Consistent Pricing of Equity and Volatility Derivatives

Ser-Huang Poon (Manchester Business School & RMI)

Joint work with He Xue Fei (RMI), Simon Acomb (MBS)

September 4, 2009

Who Trades Volatility?

- Traders of Derivative Contracts
  - Hedging vega exposure
  - Hedging correlation exposure using dispersion trading
- Asset Managers
  - Volatility has become a new asset class
  - Low correlation with other assets to produce portfolio diversification
- Proprietary Traders
  - Hedge funds and trading desks taking position in volatility
- Proprietary traders of dispersion
  - As a hedge against credit due to the negative relationship between credit and equity volatility
- People with Structure Exposure to Volatility
  - Statistical arbitrage
  - Institutions where P&L is structurally exposed to volatility

Equity and Volatility Derivatives

- Definition; any products or contracts that are sensitive to volatility specification
- S&P 500, SPX and VIX products as examples
- Demand for OTC contracts on other indices and individual stocks is strong
- Like other OTC contracts (e.g. interest rate derivatives), likely to remain as OTC and grow into trillion if it is not already there
- Many such contracts have existed for a long time.

Equity or Index Level Contracts

- Index linked notes
- SPX call and put options (CBOE), OTC warrants
- Exotic
    - Digital Options
    - Knock-in, Knock-out, Barrier
    - Asian
    - CPPI (Constant proportion portfolio insurance)

Volatility Level Contracts

- Volatility swaps
- Volatility Index linked
  - Volatility futures
  - Volatility options
- Exotic
  - Cliquet: Forward Starting Options
  - Napoleon

Use Variance and Volatility Interchangeably

Why a consistent pricing model is important?

- A typical "Spot" model for equity and volatility

$$\begin{aligned} dS &= \mu_S(S,t)\,dt + \sigma_S(S,t)\,dW_S \\ dV &= \mu_V(V,t)\,dt + \sigma_V(V,t)\,dW_V \\ V &= f(S,\sigma_V) \\ \langle dS, dV \rangle &= \rho\,\sigma_S\,\sigma_V\,dt \end{aligned}$$

- "Spot" vs. "Forward" as in the interest rate derivatives literature
- Omit jumps in this entire study; if necessary use change (of business) time
- Assume all dynamics are under risk neutral measure

Volatility Consistent Pricing Model

$$dS = \mu_S(S,t)\,dt + \sigma_S(S,t)\,dW_S \quad (1)$$

$$dV = \mu_V(V,t)\,dt + \sigma_V(V,t)\,dW_V \quad (2)$$

- Typically, (1) and (2) are used to price equity or index level contracts eventhough volatility contracts (e.g. VIX options) suggests a dynamics very different from (2).
- Volatility level contracts are priced using (2) only without considering the implications on (1).
- Inconsistency is pervasive and yet volatility level products are hedged or replicated using equity level products.

Example: S&P 500, SPX and VIX

They all share the common root

S&P 500 (Physical vs. Risk Neutral Measure)

$\downarrow$

SPX call and put options

$\downarrow$

Variance Swap and VIX

$\downarrow$

Volatility futures and options

Variance Swap

Forward contract on realised variance with payoff at maturity T:

$$
\begin{aligned}
VarSwap_T &= N\left(\sigma^2 - K_{var}\right) \\
\sigma^2 &= 252 \sum_{t=1}^{n} \frac{r_t^2}{n}
\end{aligned}
$$

where

- σ_R^2 is the realised variance (or annualised variance) of returns over the life of the contract,
- The strike price K_{var} may be quoted as K_{vol}^2.
- N is the notional amount of the swap contract (typically \$100,000 on OTC)

Variance Swap

Demeterfi, Derman, Kamal and Zou (1999)

- Assume stock prices evolve without jump

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu(t, \cdots) dt + \sigma(t, \cdots) dW_t \\ &= \mu_t dt + \sigma_t dW_t \end{aligned} \quad (3)$$

