Dynamic Portfolio Choice and Consumption Plan under Inflation with Nominal and Indexed Bonds

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   a) Nominal Bond Market
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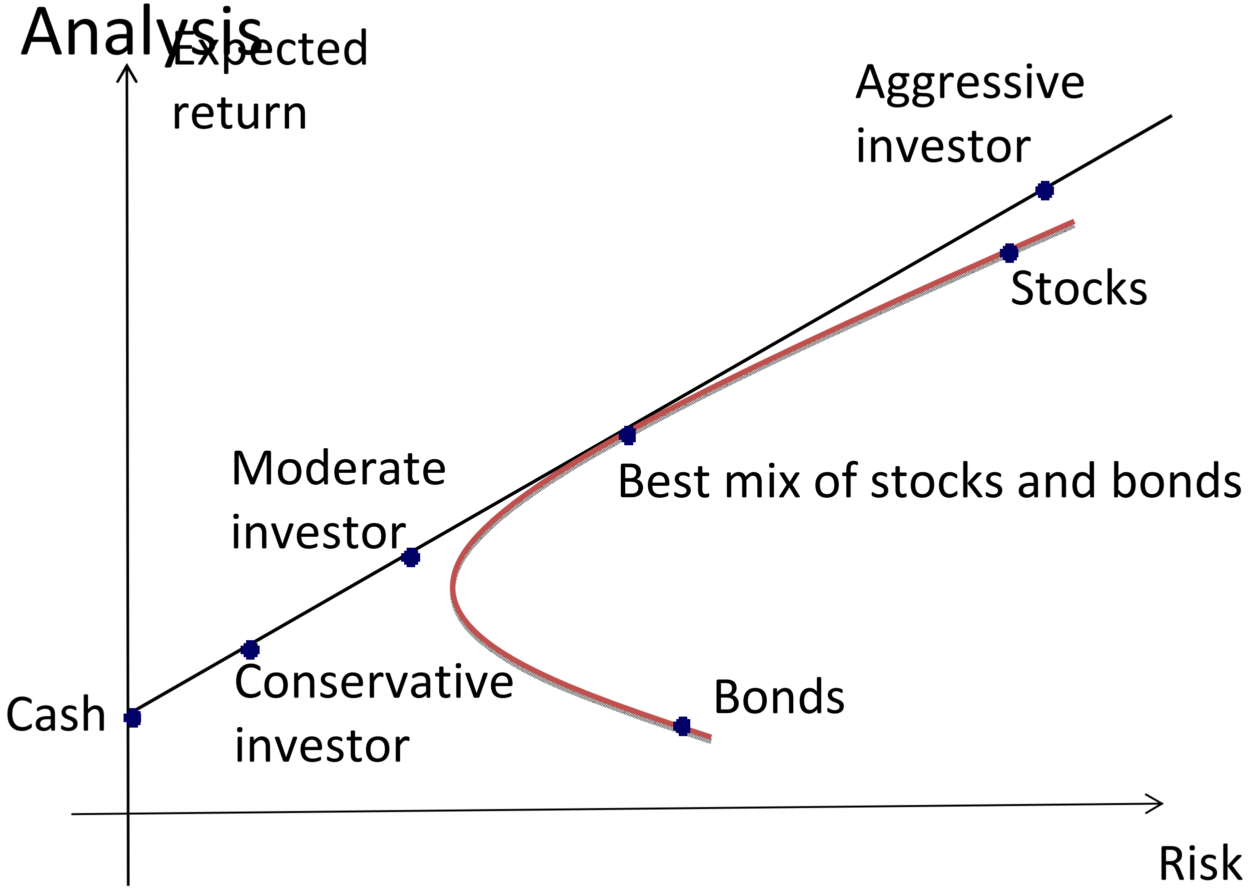
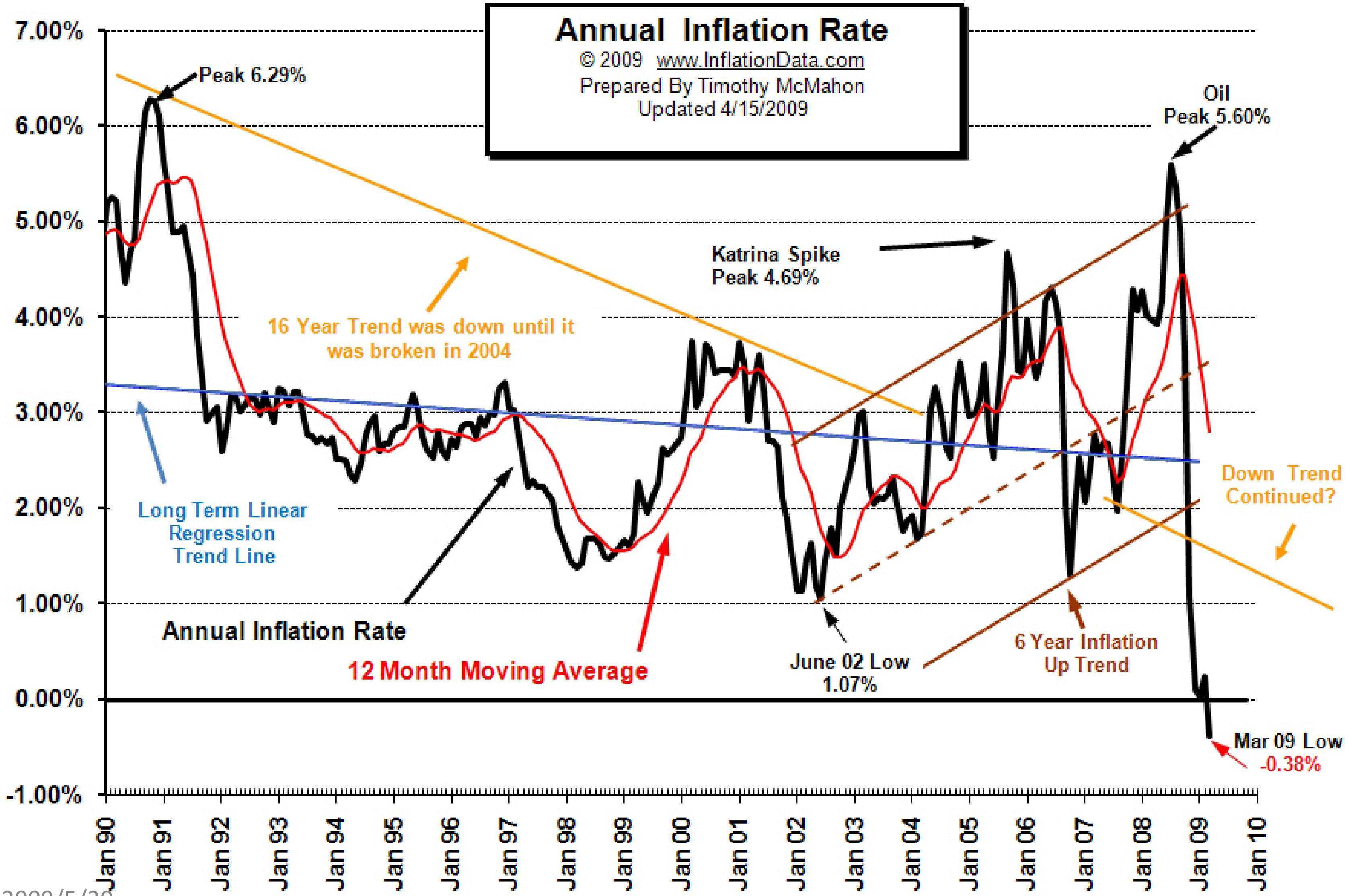
Introduction

