On the Number and Dynamic Features of State Variables in Options Pricing

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Plan of the Talk

Plan of the Talk

1. Introduction

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1. Introduction
2. Methodology

Plan of the Talk

1. Introduction
2. Methodology
3. Data and Preliminary Analysis

Plan of the Talk

1. Introduction
2. Methodology
3. Data and Preliminary Analysis
4. The Number of State Variables

Plan of the Talk

1. Introduction
2. Methodology
3. Data and Preliminary Analysis
4. The Number of State Variables
5. Dynamic Features of State Variables

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1. Introduction
2. Methodology
3. Data and Preliminary Analysis
4. The Number of State Variables
5. Dynamic Features of State Variables
6. Robustness Checks

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Two sources of mis-specification: (1) the number of state variables and (2) dynamic features captured in the drift, diffusion, jump intensity, jump size and risk premiums. The two sources are related.

2. Methodology

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Nonparametric approach with observable proxies of volatility state variables.

The set-up: Suppose the log price at time t, $s_t = \log S_t$, of the underlying security of a number of European options is driven by the stochastic process under the actual probability, P, as follows,

$$
\begin{aligned}
ds_t &= \mu_s(x_t)dt + \sigma_s(x_t)d\omega_t + z(x_t)dJ_t - \mu_z(x_t)\lambda(x_t)dt, \\
dx_t &= \mu_x(x_t)dt + \sigma_x(x_t)d\omega_t + y(x_t)dJ_t - \mu_y(x_t)\lambda(x_t)dt,
\end{aligned}
$$

where x_t is a k-dimensional state variable, ω_t is a standard vector Brownian motion with dimension greater than or equal to k, J_t is a vector counting process with jump intensity $\lambda(x_t)$, μ_s and μ_x are the conditional mean of ds and dx, $\sigma_s\sigma_s'$ and $\sigma_x\sigma_x'$ are the conditional variance of the diffusive component of ds_t and dx_t, z and y are the conformable matrices of jump sizes of ds_t and dx_t with means μ_z and μ_y, respectively, independent of ω_t and J_t.

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Given x_t, the prices of European call and put options are assumed to be homogeneous functions of degree one of the underlying price, S_t (or its forward price, F_t), and the strike price, K.

Variance-swap Prices

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A variance swap is a forward contract between two parties to exchange at $t+\tau$ a value $V_{\tau,t}^2$ pre-determined at t and the realized quadratic variation of s_u between t and $t+\tau$, $\frac{1}{\tau}\int_t^{t+\tau}\langle s_u, s_u\rangle du$. $V_{\tau,t}^2$ is known as the variance-swap price.

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Theoretically, $V_{\tau,t}^2 = E_t^Q \frac{1}{\tau}\int_t^{t+\tau}\langle s_u, s_u\rangle du$, a measurable function of (x_t, τ).

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Suppose, under the risk-neutral probability,

$$
\begin{aligned}
ds_t &= \mu(x_t)dt + \sum_{j=1}^{k} \sqrt{\sigma_{0j} + \sigma_{1j} x_{jt}}\, d\omega_{jt} + z_t dJ_t - \mu_z(\lambda_0 + \lambda_1' x_t)dt, \\
dx_t &= K(\theta - x_t)dt + \Omega(x_t)d\omega_t^* + y_t dJ_t^* - l(x_t)\mu_y dt,
\end{aligned}
$$

where $\omega_t = (\omega_{1t}, \cdots, \omega_{kt})$ is a k-vector of independent standard Brownian motions, ω_t^* is a k-vector of independent standard Brownian motions, possibly correlated with ω_t, K is a $k \times k$ matrix, J_t is independent of ω_t, the jump size of J_t has mean μ_z and variance σ_z^2, J_t^* has an intensity $l(x_t)$ and jump size with mean μ_y, $\mu(x_t)$ satisfies $E^Q[dS_t/S_t] = rdt$ where r is the instantaneous riskfree rate, $\Omega(x_t)$ and $l(x_t)$ are unspecified except for the usual regularity conditions.