- Then realised variance is the continuous integral

$$\begin{aligned} RV &= \left\langle \frac{dS_t}{S_t} \right\rangle = \frac{1}{T} \int_0^T \sigma_t^2 dt \\ \frac{T}{2} RV &= \frac{1}{2} \int_0^T \sigma_t^2 dt. \end{aligned}$$

- From Ito's lemma,

$$d \ln S_t = \left(\mu_t - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dW_t. \quad (4)$$

Subtracting (4) from (3), we get

$$\frac{dS_t}{S_t} - d\ln S_t = \frac{1}{2}\sigma_t^2$$

So

$$RV = \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t}dt - \ln\frac{S_T}{S_0}\right] \quad (5)$$

- The first term of (5), $\int_0^T \frac{dS_t}{S_t}dt$, is a long position in $\frac{1}{S_t}$ share continuously rebalanced.
- The second term of (5), $\ln\frac{S_T}{S_0}$, is the payoff of a log contract.

Log contract

Neuberger (1994)

- The log contract is a futures style contract whose settlement price is equal to the logarithmic of the price of the asset at T or $\ln \frac{S_T}{S_0}$.
- Take

$$
\begin{aligned}
\ln \frac{b}{a} &= \ln \frac{b}{a} - b\left(\frac{1}{a} - \frac{1}{b}\right) + \left(\frac{b}{a} - 1\right) \\
&= \int_a^b \frac{1}{x} dx + \int_a^b b\left(-\frac{1}{x^2}\right) dx + \left(\frac{b}{a} - 1\right) \\
&= -\int_a^b (b - x)\frac{1}{x^2} dx + \left(\frac{b}{a} - 1\right) \\
&= -1_{b>a}\left[\int_a^b (b - x)\frac{1}{x^2} dx\right] - 1_{a>b}\left[\int_b^a (x - b)\frac{1}{x^2} dx\right] \\
&\quad + \left(\frac{b}{a} - 1\right)
\end{aligned}
$$

Log contract

- For $x \in [0, \infty)$

$$\ln \frac{b}{a} = \left(\frac{b}{a} - 1\right) - \left[\int_a^{\infty} (b - x)^+ \frac{1}{x^2} dx\right] - \left[\int_0^a (x - b)^+ \frac{1}{x^2} dx\right].$$

- Now let

$$\begin{aligned} b &= \widetilde{S}_T \\ a &= S_0 \\ x &= K \end{aligned}$$

- Then

$$\begin{aligned} \ln \frac{\widetilde{S}_T}{S_0} &= \left(\frac{\widetilde{S}_T}{S_0} - 1\right) - \left[\int_{S_0}^{\infty} (S_T - x)^+ \frac{1}{K^2} dK\right] \\ &\quad - \left[\int_0^{S_0} (x - S_T)^+ \frac{1}{K^2} dK\right] \end{aligned}$$

Realised Variance

Substitute the definition of log contract

$$\frac{T}{2}RV = \int_0^T \frac{dS_t}{S_t}dt - \left(\frac{\widetilde{S}_T}{S_0} - 1\right) + \left[\int_{S_0}^{\infty} (S_T - x)^+ \frac{1}{K^2}dK\right] + \left[\int_0^{S_0} (x - S_T)^+ \frac{1}{K^2}dK\right]$$

- The first term, as mentioned before, is a futures contract on $\frac{1}{S_t}$.
- The second term, $\left(\frac{\widetilde{S}_T}{S_0} - 1\right)$ is a short position of $\frac{1}{S_0}$ amount of forward contract struck at $S_0 = K_{ATM}$.
- The third term a long position in $\frac{1}{K^2}$ put struck at K for a continum of K from 0 to S_0.
- The forth term is a long position in $\frac{1}{K^2}$ call struck at K for a continum of K from S_0 to ∞.
- Under risk neutral expectation, the first and second terms are zero.