Traditional Static Mean-Variance Analysis



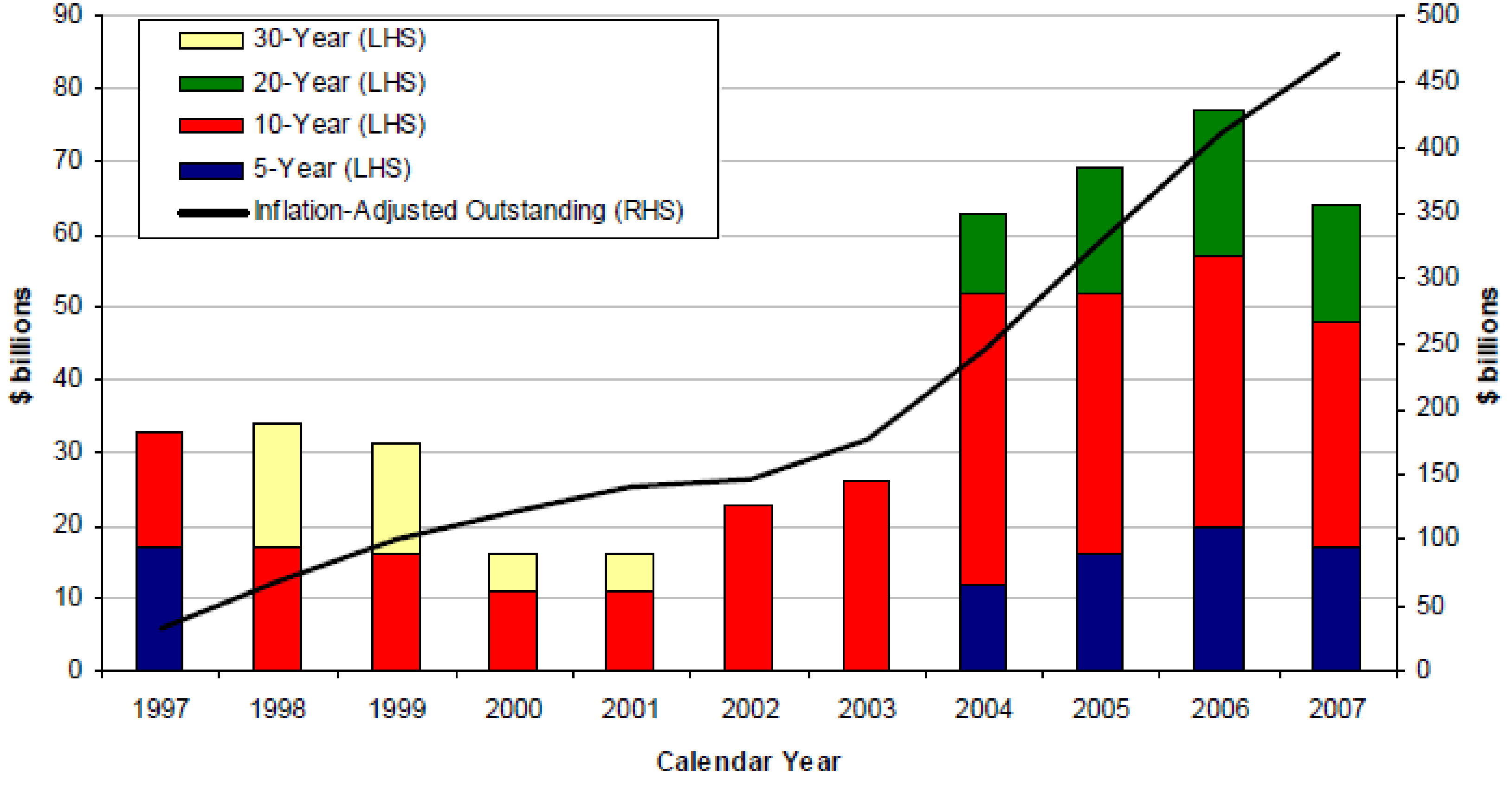
Introduction



Introduction

- The indexed-bond market is growing fast since last decade. The U.S. Treasury has been issuing the Treasury Inflation Protected Securities (TIPS) since January 1997. Next graph shows TIPS Annual Issuance and Outstanding.

Introduction



TIPS Annual Issuance and Outstanding

Introduction

- The principal of a TIPS increases with inflation and decreases with deflation, as measured by the Consumer Price Index, but the inflation-adjusted principal will not be paid until maturity.