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It is not difficult to deduce that

$$\begin{aligned} V^2_{\tau,t} &= \sigma_0 + \sigma_1'\theta + \sigma_1'(\tau K)^{-1}(I_k - e^{-\tau K})(x_t - \theta) \\ &\quad + \left[\lambda_0 + \lambda_1'\theta + \lambda_1'(\tau K)^{-1}(I_k - e^{-\tau K})(x_t - \theta)\right](\mu_z^2 + \sigma_z^2), \end{aligned}$$

where I_k is the identity matrix, $\sigma_0 = \sum_{j=1}^{k} \sigma_{0j}$ and $\sigma_1 = (\sigma_{11}, \cdots, \sigma_{1k})'$. In this example, a variance-swap price is an affine function of the underlying state variables. Let $\tau_1 < \tau_2 < \cdots < \tau_k$. From k equations for $V^2_{\tau_j,t}$, $j = 1, \cdots, k$, it is easy to see that x_{jt}s can be solved as affine functions of $V^2_{\tau_j,t}$s.

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This example serves as a benchmark to be tested.

Constructing Variance-swap Prices

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Model-free variance-swap prices can be inferred from options prices:

$$V_{\tau,t}^2 \approx \frac{2}{\tau} e^{r\tau} \left(\int_0^{\bar{K}} \frac{1}{K^2} p(K) dK + \int_{\bar{K}}^{\infty} \frac{1}{K^2} c(K) dK \right),$$

where $\bar{K}$ is the $t+\tau$-forward price of S_t and $c(K)$ and $p(K)$ are prices of European calls and puts with strike price K.

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Nonparametric options pricing is done through the BS-implied volatility:

$$\sigma_i = \sigma(K_i/F_i, T_i, V_{\tau_1}, \cdots, V_{\tau_k}) + \varepsilon_i.$$

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Six variance-swap prices series are constructed for $\tau_j = 1, 2, 6, 9, 12, 18$.

Table 1. Summary statistics of implied volatility

A. Number of contracts

	T						
K/F	$1m$	$2m$	$3-6m$	$7-9m$	$10-12m$	$>12m$	Total
(0.00, 0.94]	5433	5258	11895	6821	5880	11160	46447
(0.94, 0.97]	2516	1377	1909	814	677	1189	8482
(0.97, 1.00]	2927	1636	2079	800	651	1138	9231
(1.00, 1.03]	2877	1638	2053	796	709	1083	9156
(1.03, 1.06]	2156	1288	1763	797	645	982	7631
$(1.06, \infty)$	1672	2149	5888	4307	3816	6795	24627
Total	17581	13346	25587	14335	12378	22347	105574

B. Average implied volatility

	T						
K/F	$1m$	$2m$	$3-6m$	$7-9m$	$10-12m$	$>12m$	Total
(0.00, 0.94]	0.3062	0.2932	0.2844	0.2702	0.2607	0.2556	0.2759
(0.94, 0.97]	0.2113	0.2077	0.2073	0.2093	0.2101	0.2095	0.2093
(0.97, 1.00]	0.1859	0.1883	0.1962	0.2020	0.2039	0.2016	0.1933
(1.00, 1.03]	0.1680	0.1733	0.1833	0.1939	0.1947	0.1957	0.1800
(1.03, 1.06]	0.1672	0.1625	0.1769	0.1863	0.1902	0.1921	0.1758
$(1.06, \infty)$	0.2199	0.1904	0.1809	0.1765	0.1758	0.1792	0.1823
Total	0.2358	0.2282	0.2322	0.2269	0.2226	0.2234	0.2288

Table 2. Principal components of implied volatility

	P1	P2	P3	P4	P5	P6
VAR($\times 10^{-2}$)	8.0560	0.4846	0.1269	0.0607	0.0337	0.0260
PVAR	0.9066	0.0545	0.0143	0.0068	0.0038	0.0029
	P7	P8	P9	P10	P11	P12
VAR($\times 10^{-2}$)	0.0219	0.0203	0.0094	0.0090	0.0072	0.0056
PVAR	0.0025	0.0023	0.0011	0.0010	0.0008	0.0006
	P13	P14	P15	P16	P17	P18
VAR($\times 10^{-2}$)	0.0041	0.0032	0.0023	0.0022	0.0019	0.0015
PVAR	0.0005	0.0004	0.0003	0.0002	0.0002	0.0002
	P19	P20	P21	P22	P23	P24
VAR($\times 10^{-2}$)	0.0012	0.0012	0.0010	0.0007	0.0007	0.0006
PVAR	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
	P25	P26	P27	P28	P29	P30
VAR($\times 10^{-2}$)	0.0006	0.0005	0.0005	0.0004	0.0003	0.0003
PVAR	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000
	P31	P32	P33	P34	P35	P36
VAR($\times 10^{-2}$)	0.0003	0.0002	0.0002	0.0002	0.0002	0.0001
PVAR	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

