the consumption plan relies on the elasticity of intertemporal substitution. In contrast with other related works which adopt the one-factor real interest model and turn into the two-factor nominal interest rate under uncertain inflation, we think that the real interest rate is an unobservable state variable and we adopt a one-factor nominal interest rate which is observable to the investor and hence a two-factor real interest rate. We also argue that the expected inflation is never observable to the investor. The investor could only infer the value of expected inflation from the realized data of nominal price and suffers from the estimation risk to some extent. With the inclusion of indexed bonds in the portfolio set, we mainly find that the risk of nominal interest rate is perfectly hedged by the holdings of nominal bonds while the inflation and price risks is hedged by the holdings of indexed bonds. The demand of nominal bonds is crowded out proportionally to the demand of indexed bonds. When the maturities of the nominal and indexed bonds are identical, the demand of the indexed bond is perfectly financed by shorting the corresponding amounts of the nominal bond. The estimation risk of inflation also partly account for the demand of hedging bond portfolio.

As to the consumption, we find that the consumption-wealth ratio is obtained as an exponentially affine function of the nominal interest rate and expected inflation rate. The value of elasticity of intertemporal substitution decides the direction in which the nominal interest rate and the expected inflation affect the consumption-wealth ratio. When the elasticity of intertemporal substitution is greater than one, the substitution effect dominates. The consumption-wealth ratio falls as the nominal interest rate rises or the inflation falls. The income effect would dominate when the elasticity of intertemporal substitution is less than one. In this case the consumption-wealth ratio rises as the nominal interest rate rises or the expected inflation falls. It is also noted that the consumption-wealth ratio is not perfectly decided by the difference of the nominal interest rate and the inflation, say the real interest rate. The consumption-wealth ratio also varies with the absolute levels of the nominal interest rate and the expected inflation rate given the real interest rate unchanged. The effects of the nominal variables on the consumption-wealth ratio depend on the relative persistency of the disturbance of the nominal interest rate and the expected inflation rate. The expected growth rates of real and nominal consumption are also derived. We show that the nominal consumption growth is decided by the sum of the real consumption growth and the expected inflation rate.

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