- The interest rate, which is set at auction, remains fixed throughout the term of the security.

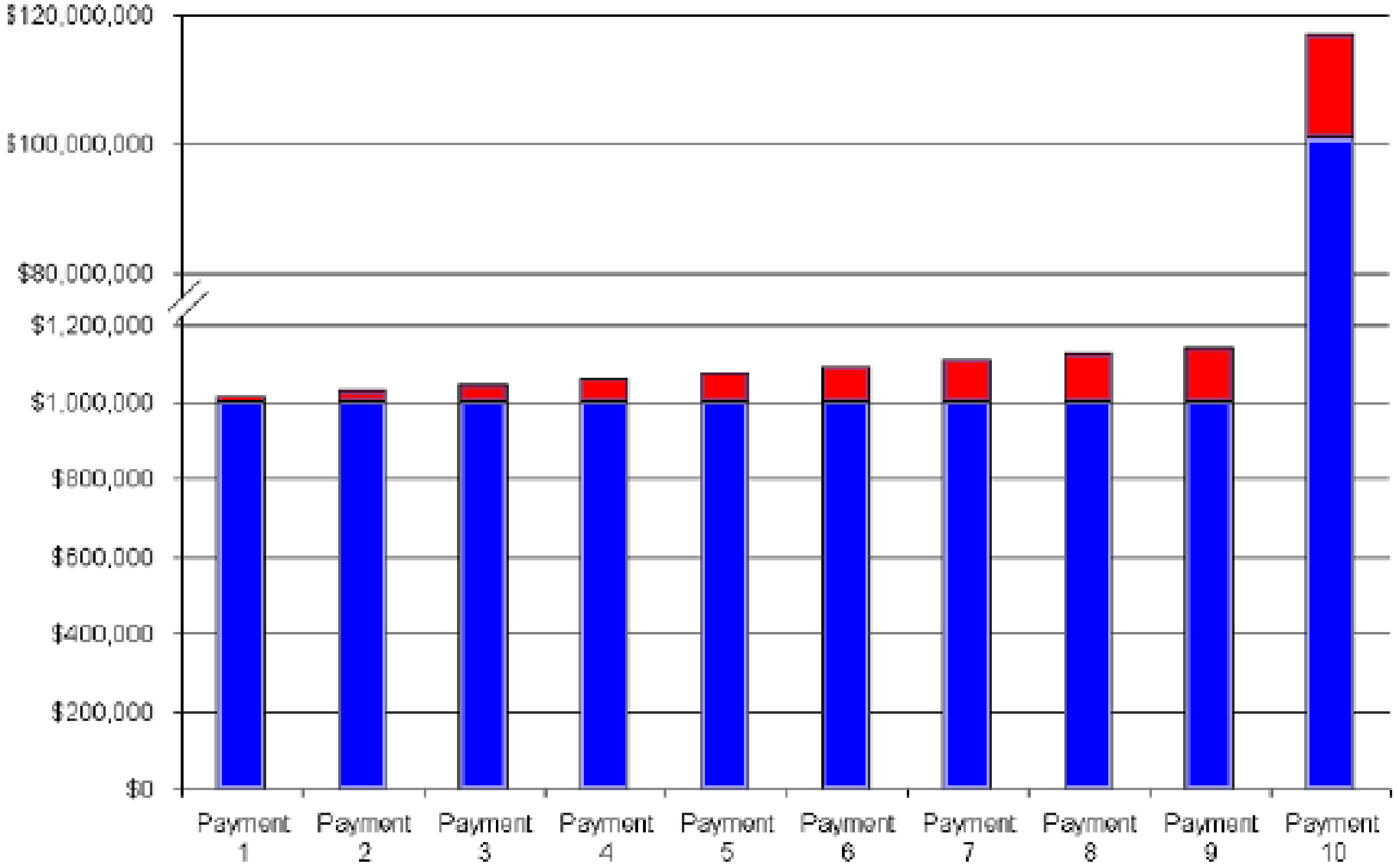
Introduction

- Semiannual interest payments are based on the inflation-adjusted principal at the time the interest is paid.
- At maturity, the securities will be redeemed at the greater of their inflation-adjusted principal or par amount at original issue.
- TIPS are issued in terms of 5, 10, and 20 years, and are offered in multiples of $100.