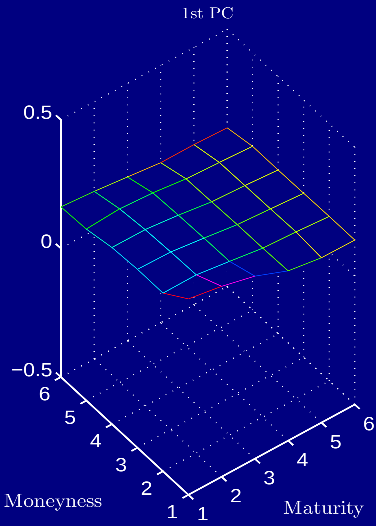

Figure 1. Loadings on principal components

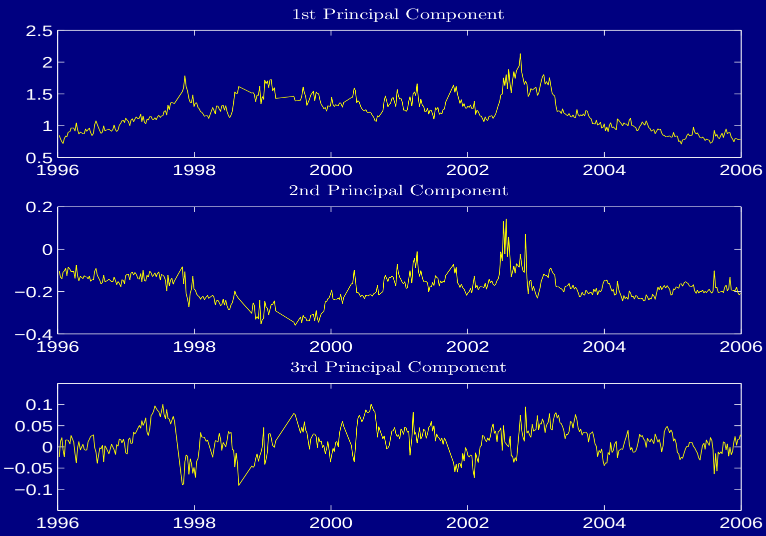

Figure 2. The principal components of the implied volatilities

Table 3. Summary statistics of the V_τs

A. Summary statistics	Mean	Std	Skew	Kurt	5P	95P
V_1	0.2153	0.0639	0.7659	0.6286	0.1240	0.3361
V_2	0.2115	0.0578	0.6728	0.5050	0.1278	0.3210
V_6	0.2135	0.0464	0.1979	-0.6726	0.1438	0.2908
V_9	0.2103	0.0429	0.2385	-0.6098	0.1472	0.2857
V_{12}	0.2107	0.0432	0.2482	-0.7033	0.1458	0.2894
V_{18}	0.2069	0.0402	0.1646	-0.7260	0.1458	0.2741
B. Principal components analysis						
Variance	P1	P2	P3	P4	P5	P6
VAR($\times 10^{-2}$)	1.2805	0.0899	0.0064	0.0047	0.0018	0.0013
PVAR	0.9247	0.0649	0.0046	0.0034	0.0014	0.0010
Loadings	P1	P2	P3	P4	P5	P6
V_1	0.5001	-0.5986	0.5231	-0.3363	0.0005	0.0679
V_2	0.4762	-0.3483	-0.4670	0.6155	0.2243	0.0671
V_6	0.4031	0.1374	-0.2604	-0.1986	-0.4209	-0.7308
V_9	0.3708	0.2866	-0.2684	-0.2407	-0.4422	0.6743
V_{12}	0.3487	0.4230	-0.0754	-0.3576	0.7513	-0.0366
V_{18}	0.3183	0.4902	0.6021	0.5316	-0.1115	-0.0270

