Revisiting Asset Pricing Anomalies in an Exchange Economy

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Background

Several asset pricing anomalies in standard one-tree consumption based exchange economy with CRRA utility and i.i.d. consumption growth, e.g.,

- Equity premium puzzle
- Excess volatility puzzle
- Risk-free rate puzzle

Vast literature on explaining these puzzles

- Usually quite drastic changes of standard model: E.g., change preferences (habit formation, catching up with the Joneses), bubbles, etc.
- Often focus on one puzzle at a time.

Standard model

Variable	Value
Risk-free rate, r_s	$\rho + \gamma\left(\mu + \frac{\sigma^2}{2}\right) - \gamma(\gamma+1)\frac{\sigma^2}{2}$
Long rate, r_l	$\rho + \gamma\left(\mu + \frac{\sigma^2}{2}\right) - \gamma(\gamma+1)\frac{\sigma^2}{2}$
Market return, r_e	$\rho + \gamma\left(\mu + \frac{\sigma^2}{2}\right) - \gamma(\gamma-1)\frac{\sigma^2}{2}$
Dividend yield, $\eta \stackrel{\text{def}}{=} D/P$	$\rho + (\gamma-1)\mu - (\gamma-1)^2\frac{\sigma^2}{2}$
Market risk premium, $r_e - r_s$	$\gamma\sigma^2$
Consumption volatility	σ
Dividend volatility	σ
Price volatility	σ
Market Sharpe ratio	$\gamma\sigma$

Table: γ is risk aversion parameter (<15), σ is consumption volatility (<4%), ρ is personal discount rate (<10%), μ is consumption growth rate ($\approx$ 1%).

Contribution of paper

We introduce the Minimum Consumption (MC) model. By including an arbitrarily small risk-free tree in the economy, puzzles are *significantly* reduced:

- Market price of risk increases: New bound, $\gamma^2\sigma^2$. Risk premium higher, although less risky!
- Return volatility increases, New bound, $\gamma\sigma$.
- Risk-free rate at longer horizons is independent of risk-aversion: New formula, $r_l = \rho + \frac{\mu^2}{2\sigma^2}$.

Example from Weitzman (2007): Market risk premium of 6% and consumption volatility of 2%.

- Standard model: Risk-aversion coefficient: $\gamma = 150$, Price volatility: $\sigma_p = 2\%$, Risk-free rate: $r < -300\%$.
- MC model: Risk-aversion coefficient: $\gamma = 12$, Price volatility: $\sigma_p = 24\%$, Long risk-free rate: $r_l = 3\%$.

Related Literature

- Cochrane, Longstaff and Santa Clara (2007)
- Martin (2009)
- Parlour, Stanton and Walden (2009)
- Weitzman (1998,2001,2007)
- Barro (2005)
- Rietz (1988)
- Abel (1990)
- Campbell and Cochrane (1999)
- Blanchard (1979)
- Froot and Obstfeld (1991)
- Weil (1989)
- Le Roy and Porter (1981)
- Mehra and Prescott (1985)

Model (1/2)

Price taking representative agent:

- CRRA utility: $U = E\left[\int_0^\infty e^{-\rho t}\frac{(D_t+B_t)^{1-\gamma}}{1-\gamma}\,dt\right]$

Risky sector, size D:

- Fruits: Instantaneous Ddt.
- Growth $dD = \hat{\mu}D\,dt + \sigma D\,d\omega$.

Risk-free sector, size B.

- Fruits: Instantaneous Bdt.
- Growth $dB = rBdt$.
- For simplicity, $r = 0$.

Complete competitive market:

- Price of asset paying dividend stream, ξ_s:

$$P_0 = (B + D_0)^\gamma E_0\left[\int_0^\infty e^{-\rho s}\frac{1}{(B + D_s)^\gamma}\xi_s\,ds\right].$$

Model (2/2)

A small, but positive, B creates a lower bound on consumption. Motivation:

- Technological memory — subsistence farming
- Transfer across generations

Note: If $B = 0$, model reduces to standard model

- CRRA utility: $U = E\left[\int_0^\infty e^{-\rho t} \frac{D_t^{1-\gamma}}{1-\gamma}\, dt\right]$

What if B is very close to 0?

- Real dynamics indistinguishable from standard model
- What about asset prices?

The break-point (1/2)

Proposition

In the MC model, the normalized value function of the representative agent, $w(z)$, is given by

$$w(z) = \frac{z^{-\kappa}(1-z)^{1-\gamma-\kappa}}{q(1-\gamma)}\Bigg[V\left(\frac{1-z}{z},\kappa,2-\gamma\right) + \left(\frac{1-z}{z}\right)^{\frac{2q}{\sigma^2}} V\left(\frac{z}{1-z},\alpha+\frac{2q}{\sigma^2}-1,2-\gamma\right)\Bigg].$$

Here,

$$V(y,a,b) \stackrel{\text{def}}{=} \int_0^y t^{a-1}(1+t)^{b-1}\,dt$$

is defined for $a > 0$. Also, $w(0) = \frac{1}{\rho(1-\gamma)}$.