Example

Assumptions: Annual inflation = 3%; Coupon Rate = 2%;
Size Purchased at Auction = $100,000,000 ; Maturity = 5 Years

Blue = Payment before Inflation adjustment **Red** = Payment from inflation adjustment



Introduction

- The goal of this paper is to solve an intertemporal portfolio choice problem with interim consumption for an infinitely lived investor under uncertain inflation.
- The investor has the stochastic differential utility of Duffie and Epstein (1992).
- We try to find out the optimal consumption plan and the optimal portfolio rule for stocks, nominal bonds as well as inflation-indexed bonds.

The Bond Markets

The Nominal Bond Market

We assume that the investor could directly observe the nominal interest rate which could be described as an Ornstein-Uhlenbeck process of Vasicek (1977) type:

$$dR_t = \kappa\left(\overline{R} - R_t\right)dt + \sigma_R dZ_R \qquad (10)$$

Let $N\left(R_t, T-t\right)$ denote the price of the nominal zero-coupon bond which pays one money unit when it matures at time T.

The Nominal Bond Market

$N\left(R_t, T-t\right)$ would satisfy the following partial differential equation with the boundary condition $N(R,0)=1$:

$$N_R \kappa\left(\bar{R}-R_t\right)+\frac{\partial N}{\partial t}+\frac{1}{2} N_{RR}\sigma_R^2 = R_t N - \lambda_R N_R \sigma_R \qquad (11)$$

Where the constant λ_R represents the risk premium of the interest rate risk

N_R is the first-order partial derivative of $N(\cdot)$ with respect to R and N_{RR} is the second-order partial derivative.

The Nominal Bond Market

The solution of Equation (11) gives the dynamics of the return of the nominal zero-coupon bond:

$$\frac{dN\left(R_t, T-t\right)}{N\left(R_t, T-t\right)} = \left(R_t + \lambda_R b_1(T-t)\sigma_R\right)dt - b_1(T-t)\sigma_R dZ_R \quad (12)$$

Where $b_1(T-t) = \kappa^{-1}[1 - \exp\left(-\kappa(T-t)\right)]$ is a function of the time to maturity $T-t$.

The Nominal Bond Market

Since the infinitely-lived investor has to roll over her investment of bonds whenever the bonds in the portfolio expire, we could assume that investor always sells the expiring bonds and buys the newly-issued bonds continuously to keep the maturity of the bonds in her portfolio a constant. Thus, we would take the value of $b_1\left(T-t\right)$ as a constant b for a given τ^* such that $T-t=\tau^*$ and $b=b_1(\tau^*)$. For this sake, we simplify the expression of Equation (12):

.

$$\frac{dN_t}{N_t}=\left(R_t+\eta_N\right)dt-\sigma_N dZ_R \qquad (13)$$

Where $\eta_N \equiv \lambda_R \sigma_N$ and $\sigma_N \equiv b\sigma_R$

The Dynamics of Price Level and Expected Inflation

Let P_t denote the nominal price level per unit of the consumption good at time t

$$\frac{dP_t}{P_t} = \tilde{\pi}_t dt + \sigma_p dZ_1 \qquad (1)$$

Where $\tilde{\pi}_t$ is expected inflation rate at time t

dZ_1 is the increment of a standard Brownian motion representing the shock to the instantaneous unexpected inflation

The Dynamics of Price Level and Expected Inflation

Following Brennan and Xia (2002), we assume that the expected inflation rate $\tilde{\pi}_t$ is also stochastic and follows an Ornstein-Uhlenbeck process:

$$d\tilde{\pi}_t = \ell(\bar{\pi} - \tilde{\pi}_t)dt + \sigma_1 dZ_1 + \sigma_2 dZ_2 \quad (2)$$

We argue that $\tilde{\pi}_t$ is unobservable to the investor and is not necessarily perfectly correlated to the instantaneous price level. In Equation (2), dZ_2 represents the remaining part of innovation that is uncorrelated to the innovation of the price level.

The Dynamics of Price Level and Expected Inflation

- According to Liptser and Shiryayev (1977), if at time $t=0$ the distribution of $\tilde{\pi}_0$ is conditionally Gaussian, the conditional distribution of $\tilde{\pi}_t$ would also be Gaussian $N\left(\pi_t, v(t)\right)$.
- The conditional mean π_t is the optimal estimator of $\tilde{\pi}_t$, and the conditional variance $v(t)$ could be viewed as the estimation error.
- The investor would then substitute the estimator π_t for the unobservable $\tilde{\pi}_t$ in Equation (1):

The Dynamics of Price Level and Expected Inflation

According to Liptser and Shiryayev (1977),

$$\frac{dP_t}{P_t} = \pi_t dt + \sigma_p dZ_p \qquad (3)$$

Where

$$d\pi_t = \ell\left(\bar{\pi} - \pi_t\right)dt + \left(\sigma_1 + \frac{v(t)}{\sigma_p}\right)dZ_p \qquad (4)$$

$$\frac{dv(t)}{dt} = -2\ell v(t) + \sigma_1^2 + \sigma_2^2 - \left(\sigma_1 + \frac{v(t)}{\sigma_p}\right) \qquad (5)$$