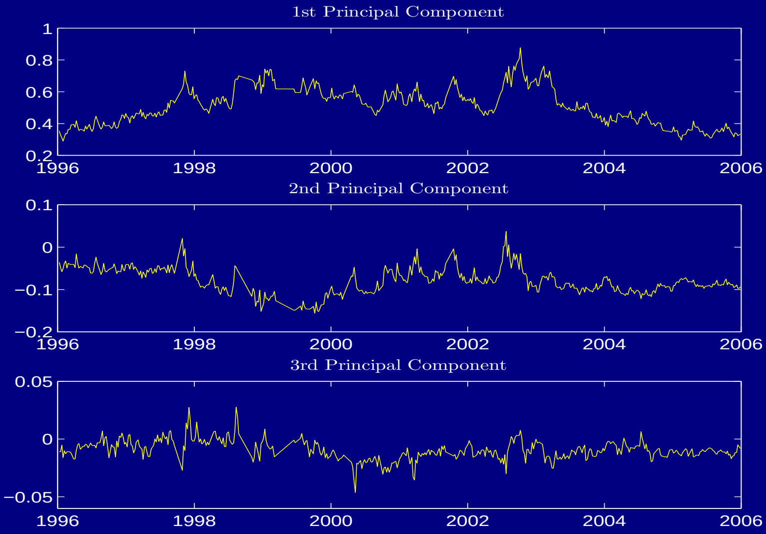

Figure 3. The principal components of the square root of variance-swap prices

4. The Number of State Variables

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Nonparametric estimation using the classical Nadaraya-Waston kernel estimator:

$$\widehat{\sigma}(K/F, T, V_{\tau_1}, \ldots V_{\tau_k}) = \frac{\sum_{i=1}^{N} \phi(\frac{K_i/F_i - K/F}{h_{K/F}})\phi(\frac{T_i - T}{h_T}) \prod_{j=1}^{k} \phi(\frac{V_{\tau_j,i} - V_{\tau_j}}{h_{V_{\tau_j}}})\sigma_i}{\sum_{i=1}^{N} \phi(\frac{K_i/F_i - K/F}{h_{K/F}})\phi(\frac{T_i - T}{h_\tau}) \prod_{j=1}^{k} \phi(\frac{V_{\tau_j,i} - V_{\tau_j}}{h_{V_{\tau_j}}})},$$

where h is the bandwidth, N is the number of observations, and $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}(\frac{3}{2} - \frac{z^2}{2})$, the fourth-order Gaussian kernel.

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Mean squared errors (MSE) to gauge the absolute performance and R^2 defined as

$$R^2 = 1 - \frac{\mathrm{MSE}(\mathrm{M}_{k+1}(\tau_1, \cdots, \tau_k, \tau_{k+1}))}{\mathrm{MSE}(\mathrm{M}_k(\tau_1, \cdots, \tau_k))},$$

to gauge the improvement of the model with an addition of a state variable, $V_{\tau_{k+1}}$.

The Number of State Variables (cont'd)

Nonparametric testing:

$$H_0 : \sigma(K/F, T, V_{\tau_1}, ..., V_{\tau_k}) = \sigma(K/F, T, V_{\tau_1}, ..., V_{\tau_{k+1}})$$

against the alternative

$$H_1 : \sigma(K/F, T, V_{\tau_1}, ..., V_{\tau_k}) \neq \sigma(K/F, T, V_{\tau_1}, ..., V_{\tau_{k+1}}),$$

where $\sigma(K/F, T, V_{\tau_1}, ..., V_{\tau_k})$ is the restricted model, and $\sigma(K/F, T, V_{\tau_1}, ..., V_{\tau_{k+1}})$ is the unrestricted model.