Moreover, recall that the dividend yield, $\eta = \rho + (\gamma-1)\mu - (\gamma-1)^2\frac{\sigma^2}{2}$. Then, if $\eta > 0$, $w(1) = \frac{1}{\eta(1-\gamma)}$. If, on the other hand, $\eta \leq 0$, then $w(1) = -\infty$.

The break-point (2/2)

Define: $q = \sqrt{\mu^2 + 2\rho\sigma^2}$, $\kappa = \frac{\mu+q}{\sigma^2}$, $\alpha = \gamma - \kappa$.

Equivalent definitions of the break-point

- $\eta < 0$
- $\alpha > 1$
- $\gamma > 1 + \kappa$

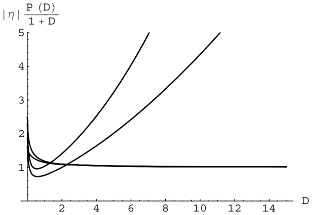
Price-dividend ratios

Proposition

The price of the market $P(D)$ is

$$P(D) = (1+D)^{\gamma} \frac{D^{-\kappa}}{q} \left[V(D, \kappa, 2-\gamma) + D^{\frac{2q}{\sigma^2}} V\left(\frac{1}{D}, \alpha + \frac{2q}{\sigma^2} - 1, 2-\gamma\right)\right]$$

where $q = \sqrt{\mu^2 + 2\rho\sigma^2}$, $\kappa = \frac{\mu+q}{\sigma^2}$, $\alpha = \gamma - \kappa$.



Knock-in option

Pays \$1 if $D = \epsilon$ is reached. Price $K(D_0, \epsilon)$.

$$K(D_0, \epsilon) = \underbrace{\left(\frac{1 + D_0}{1 + \epsilon}\right)^{\gamma}}_{\text{marginal utility}} \underbrace{E_0\left[e^{-\rho \tau_f}\right]}_{\text{timing}},$$

where τ_f is the stopping time

$$\tau_f \stackrel{\text{def}}{=} \inf_t \{t : D_t \leq \epsilon\}.$$

Above breakpoint

$$K(D_0, \epsilon) = \left(\frac{1 + 1/D_0}{1 + \epsilon}\right)^{\gamma} \frac{D_0^{\alpha}}{\log[D_0]}.$$

Asymptotic price dividend ratios

Proposition

The asymptotic price-dividend ratio in the MC model depends on the parameter region. Specifically,

(i) *Below the breakpoint, for large D the price-dividend ratio converges to* $\frac{P(D)}{1+D} = \frac{1}{\eta}$.

(ii) *Above the breakpoint, for large D the price-dividend ratio converges to* $c\left(\frac{D}{B}\right)^{\alpha-1}$, *for some constant* $c > 0$, *where* $\alpha = \gamma - \frac{\mu+\sqrt{\mu^2+2\rho\sigma^2}}{\sigma^2}$.

Convexity parameter, max$(\alpha, 1)$

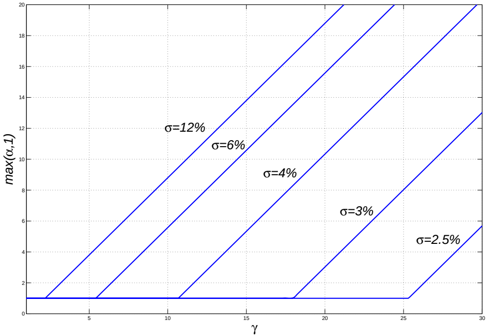


Figure: **Convexity parameter,** max$(\alpha, 1)$**, as a function of risk aversion** γ**.**
Parameters: $\hat{\mu} = 0.75\%$**,** $\rho = 1\%$**,** $\sigma = 2.5\%, 3\%, 4\%, 6\%, 12\%$**.**

Market risk premium (1/3)

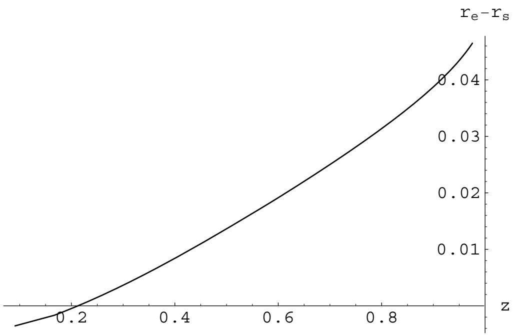


Figure: **Market risk-premium as a function of z, for fixed risk aversion. Parameters: $\hat{\mu} = 0.75\%$, $\rho = 1\%$, $\sigma = 4\%$, $\gamma = 12.25$.**

Market risk premium (2/3)

Proposition

For large D

(i) *Below the breakpoint, the market risk premium is the same as in the standard model:* $r_e - r_s = \gamma\sigma^2$.