The Dynamics of Price Level and Expected Inflation

With an infinitely-lived investor, Equation (4) can be rewritten as

$$d\pi_t = \ell\left(\bar{\pi} - \pi_t\right)dt + \sigma_\pi dZ_p \qquad (8)$$

Where

$$\sigma_\pi \equiv \sigma_1 + \frac{\nu}{\sigma_p}$$

$$\nu = \lim_{t \to \infty} \nu(t)$$

The Indexed Bond Market

The price of the indexed bond, $I(R_t,\pi_t,P_t,T-t)$, could be decided by the following partial differential equation with the boundary condition $I(R_t,\pi_t,P_t,0)=P_T$:

$$\begin{aligned}&I_R\kappa\left(\bar{R}-R\right)+I_P P\pi+I_\pi\ell\left(\bar{\pi}-\pi\right)+\frac{\partial I}{\partial t}+\frac{1}{2}I_{RR}\sigma_R^2+\frac{1}{2}I_{PP}\sigma_p^2+\frac{1}{2}I_{\pi\pi}\sigma_\pi^2\\&+I_{RP}\sigma_{RP}P+I_{R\pi}\sigma_{R\pi}+I_{P\pi}P\sigma_p\sigma_\pi=R_tI-\lambda_R I_R\sigma_R+\lambda_p\left(I_P P\sigma_p+I_\pi\sigma_\pi\right)\end{aligned} \qquad (14)$$

Where $\sigma_{XY}\equiv \mathrm{E}\left(dXdY\right)$ denotes the covariance of the state variables X Y

λ_p and .

represents the measure of the risk dZ_p premium with respect to the innovation .

The Indexed Bond Market

Solving Equation (14), we then derive the return of the indexed bond as the following process:

$$\frac{dI\left(R_t,\pi_t,P_t,T-t\right)}{I\left(R_t,\pi_t,P_t,T-t\right)}=\left(R_t+\lambda_R b_1(T-t)\sigma_R+\lambda_p(\sigma_p+b_2(T-t)\sigma_\pi)\right)dt \\ -b_1(T-t)\sigma_R dZ_R+\left(\sigma_p+b_2(T-t)\sigma_\pi\right)dZ_p \qquad (15)$$

Where $b_1(T-t)=\kappa^{-1}[1-\exp\left(-\kappa(T-t)\right)]$

and $b_2(T-t)=\ell^{-1}\left[1-\exp(-\ell(T-t))\right]$.

The Indexed Bond Market

We also assume that, for an infinitely-lived investor, he would adopt the trading strategy of substituting the newly-issued bonds for the expiring bonds continuously to keep the time to maturity of the indexed bond to be a constant τ^{**} such that $b_1 \equiv b_1(\tau^{**})$ and $b_2 \equiv b_2(\tau^{**})$ would be two constants. If τ^{**} equals τ^{*}, the time to maturity of the nominal bond, the values of b and b_1 would be equal. For brevity, we rewrite Equation (15) to be

$$\frac{dI_t}{I_t} = \left(R_t + \eta_I \right) dt - \sigma_{I1} dZ_R + \sigma_{I2} dZ_p \qquad (16)$$

Where $\eta_I \equiv \lambda_R \sigma_{I1} + \lambda_p \sigma_{I2}$, $\sigma_{I1} \equiv b_1 \sigma_R$

and $\sigma_{I2} \equiv \sigma_p + b_2 \sigma_\pi$

The Real Bond Market

We define I_t / P_t as the real bond. According to Equation (3), (15) and Itô's lemma, the return of the real bond is

$$\frac{d(I_t / P_t)}{I_t / P_t} = \left(R_t - \pi_t + \lambda_p \sigma_p + \lambda_R b_1 \sigma_R + \lambda_P b_2 \sigma_\pi + b_1 \sigma_{RP} - b_2 \sigma_p \sigma_\pi\right) dt \qquad (17)$$

$$-b_1 \sigma_R dZ_R + b_2 \sigma_\pi dZ_P$$

The Real Bond Market

The instantaneous real risk-free interest rate, r_t, is obtained by taking the limit of the return of the real bond in Equation (17) when $T - t \to 0$:

$$r_t = R_t - \pi_t + \lambda_p \sigma_p \qquad (18)$$

The real interest rate equals the difference of the nominal interest rate and the inflation rate plus the risk premium of the nominal price level for consumption goods. The Fisher equation is not hold unless the risk premium of the price risk is zero.

Optimization Problem

The Optimization Problem

We assume that the investor's preference is represented by the stochastic differential utility proposed by Duffie and Epstein (1992):

$$J_t = \mathrm{E}_t\left[\int_t^{\infty} f\left(C_s, J_s\right) ds\right] \qquad (19)$$

$$f\left(C, J\right) = \beta\left(1 - \frac{1}{\varphi}\right)^{-1}\left(1-\gamma\right) J\left[\left(\frac{C}{\left((1-\gamma)J\right)^{1/(1-\gamma)}}\right)^{1-(1/\varphi)} - 1\right] \qquad (20)$$

$f\left(C, J\right)$ is called the normalized aggregator of the investor's current consumption and utility,

β is the time preference,

γ is the measure of the relative risk aversion for the investor,

φ stands for the elasticity of intertemporal substitution of consumption.

The Optimization Problem

- Obviously, the stochastic differential utility is a more generalized setting and the standard time-additive power utility could be viewed as a special case when $\varphi = 1/\gamma$ in Equation (20).

The Optimization Problem

There are three kinds of risky assets to be traded in the financial market:

1. The stock with nominal price S_t following

$$\frac{dS_t}{S_t} = \left(R_t + \eta_s\right)dt + \sigma_s dZ_s \qquad (21)$$

η_s is the excess return of stock

dZ_s is the unexpected disturbance of stock return.