The Number of State Variables (cont'd)

The two-point wild bootstrap procedure: fit the implied volatility, σ_i, using the restricted model and calculate the residuals from the model as

$$\varepsilon_i = \sigma_i - \widehat{\sigma}_i(K_i/F_i, T_i, V_{\tau_1}, ..., V_{\tau_k}).$$

Using the residuals from the restricted model, construct the bootstrap samples under the null hypothesis,

$$\sigma_i^* = \widehat{\sigma}_i(K_i/F_i, T_i, V_{\tau_1}, ..., V_{\tau_k}) + \varepsilon_i^*,$$

where $\varepsilon_i^* = \frac{1-\sqrt{5}}{2}\varepsilon_i$ with probability $\frac{1+\sqrt{5}}{2\sqrt{5}}$, and $\varepsilon_i^* = \frac{1+\sqrt{5}}{2}\varepsilon_i$ with probability $\frac{-1+\sqrt{5}}{2\sqrt{5}}$.

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We calculate the R^2 from many bootstrap samples and obtain the empirical distribution of R^2 under the null hypothesis.

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Comparing the R^2 from the original sample with the empirical distribution of R^2 from the bootstrap samples.

Table 4. Testing the number of state variables

Model	MSE($\times 10^{-3}$)	Restricted	R^2	p-value
M_0	2.3037			
$M_1(1)$	0.3894	M_0	0.8310	0.00
$M_2(1,18)$	0.2070	$M_1(1)$	0.4683	0.00
$M_3(1,6,18)$	0.1945	$M_2(1,18)$	0.0604	0.18
$M_3(1,9,18)$	0.2010	$M_2(1,18)$	0.0290	0.30
$M_3(1,12,18)$	0.1986	$M_2(1,18)$	0.0407	0.11