Fair Value Pricing Principle

- At contract initialisation, the fair value of volatility (or variance) is the delivery price that makes the swap has zero value at initialisation such that

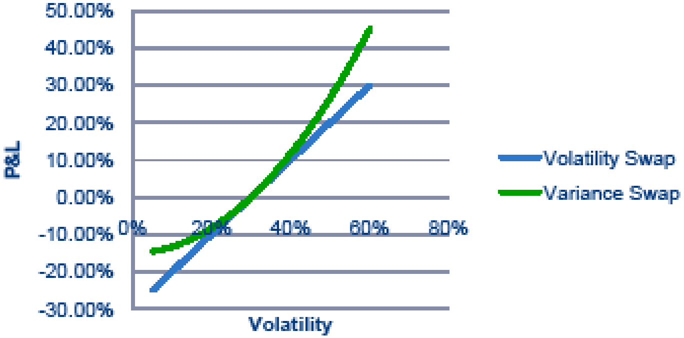
$$E_0^Q\left(VarSwap_T\right) = E_0^Q\left(\sigma_R^2 - K_{var}\right) = 0$$

$$\begin{aligned} K_{var} &= E_0^Q\left(\sigma_R^2\right) \\ &= \frac{2}{T}\left[\int_{S_0}^{\infty}\left(S_T - x\right)^+ \frac{1}{K^2} dK\right] + \left[\int_0^{S_0}\left(x - S_T\right)^+ \frac{1}{K^2} dK\right] \end{aligned}$$

- This means variance swap can be replicated by trading a continuum of options each strike weighted by $1 \big/ K^2$.

Volatility Swap

Since $\sqrt{E\left(x^2\right)} > E\left(x\right)$, variance swap is more expensive than volatility swap, and the difference is due to convexity adjustment with respect to the volatility of volatility.



VIX Futures and Forward Variance

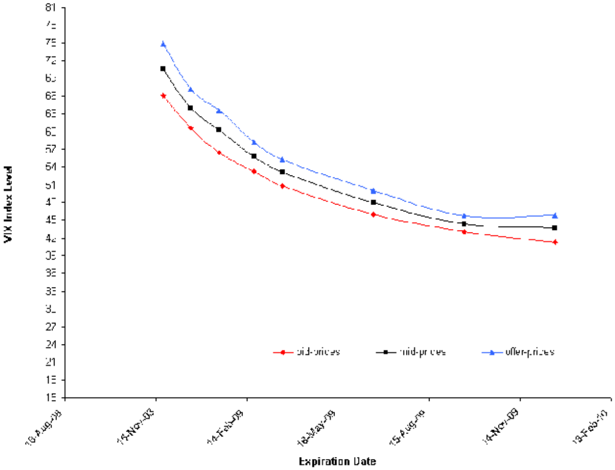
A forward starting variance swap is just the difference between two variance swaps.

$$
\begin{aligned}
\int_0^{T_2} \sigma_t^2 dt &= \int_0^{T_1} \sigma_t^2 dt + \int_{T_1}^{T_2} \sigma_t^2 dt \\
\int_{T_1}^{T_2} \sigma_t^2 dt &= \int_0^{T_2} \sigma_t^2 dt - \int_0^{T_1} \sigma_t^2 dt \\
E\left[\sigma_{T_1,T_2}^2\right] &= T_2 \times E\left[\sigma_{T_2}^2\right] - T_1 \times E\left[\sigma_{T_1}^2\right]
\end{aligned}
$$

- A VIX future is the expectation today, of the square root of the future expectation of variance. Hence, a VIX future is bounded above by the fair volatility of a forward starting variance swap.
- Value of the future reflects the SPX volatility term structure; an upward sloping vol surface will produce bigger value of VIX futures through time.
- Futures de-correlate with time. Near dated futures are more correlated than far dated futures.

VIX Term Structure

VIX Term Structure data calculated using SPX option prices on Wednesday 29 October, 2008.



VIX Options

- Multiplier of \$100 a point (compared with \$1000 for futures)
- Range of strikes traded at 2.5pt intervals
- Maturities of the next two VIX futures expiries + quarterly cycle
- Expire into the same VIX calculation.
- European style exercise, call and puts.
- Facilitates range of strategies. What combination would be appropriate for the following ?
  - Protection from rise in volatility
  - Believe that volatility is going to become more unstable
  - Volatility term structure is going to become less steep.