(ii) *Above the breakpoint, the market risk premium is* $r_e - r_s = \alpha\gamma\sigma^2$.

Market risk premium (3/3)

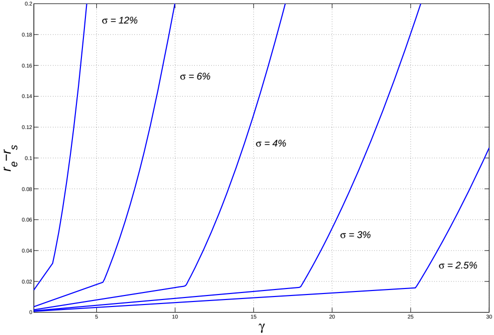


Figure: **Market risk premium for large *D*, as a function of risk aversion, γ. Parameters: $\hat{\mu} = 0.75\%$, $\rho = 1\%$, $\sigma = 2.5\%, 3\%, 4\%, 6\%, 12\%$.**

Discount rates

Proposition

In the MC economy, the short-term rate is

$$r^s = \rho + \gamma z \left(\mu + \frac{\sigma^2}{2}\right) - \gamma(\gamma+1)\frac{\sigma^2}{2}z^2.$$

For $z \in (0,1)$, if $\mu \leq \gamma\sigma^2$, the long-term rate is

$$r^l = \rho + \frac{1}{2} \times \frac{\mu^2}{\sigma^2}.$$

If, on the other hand, $\mu > \gamma\sigma^2$, the long-term rate is

$$r^l = \rho + \gamma \left(\mu + \frac{\sigma^2}{2}\right) - \gamma(\gamma+1)\frac{\sigma^2}{2}.$$

Term structure

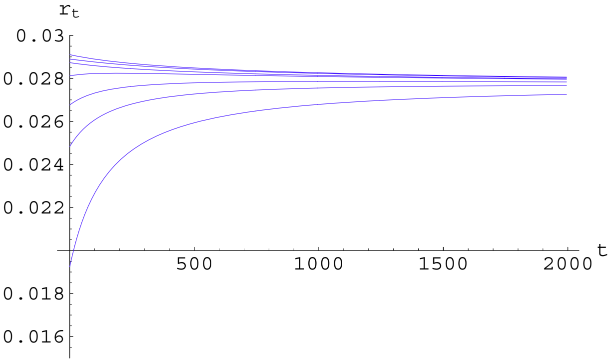


Figure: **Term structure of interest rates in the MC model. Parameters, $z = 0.7$, γ varies between 6 (highest curve) to 12 (lowest curve). Other parameters: $\hat{\mu} = 0.75\%$, $\rho = 1\%$.**

Short rate

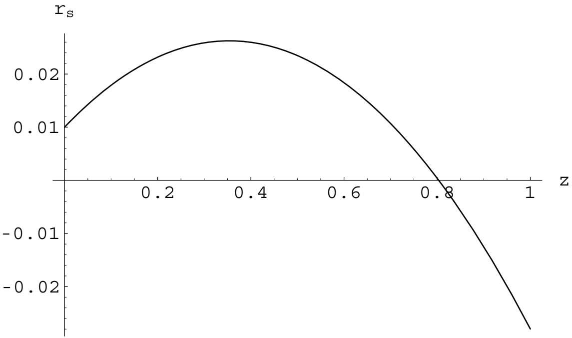


Figure: **Short rate as a function of z. parameters: $\hat{\mu} = 0.75\%$, $\rho = 1\%$, $\gamma = 12.25$.**

Volatility

$$vol\left(\frac{dP}{P}\right) = \max(\alpha, 1)\sigma.$$

Example: $\hat{\mu} = 0.75\%$, $\sigma = 4\%$, $\rho = 1\%$, $\gamma = 12.2 \Rightarrow vol\left(\frac{dP}{P}\right) = 10.6\%$

Thus, disconnect between consumption and price volatility.