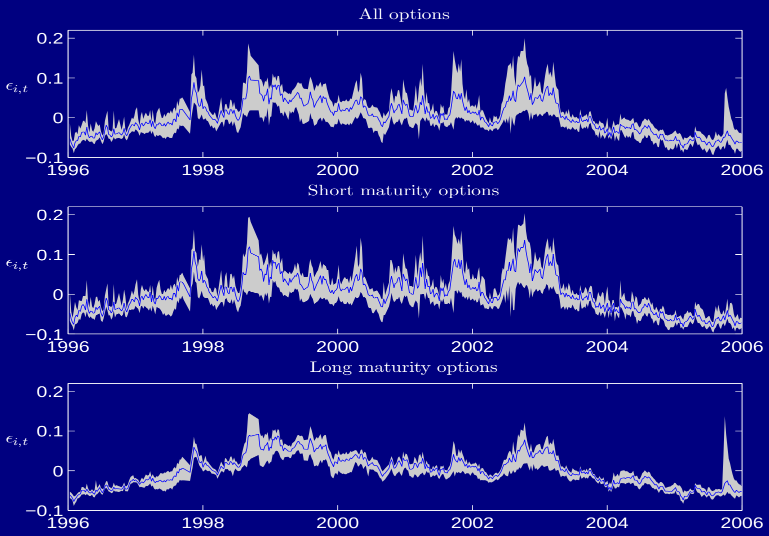

Figure 4. Residuals from model M_0

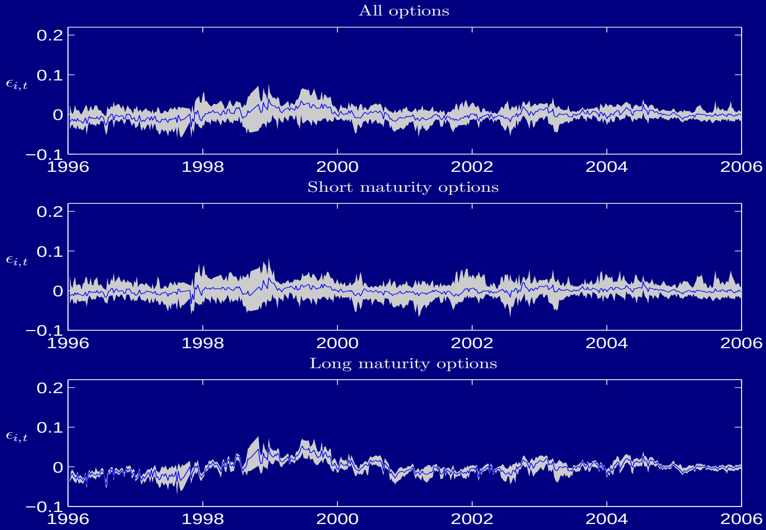

Figure 5. Residuals from model $M_1(1)$

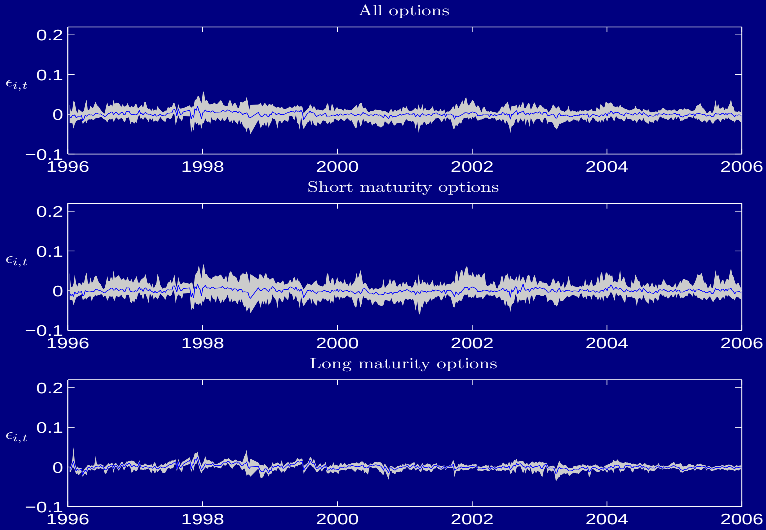

Figure 6. Residuals from model $M_2(1, 18)$

Table 5. Principal components analysis of residuals

A. M_0	P1	P2	P3	P4	P5	P6
VAR($\times 10^{-2}$)	7.2993	0.3908	0.0917	0.0316	0.0251	0.0171
PVAR	0.9233	0.0494	0.0116	0.0040	0.0032	0.0022
B. $M_1(1)$	P1	P2	P3	P4	P5	P6
VAR($\times 10^{-2}$)	0.5888	0.1289	0.0391	0.0317	0.0188	0.0132
PVAR	0.6845	0.1499	0.0454	0.0369	0.0219	0.0153
C. $M_2(1, 18)$	P1	P2	P3	P4	P5	P6
VAR($\times 10^{-2}$)	0.1048	0.0825	0.0406	0.0349	0.0246	0.0175
PVAR	0.2985	0.2350	0.1155	0.0995	0.0702	0.0499
D. $M_3(1, 6, 18)$	P1	P2	P3	P4	P5	P6
VAR($\times 10^{-2}$)	0.1061	0.0569	0.0320	0.0294	0.0217	0.0090
PVAR	0.3629	0.1946	0.1094	0.1004	0.0741	0.0307
E. $M_3(1, 9, 18)$	P1	P2	P3	P4	P5	P6
VAR($\times 10^{-2}$)	0.1000	0.0577	0.0329	0.0296	0.0201	0.0086
PVAR	0.3489	0.2015	0.1149	0.1035	0.0702	0.0300
F. $M_3(1, 12, 18)$	P1	P2	P3	P4	P5	P6
VAR($\times 10^{-2}$)	0.1064	0.0569	0.0379	0.0280	0.0199	0.0090
PVAR	0.3601	0.1925	0.1283	0.0947	0.0675	0.0305

Table 6. Simulation result of the bootstrap test

A. M_0	$c = 0.85$		$c = 1$		$c = 1.2$	
	95%	90%	95%	90%	95%	90%
$N = 500$	10	13	5	7	7	7
$N = 1000$	3	6	8	8	8	9
$N = 2000$	7	10	8	9	5	9
B. $M_1(1)$	$c = 0.85$		$c = 1$		$c = 1.2$	
	95%	90%	95%	90%	95%	90%
$N = 500$	17	25	12	15	7	7
$N = 1000$	18	25	9	13	3	5
$N = 2000$	13	20	6	9	3	10
C. $M_2(1, 18)$	$c = 0.85$		$c = 1$		$c = 1.2$	
	95%	90%	95%	90%	95%	90%
$N = 500$	11	21	4	7	4	5
$N = 1000$	19	29	13	14	2	6
$N = 2000$	18	28	8	10	3	5