Pricing VIX Options with Black-Scholes (or Black)

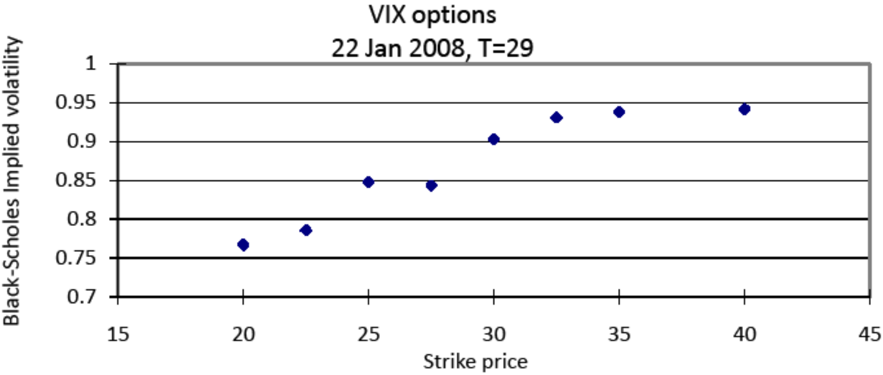
Carr and Wu (2006) treat VIX like a commodity with known forward price. (i.e. VIX futures price in this case) Then

$$
\begin{aligned}
c &= e^{-rT}\left[FN(d_1) - KN(d_2)\right] \\
d_1 &= \frac{\ln\frac{F}{K} + 0.5\eta T}{\eta\sqrt{T}}, \quad d_2 = d_1 - \eta\sqrt{T}
\end{aligned}
$$

However, Greeks become difficult

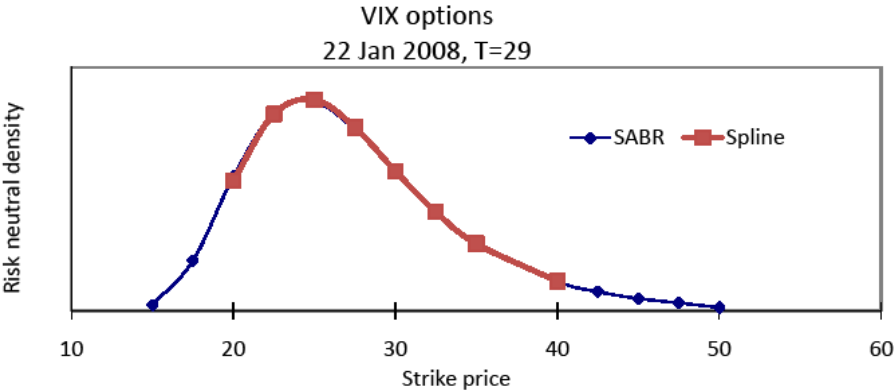
- Delta is sensitivity to VIX (a lot like SPX vega), whereas Vega is sensitivity to Vol of Vol, η.
- There is no delta to the S&P (but it surely exists!!)
- There is no mechanism to pricing more exotic instruments – e.g. S&P option with expiries with VIX reaches 40.
- There is no insight into the combined dynamics of S&P and volatility

VIX Options "Volatility" Smile/Skew



The Black-Scholes (Black) cannot be the right model!!!

VIX Options Risk Neutral Density



Heston for the Vol Surface

- This is the model underlying Bloomberg's variance swap platform!!!
- Heston Process

$$\begin{aligned} ds &= rsdt + s\sqrt{v}dw_t^s && (6) \\ dv &= k\left(\theta - v\right)dt + \eta\sqrt{v}dw_t^v && (7) \\ corr\left(w_t^s, w_t^v\right) &= \rho \end{aligned}$$

- The distribution of v_t is non-centred chi-squared under Heston.