2. The nominal zero coupon : Equation (13)

3. The indexed bonds : Equation (16)

The Optimization Problem

Investor's problem: to maximize the utility represented in Equation (19) subjected to the intertemporal budget constraint:

$$dW_t = \left[\left(x^{\mathrm{T}}\eta + R_t\right)W_t - C_t P_t\right]dt + x^{\mathrm{T}}\Gamma\,\mathrm{dZ} \qquad (22)$$

C_t is the instantaneous consumption at time t in terms of real units.

x^{T} is the transpose of the vector of the portfolio weights x

$$x = \begin{pmatrix} x_S \\ x_N \\ x_I \end{pmatrix} \qquad (23)$$

The Optimization Problem

η is the vector of excess return

$$\eta = \begin{pmatrix} \eta_S \\ \eta_N \\ \eta_I \end{pmatrix} \tag{24}$$

and

$$\Gamma = \begin{pmatrix} \sigma_s & 0 & 0 \\ 0 & -\sigma_N & 0 \\ 0 & -\sigma_{I1} & \sigma_{I2} \end{pmatrix} \tag{25}$$

$$\mathrm{dZ} = \begin{pmatrix} dZ_s \\ dZ_R \\ dZ_p \end{pmatrix} \tag{26}$$

The Optimization Problem

The Bellman equation for this problem is

$$\max_{x,C}\Big[\, f\left(C,J\right)+J_W W\left(x^{\mathrm{T}}\eta+R\right)-J_W CP+J_R\kappa\left(\bar{R}-R\right)+J_P P\pi+J_\pi\ell\left(\bar{\pi}-\pi\right)$$

$$+\frac{1}{2}\left(J_{WW}W^2x^{\mathrm{T}}\Sigma x+J_{RR}\sigma_R^2+J_{PP}P^2\sigma_p^2+J_{\pi\pi}\sigma_\pi^2\right)+J_{WR}Wx^{\mathrm{T}}\Gamma\rho e_2\sigma_R+J_{WP}WPx^{\mathrm{T}}\Gamma\rho e_3\sigma_p$$

$$+J_{W\pi}Wx^{\mathrm{T}}\Gamma\rho e_3\sigma_\pi+J_{RP}P\sigma_{RP}+J_{R\pi}\sigma_{R\pi}+J_{P\pi}P\sigma_p\sigma_\pi\,\Big]=0 \qquad (27)$$

Where e_2 is $\left(0 \quad 1 \quad 0\right)^{\mathrm{T}}$ and e_3 is $\left(0 \quad 0 \quad 1\right)^{\mathrm{T}}$,

$\Sigma\equiv\Gamma\rho\Gamma^{\mathrm{T}}$ represents the variance-covariance matrix of the nominal risky asset returns,

The Optimization Problem

and $\Gamma\rho e_2\sigma_R$ is the vector of covariance between the risky asset returns and the nominal interest rate (and so forth for the similar terms) where ρ (3 x 3) is the matrix of correlation coefficients

$$\rho \equiv \begin{pmatrix} 1 & \rho_{SR} & \rho_{SP} \\ \rho_{SR} & 1 & \rho_{RP} \\ \rho_{SP} & \rho_{RP} & 1 \end{pmatrix} \tag{28}$$

ρ_{SR} is defined by $E(dZ_s dZ_R) = \rho_{SR}dt$, and others are defined in a similar fashion.

The Optimization Problem

It can be shown that the first-order conditions for the Bellman equation are:

$$C^* = \beta^{\varphi}\left(J_W P\right)^{-\varphi}\left[\left(1-\gamma\right)J\right]^{(1-\gamma\varphi)/1-\gamma} \qquad (29)$$

$$x^* = \frac{-J_W}{J_{WW}W}\Sigma^{-1}\eta + \frac{-J_{WR}}{J_{WW}W}\Sigma^{-1}\Gamma\rho e_2\sigma_R + \frac{-J_{WP}P}{J_{WW}W}\Sigma^{-1}\Gamma\rho e_3\sigma_p + \frac{-J_{W\pi}}{J_{WW}W}\Sigma^{-1}\Gamma\rho e_3\sigma_\pi \qquad (30)$$

Results

The Approximate Solution with Log-Linearization

The partial differential equation after we substitute Equation (29) and (30) back into (27) is complicated to solve. However, by conjecture, it could be verified that the value function would have the following form:

$$J\left(W_t, R_t, P_t, \pi_t\right) = \left[H\left(R_t, \pi_t\right)\right]^{(\gamma-1)/(1-\varphi)} \frac{\left(W_t / P_t\right)^{1-\gamma}}{1-\gamma} \qquad (31)$$