Summary

Variable	Formula	Value-MC	Value-standard
Short rate, r_s	$\rho+\gamma\left(\mu+\frac{\sigma^2}{2}\right)-\gamma(\gamma+1)\frac{\sigma^2}{2}$	-2.8%	-2.8%
Long rate, r_l,when $\mu<\gamma\frac{\sigma^2}{2}$	$\rho+\frac{\mu^2}{2\sigma^2}$	2.4%	-2.8%
Long rate, r_l, when $\mu>\gamma\frac{\sigma^2}{2}$	$\rho+\gamma\left(\mu+\frac{\sigma^2}{2}\right)-\gamma(\gamma+1)\frac{\sigma^2}{2}$		
Market return, r_e, when $\alpha>1$	$\alpha\mu+\alpha^2\frac{\sigma^2}{2}$	2.2%	-0.8%
Market return, r_e, when $\alpha<1$	$\rho+\gamma\left(\mu+\frac{\sigma^2}{2}\right)-\gamma(\gamma-1)\frac{\sigma^2}{2}$		
Risk premium, r_e-r_s	$\gamma\max(\alpha,1)\sigma^2$	5.0%	2.0%
Consumption volatility	σ	4%	4%
Dividend volatility	σ	4%	4%
Price volatility	$\max(\alpha,1)\sigma$	10.3%	4%
Market Sharpe ratio	$\gamma\sigma$	0.49	0.49

Table: Properties of the MC model for large D, and an example with parameters: $\hat{\mu}=0.75\%$, $\sigma=4\%$, $\rho=1\%$, with $\gamma=12.25$, implying that $\alpha=2.58$.

Additional comments

- Finite time horizons. For $T < \infty$, standard model is defined above break-point and market risk-premium is $\gamma\sigma^2$. In MC model, risk premium is still $\alpha\gamma\sigma^2$. How?
- Convex price function also occurs in rational bubble literature. Is there a bubble?
- Relationship to Hansen-Jagannathan bounds.

The general case

Are results specific for the two-trees model? No!

Proposition

Consider an exchange economy, in which the consumption is $C_t = f(D_t)$, *where* $D_t = e^{s_t}$, *and where* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is a continuous, increasing function, such that for large* d, $c_0 d \leq f(d) \leq c_1 d$, *for some constants* $0 < c_0 \leq c_1 < \infty$.
For the stochastic process, $s_t \in \mathbb{R}$, *define the c.d.f.* $F(s, t | s_*, \mathcal{I}) = \mathbb{P}(s_t \leq s | s_0 = s_*, \mathcal{I})$, *where* $\mathcal{I}$ *captures the information known about* s_t *at* $t = 0$. *Assume that the following condition is satisfied:*

$$\exists \mu, \exists \sigma > 0,\ \exists \bar{t} \geq 0,\ \exists \underline{s},\ \exists \bar{s},\ \forall t \geq \bar{t},\ \forall s_* \geq \bar{s},\ \forall s < \underline{s} :\ \ F(\underline{s}, t | s_*, \mathcal{I}) \geq \Phi\left(\frac{s - s_* - \mu t}{\sigma \sqrt{t}}\right).$$

Further, assume that the economy is beyond the breakpoint, i.e., that $\alpha = \gamma - \frac{\mu + \sqrt{\mu^2 + 2\rho\sigma^2}}{\sigma^2} > 1$.
Then

(i) *If* $f(x) \leq c_2 x$ *in a neighborhood of* $x = 0$, *for some constant* $c_2 \geq 0$, *then there is no equilibrium in the economy.*

(ii) *If* $f(0) > 0$, *then in any equilibrium the price of the market satisfies* $P(C_0) \geq c_3 C_0^{\alpha}$, *for some constant* $c_3 > 0$.

Examples

Example

Economy already studied: $f(x) = 1 + x$, and $s_t \sim N(s_0 + \mu t, \sigma^2 t)$.

Example

Economy with $f(x) = 1 + x$, mean-reverting growth process,

$$\begin{aligned} ds_t &= \mu_t\, dt + \hat{\sigma}\, d\omega_1, \\ d\mu_t &= \beta(\mu - \mu_t)\, dt + \sigma_\mu\, d\omega_2, \end{aligned}$$

where μ, $\hat{\sigma}$, σ_μ, and β are positive constants and where $cov(d\omega_1, d\omega_2) = \rho\, dt$. Satisfies proposition with $\sigma^2 \stackrel{\text{def}}{=} \frac{\sigma_\mu^2 + 2\sigma_\mu \beta \rho \hat{\sigma} + \beta^2 \hat{\sigma}^2}{\beta^2}$.

Conclusions

- Rare events crucial in MC-model
  - Impact can be understood by knock-in option analysis: Above the break-point marginal utility effect outweighs discount effect.
  - Results seem to be the opposite of Barro (2005), Weitzman (2007) and Rietz (1988).
  - However, the two approaches are actually similar
- Intriguing that dynamics completely change in an ϵ-modification of the standard model.
  - Standard setting highly sensitive to perturbations.
  - Real variables empirically indistinguishable from standard model.
  - Asset pricing puzzles significantly decreased — but not completely solved.