Realized variance under Heston

Realized variance $J(t, T)$:

$$J(t, T) = \frac{1}{T - t} \int_t^T v_\tau d\tau \tag{8}$$

$I(t, T)$ is the time t expectation of $J(t, T)$,

$$I(t, T) = E_t [J(t, T)] = \frac{1}{T - t} \int_t^T E(v_\tau)\, d\tau \tag{9}$$

$$
\begin{aligned}
I(t,T) &= \theta\left[1-\left(\frac{1-e^{-k(T-t)}}{k(T-t)}\right)\right]+v_t\left(\frac{1-e^{-k(T-t)}}{k(T-t)}\right) && (10)\\
&= A(t,T)+B(t,T)v_t && (11)
\end{aligned}
$$

$$
V_t^{swap}=\left(K-\frac{1}{T}\int_0^t v_\tau d\tau-\frac{T-t}{T}\cdot I(t,T)\right)\cdot Not\cdot e^{-r(T-t)}
$$

At inception, $V_0^{swap}=0$

$$
K=I(0,T)=\theta\left(1-\frac{1-e^{-kT}}{kT}\right)+v_0\left(\frac{1-e^{-kT}}{kT}\right)
$$

See Lipton and Pugachevsky (1998), Chatsangar and Poon (2009)

Variance Swaption

If T is the maturity date of the swaption and T' is the maturity of the swap and $T < T'$,

$$V_t^{swaption} = E_t \max\left[\left(K - I(T, T')\right), 0\right] \cdot Not \cdot e^{-r(T-t)}$$

Define

$$\begin{aligned} \widetilde{K} &= \frac{K - A(T, T')}{B(T, T')} \\ V_t^{swaption} &= B(T, T') E_t \max\left[\left(\widetilde{K} - v_T\right), 0\right] \cdot Not \cdot e^{-r(T-t)} \end{aligned}$$

Variance Swaption under Heston: A "closed form" solution

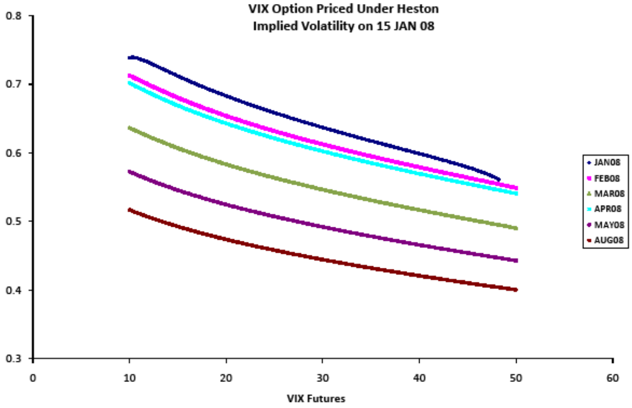
Lipton and Pugachevsky (1998), Chatsangar and Poon (2009)

$$
\begin{aligned}
E_t\left[\left(\widetilde{K}-v_T\right)^+\right] &= \left[\widetilde{K}-E(v_T)\right]\cdot F\left(2c\widetilde{K};\frac{4k\theta}{\eta^2},2cv_t e^{-k(T-t)}\right)\\
&\quad +E(v_T)\cdot f\left(2c\widetilde{K};\frac{4k\theta}{\eta^2}+2,2cv_t e^{-k(T-t)}\right)\\
&\quad +v_t e^{-k(T-t)}\cdot f\left(2c\widetilde{K};\frac{4k\theta}{\eta^2}+4,2cv_t e^{-k(T-t)}\right)
\end{aligned}
$$

Fair value of volatility swap based on Taylor's series approximation

$$
E\sqrt{J(0,T)}\approx\sqrt{E\left[J(0,T)\right]}-\frac{Var\left[J(0,T)\right]}{8\left(E\left[J(0,T)\right]\right)^{3/2}}
$$

Heston "Implied Vol" as of March 5, 2008



Heston cannot be the right model!!!