The Approximate Solution with Log-Linearization

This guess implies a second-order ODE for $H\left(R,\pi\right)$

$$
\begin{aligned}
&\beta^{\varphi}\frac{1}{H}-\beta\varphi+(\varphi-1)\left(x^{\mathrm{T}}\eta+R\right)+\frac{H_R}{H}\kappa\left(\overline{R}-R\right)-(\varphi-1)\pi+\frac{H_\pi}{H}\ell\left(\overline{\pi}-\pi\right)\\
&+\frac{1}{2}\left[\frac{\gamma+\varphi-2}{1-\varphi}\left(\frac{H_R}{H}\right)^2+\frac{H_{RR}}{H}\right]\sigma_R^2+\frac{1}{2}\left[\frac{\gamma+\varphi-2}{1-\varphi}\left(\frac{H_\pi}{H}\right)^2+\frac{H_{\pi\pi}}{H}\right]\sigma_\pi^2\\
&-\frac{1}{2}\left(\gamma-2\right)\left(\varphi-1\right)\sigma_P^2-\frac{\gamma(\varphi-1)}{2}x^{\mathrm{T}}\Sigma x-(\gamma-1)\frac{H_R}{H}x^{\mathrm{T}}\Gamma\rho e_2\sigma_R\\
&-(\gamma-1)\frac{H_\pi}{H}x^{\mathrm{T}}\Gamma\rho e_3\sigma_\pi+(\gamma-1)(\varphi-1)x^{\mathrm{T}}\Gamma\rho e_3\sigma_P+(\gamma-1)\frac{H_R}{H}\sigma_{RP}\\
&+\left[\frac{\gamma+\varphi-2}{1-\varphi}\left(\frac{H_R}{H}\right)\left(\frac{H_\pi}{H}\right)+\frac{H_{R\pi}}{H}\right]\sigma_{R\pi}+(\gamma-1)\frac{H_\pi}{H}\sigma_p\sigma_\pi=0
\end{aligned}
\tag{32}
$$

The Approximate Solution with Log-Linearization

To solve Equation (32), we employ the method of log-linear approximation to find an approximate analytical solution to our problem.

The solution is based on a loglinear expansion of the consumption-wealth ratio around its unconditional mean.

First, we substitute Equation (31) into Equation (29) and find that the envelope condition would be expressed as

$$\frac{P_t C_t}{W_t} = \beta^{\varphi} \frac{1}{H(R_t, \pi_t)} \tag{33}$$

The Approximate Solution with Log-Linearization

Denoting $\left(c_t - w_t\right) \equiv \log(P_t C_t / W_t)$ and using the first-order Taylor expansion of $\exp(c_t - w_t)$ around its unconditional mean $\mathrm{E}(c_t - w_t) \equiv \overline{c - w}$, the envelope condition could be rewritten as

$$\beta^{\varphi} \frac{1}{H} = \exp\left(c_t - w_t\right) \approx \exp(\overline{c - w}) + \exp(\overline{c - w})[c_t - w_t - (\overline{c - w})] \qquad (34)$$

$$= h_0 - h_1 \log H$$

Where $h_1 \equiv \exp(\overline{c - w})$ and $h_0 \equiv h_1(1 + \varphi \log \beta - \log h_1)$

The Approximate Solution with Log-Linearization

Second, substituting the approximate result in Equation (34) for $\beta^{\varphi}H^{-1}$ in Equation (33), it is easy to see that the solution of H would take the form of $H\left(R_t,\pi_t\right)=\exp(a_0-a_1R_t+a_2\pi_t)$. The undetermined coefficients would then be solved as following:

$$a_1=\frac{1-\varphi}{h_1+\kappa} \tag{35}$$

$$a_2=\frac{1-\varphi}{h_1+\ell} \tag{36}$$

$$\begin{aligned}a_0=\frac{1}{h_1}\Bigg[&h_0-\beta\varphi+(\varphi-1)x^{\mathrm{T}}\eta+\frac{1}{2}\frac{\gamma-1}{1-\varphi}\left(a_1^2\sigma_R^2+a_2^2\sigma_\pi^2-2a_1a_2\sigma_{R\pi}\right)\\&+(\gamma-1)x^{T}\Gamma\rho(a_1\sigma_Re_2-a_2\sigma_\pi e_3-(1-\varphi)\sigma_pe_3)-a_1\kappa\overline{R}+a_2\ell\overline{\pi}\\&-(\gamma-1)a_1\sigma_{RP}+(\gamma-1)a_2\sigma_p\sigma_\pi+\frac{1}{2}(\gamma-2)(1-\varphi)\sigma_p^2+\frac{\gamma(1-\varphi)}{2}x^{\mathrm{T}}\Sigma x\Bigg]\end{aligned} \tag{37}$$

The Optimal Policies

Proposition (1A)

The approximate analytical solution for the investor's value function is

$$J\left(W_t, R_t, P_t, \pi_t\right) = \exp\left[\frac{\gamma - 1}{1-\varphi}\left(a_0 - a_1 R_t + a_2 \pi_t\right)\right]\frac{\left(W_t / P_t\right)^{1-\gamma}}{1-\gamma} \qquad (38)$$

The Optimal Policies

Proposition (1B)

The optimal consumption is

$$\frac{P_t C_t}{W_t} = \beta^{\varphi} \exp(-a_0 + a_1 R_t - a_2 \pi_t) \qquad (39)$$

In Proposition (1B), we find that the consumption-wealth ratio $P_t C_t / W_t$ is an exponentially affine function of the interest rate R_t and the (estimated) inflation rate π_t. The exact relationship between the consumption-wealth ratio and the two state variables is decided by the value of φ, the elasticity of intertemporal substitution.

The Optimal Policies

- According to Equation (35) and (36), when $\varphi = 1$, the values of a_1 and a_2 are both identical to zero and hence the consumption-wealth ratio turn out to be a constant over time. A constant consumption-wealth ratio makes $c_t - w_t$ exactly identical to its unconditional mean. This implies that the solution of $\varphi = 1$ is an exact solution.
- When $\varphi < 1$, a_1 and a_2 are positive. The consumption-wealth ratio would rise as R_t rises or π_t falls. However, when $\varphi > 1$, the consumption-wealth ratio would fall as R_t rises or π_t falls.