$$dS(t) = \sigma_S(t) S(t) d\widetilde{W}_S$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & t & & & & T & & \\
 & & & & & & \downarrow & & & & \downarrow & & \\
0 & — & T_1 & — & T_2 & — \cdots & T_{k(t)-1} — & — & T_{k(t)} & — \cdots & T_n & — & T_{n+1}
\end{array}
$$

$$
\begin{aligned}
d\Omega_k(t) &= v_k \Omega_k^{\beta_k}(t) \, d\widetilde{Z}_k^{Q_{T_{k(t)+1}}} \\
&\cdots \\
d\Omega_i(t) &= v_i \Omega_i^{\beta_i}(t) \, d\widetilde{Z}_i^{Q_{T_{i+1}}} \\
&\cdots \\
d\Omega_n(t) &= v_n \Omega_n^{\beta_n}(t) \, d\widetilde{Z}_n^{Q_{T_{n+1}}}
\end{aligned}
$$

LMM for Forward Variance

$$\begin{aligned} dS(t) &= \sigma_S(t)\, S(t)\, d\widetilde{W}_S \\ d\Omega_i(t) &= v_i \Omega_i^{\beta_i}(t)\, d\widetilde{Z}_i^{Q_{T_{i+1}}} \end{aligned}$$

- $d\widetilde{W}_S$ is an equivalent measure of dW_S after appropriate Girsanov transform (and similarly for $dZ_i^{Q_{T_{i+1}}}$).
- Aggregate variance $\Omega_i(t)$ is a martingale under its own terminal measure with forward starting swap at T_i as its numeraire.
- Each $d\widetilde{Z}_i^{Q_{T_{i+1}}}$ is correlated with $d\widetilde{W}_S$ with coefficient ρ_i.
- Could write $v_i = v_\tau$ and $\tau = T_i - t$ to make "Vol of Vol" time homogenous.

Forward Variance under Terminal Measure

- The stock price $dS(t)$ is a driftless SV process with

$$dS(t) = \sigma_S(t)\, S(t)\, d\widetilde{W}_S$$

- Under the terminal measure for equity (derivatives)

$$\begin{aligned}\sigma_S(t) &= \overline{\sigma}_{S,T^*} = E_t\left(\frac{1}{T^*-t}\int_t^{T^*} \sigma_u du\right) \\ &= \frac{\delta}{T^*-t}\left[\frac{k(t)-t}{\delta}\Omega_k^2(t) + \cdots + \Omega_i^2(t) + \cdots + \Omega_{T^*-\delta}^2(t)\right.\end{aligned}$$

where T^* is the maturity of derivative written on $S(t)$.

Forward Variance under Periodic Risk Neutral Measure

$$dS(t) = \sigma_S(t) S(t) d\widetilde{W}_S$$

- For periodic risk neutral measure with multiple maturities and possibility of early exercise,

$$\begin{aligned} i &\leq t < i+1 \\ \sigma_S(t) &= \overline{\sigma}_{S,i} \\ &= E_t\left(\frac{1}{T_{i+1}-T_i}\int_{T_i}^{T_{i+1}} \sigma_u du\right) \end{aligned}$$

where $\overline{\sigma}_{S,i}$ is the average (or step-wise constant) variance

A Model based on Forward Variance

- Equity is a geometric process with stochastic vol

$$\frac{dS(t)}{S(t)} = (r - q)\,dt + \sigma_S(t)\,dW_S$$

$$0 \;—\; T_1 \;—\; T_2 \;—\; \cdots\; T_{k(t)-1} \;—\; \underset{\overset{\downarrow}{t}}{} \;—\; T_{k(t)} \;—\; \cdots\; \underset{T_n}{\overset{T}{\downarrow}} \;—\; T_{n+1}$$

- Calibration is based on aggregate variance

$$\Omega^2_{\underbrace{T_i,\, T_{i+1}}_{\text{[vol ref period]}}}(t) = \Omega_i^2(t) = E_t\left[\frac{1}{T_{i+1} - T_i}\int_{T_i}^{T_{i+1}} \sigma_u^2 du\right]$$

Forward Variance: Parity Relationship

$$\begin{aligned}\Omega_i^2(t) &= E_t\left[\frac{1}{T_{i+1}-T_i}\int_{T_i}^{T_{i+1}}\sigma_u^2 du\right] \\ \Omega_i^2(0) &= E_0\left[\frac{1}{T_{i+1}-T_i}\int_{T_i}^{T_{i+1}}\sigma_u^2 du\right] \\ &= \text{Fair price of a forward starting variance swap} \qquad (12)\end{aligned}$$

$$\begin{aligned}\Omega_0^2(0) &= E_0\left[\frac{1}{T_1-T_0}\int_{T_0}^{T_1}\sigma_u^2 du\right] \\ &= \text{Fair price of a spot variance swap} \qquad (13)\end{aligned}$$

From (12) and (13)

$$\Omega_{t,T_i,T_{i+1}}^2 = \Omega_{t,t,T_{i+1}}^2 - \Omega_{t,t,T_i}^2$$

Forward Variance vs. Instantaneous Volatility

$$dS(t) = \sigma_t S(t)\, d\widetilde{W}_S$$

1. $\Omega^2_{t,T_i,T_{i+1}} \to \sigma^2_t$ if $T_{i+1} \to T_i \to t$
2. Average (or step-wise constant) variance

$$\overline{\sigma}^2_{t,T_i,T_{i+1}} = \Omega^2_{t,T_i,T_{i+1}} = E_t\left[\frac{1}{T_{i+1} - T_i}\int_{T_i}^{T_{i+1}} \sigma^2_u du\right]$$

3. If $t > i + 1$, $\overline{\sigma}^2_{t,T_i,T_{i+1}}$ is the realised variance of the reference period.
4. If $i < t < i + 1$, $\overline{\sigma}^2_{t,T_i,T_{i+1}}$ is the sum of variance realised plus the expectation of the remain variance.

Dynamic of Forward Variance

- Each of the forward volatility is a SABR model

$$d\Omega_{t,T_i,T_{i+1}} = v_i \Omega_{t,T_i,T_{i+1}}^{\beta=0.5} dZ_i$$

- Homogenise "Vol of Vol"

$$\begin{aligned} v_i &= v_\tau \\ \tau &= T_i - t \end{aligned}$$

- Correlation with equity price

$$\langle dZ_{i,t}, dW_{S,t} \rangle = \rho_{i,S} \ \sigma_S \ v_i \ dt$$

- Correlation between volatility

$$\langle dZ_{i,t}, dZ_{j,t} \rangle = \rho_{i,j} \ v_{\tau_i} \ v_{\tau_j} \ dt$$

Could reduce this to a factor structure or just model adjacent variance rates.

Calibration:

- Assuming that there are only three time references, t, T_i and T_{i+1}.
- Volatility options and futures at different maturities will use used to extract $\rho_{i,j}$, v_{τ_i}, v_{τ_j}, Ω_{t,t,T_i} and $\Omega_{t,T_i,T_{i+1}}$.
- Use Ω_{t,t,T_i} to price SPX options with maturity T_i under terminal measure, with the variance swap expiring at T_i as the numeraire.
- Use Ω_{t,t,T_i} and $\Omega_{t,T_i,T_{i+1}}$ to construct Ω_{t,t,T_i+1} and repeat the previous step.
- The calibration results will be checked against the fit of SPX call and put options.
- The calibrated models can be used to price exotics on equity and volatility.

References:

- Carr Peter and Liuren Wu (2006) A tale of two indices, *Journal of Derivatives*, 13, 3, 13-29.
- Chatsangar Ratchada and Ser-Huang Poon (2009) Volatility Derivatives: Volatility/Variance Swap and VIX Option, Working Paper, Manchester Business School.
- Demeterfi Kresimir, Emanuel Derman, Michael Kamal and Joseph Zou (1999) A guide to volatility and variance swap (previously more than you ever want to know about Vol/Var swap), *Journal of Derivatives*, 6, 4, 9-32.
- Lipton Alex and Dmitry Pugachevsky (1998) Pricing of volatility-sensitive products in the Heston model framework, Working Paper, Deutch Bank.
- Neuberger Anthony (1994) The log contract, *Journal of Portfolio Management*, 20, 2, 74-90.