The Optimal Portfolio Weights

Proposition (1C)

$$x=\begin{pmatrix} x_S \\ x_N \\ x_I \end{pmatrix}=\underbrace{\frac{1}{\gamma}\Sigma^{-1}\begin{pmatrix} \eta_S \\ \eta_N \\ \eta_I \end{pmatrix}}_{x_1}+\underbrace{\left(1-\frac{1}{\gamma}\right)\frac{1}{(h_1+\kappa)}\begin{pmatrix} 0 \\ 1/b \\ 0 \end{pmatrix}}_{x_2}$$

$$+\underbrace{\left(1-\frac{1}{\gamma}\right)\frac{\sigma_\pi}{(h_1+\ell)\sigma_{I2}}\begin{pmatrix} 0 \\ -b_1/b \\ 1 \end{pmatrix}}_{x_3}+\underbrace{\left(1-\frac{1}{\gamma}\right)\frac{\sigma_p}{\sigma_{I2}}\begin{pmatrix} 0 \\ -b_1/b \\ 1 \end{pmatrix}}_{x_4} \qquad (41)$$

The term x_1 is the speculative (or myopic) demand,

x_2 is the hedging demand against the unexpected innovation of interest rate

x_3 , x_4 are to hedge against the innovation of the inflation rate and the instantaneous nominal price level.

The Optimal Portfolio Weights

- Equation (41) shows that the investor’s portfolio choice is represented by a weighted average of the speculative demand and the hedging demands.
- It is obviously that the portfolio policy depends only on the risk aversion but does not depend on the elasticity of intertemporal substitution explicitly.
- The elasticity of intertemporal substitution only affects the optimal portfolio implicitly by the unconditional mean of log consumption-wealth ratio through the coefficient h_1 .

The Optimal Portfolio Weights

- For an infinitely risk-averse investor ($\gamma \to \infty$), the speculative demand would vanish and the optimal portfolio for the investor is composed of the mix of nominal and indexed zero coupon bonds. This meets the common advice that the more conservative investor should put more weights on bonds.

Dynamics of Nominal and Real Consumptions

By Equation (22), (33), (34) and the solution of $H(R_t, \pi_t)$, the intertemporal budget constraint could be rewritten as

$$\frac{dW_t}{W_t} = \left[x^{\mathrm{T}}\eta + R_t - h_0 + h_1(a_0 - a_1 R_t + a_2 \pi_t) \right] dt + x^{\mathrm{T}}\Gamma\, \mathrm{dZ} \qquad (42)$$

where we use the approximate consumption-wealth ratio to substitute for its exact expression.

Dynamics of Nominal and Real Consumptions

According to Equation (39), (42) and Itô’s lemma we obtain the following proposition:

Proposition (2A)

The dynamics of the nominal consumption $P_t C_t$ could be expressed as following:

$$\frac{d(P_t C_t)}{P_t C_t} = \mu_{NC}(R_t, \pi_t)dt + \sigma_{NC}^{\mathrm{T}}\,\mathrm{dZ} \tag{43}$$

where $$\mu_{NC}(R_t, \pi_t) = \varphi(R_t - \beta + x^{\mathrm{T}}\eta) + (1-\varphi)\pi_t + \phi_0 \tag{44}$$

and $$\sigma_{NC} = \Gamma^{\mathrm{T}} x + a_1 \sigma_R e_2 - a_2 \sigma_\pi e_3 \tag{45}$$

β is the investor’s subjective time preference

ϕ_0 is a collection of the variance-covariance terms in our model.

Dynamics of Nominal and Real Consumptions

Proposition (2B)

The dynamics of real consumption C_t is:

$$\frac{dC_t}{C_t} = \mu_C(R_t, \pi_t)dt + \sigma_C^{\mathrm{T}}\,\mathrm{dZ} \tag{46}$$

where $$\mu_C(R_t, \pi_t) = \varphi(R_t - \pi_t - \beta + x^{\mathrm{T}}\eta) + \phi_1 \tag{47}$$

ϕ_1 represents a collection of the variance-covariance terms and

$$\sigma_C = \Gamma^{\mathrm{T}} x + a_1\sigma_R e_2 - a_2\sigma_\pi e_3 - \sigma_p e_3 \tag{48}$$

Dynamics of Nominal and Real Consumptions

- Proposition 2 shows that the elasticity of intertemporal substitution decides the sensitivity of the expected nominal and real consumption growth with respect to the nominal interest rate, estimated inflation and the excess return of the portfolio.
- The coefficient of risk aversion has no effect on consumption growth.

Dynamics of Nominal and Real Consumptions

- In contrast, the coefficient of risk aversion decides the portfolio rule while the elasticity of substitution has no effect on portfolio choice.
- This is why we should separate the elasticity of substitution from the measure of risk aversion when solving a problem involving the portfolio and consumption choice simultaneously.

Conclusion

- We have derived the optimal intertemporal portfolio-consumption choice of the investor with the stochastic differential utility under uncertain inflation.
- The optimal portfolio rule depends on the coefficient of risk aversion while the consumption plan relies on the elasticity of intertemporal substitution.

Conclusion

- With the inclusion of indexed bonds in the portfolio set, we find that the risk of nominal interest rate is perfectly hedged by the holdings of nominal bonds.
- The inflation and price risks is hedged by the holdings of indexed bonds. The demand of nominal bonds is crowded out proportionally to the demand of indexed bonds.

Conclusion

- As to the consumption, we find that the consumption-wealth ratio is obtained as an exponentially affine function of the nominal interest rate and expected inflation rate.
- Work need to be done: Calibration.