5. Dynamic Features of State Variables

Nonparametric estimation:

$$
\begin{aligned}
V_{1,t+1}^2 - V_{1,t}^2 &= \mu_1(V_{1,t}^2, V_{18,t}^2) + \eta_{1,t+1}, \\
V_{18,t+1}^2 - V_{18,t}^2 &= \mu_{18}(V_{1,t}^2, V_{18,t}^2) + \eta_{18,t+1}.
\end{aligned}
$$

And then

$$
\begin{aligned}
\hat{\eta}_{1,t+1}^2 &= \sigma_1^2(V_{1,t}^2, V_{18,t}^2) + \xi_{1,t+1}, \\
\hat{\eta}_{18,t+1}^2 &= \sigma_{18}^2(V_{1,t}^2, V_{18,t}^2) + \xi_{18,t+1}, \\
\hat{\eta}_{1,t+1}\hat{\eta}_{18,t+1} &= \sigma_{1,18}(V_{1,t}^2, V_{18,t}^2) + \xi_{1,18,t+1},
\end{aligned}
$$

where $\hat{\eta}_{1,t+1}$ and $\hat{\eta}_{18,t+1}$ are the residuals.

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\end{aligned}
$$

And then

$$
\begin{aligned}
\hat{\eta}_{1,t+1}^2 &= \sigma_1^2(V_{1,t}^2, V_{18,t}^2) + \xi_{1,t+1}, \\
\hat{\eta}_{18,t+1}^2 &= \sigma_{18}^2(V_{1,t}^2, V_{18,t}^2) + \xi_{18,t+1}, \\
\hat{\eta}_{1,t+1}\hat{\eta}_{18,t+1} &= \sigma_{1,18}(V_{1,t}^2, V_{18,t}^2) + \xi_{1,18,t+1},
\end{aligned}
$$

where $\hat{\eta}_{1,t+1}$ and $\hat{\eta}_{18,t+1}$ are the residuals.

We also estimate a univariate version.

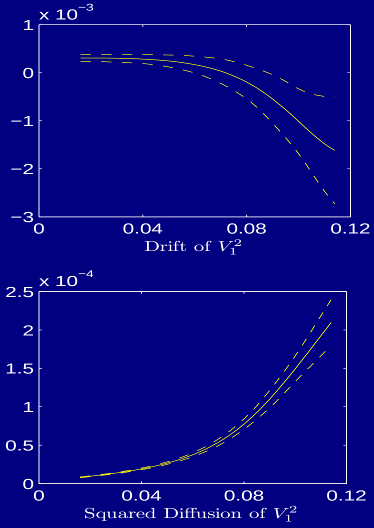

Figure 7. Estimated drift and squared diffusion (one-dimensional)

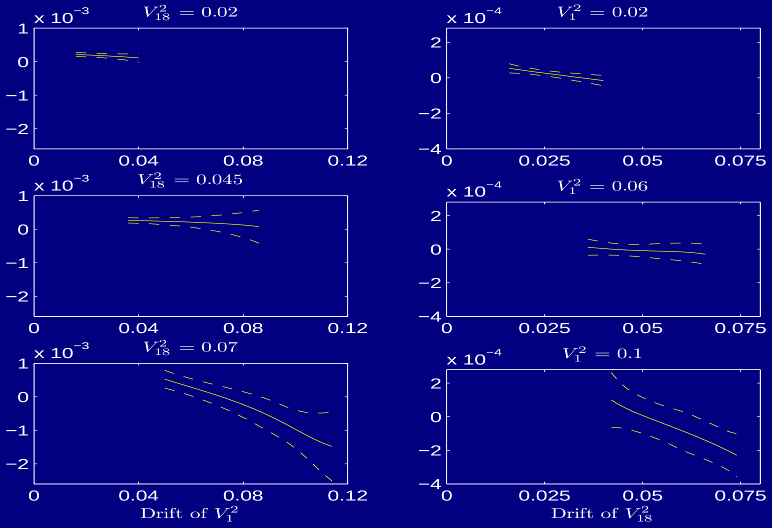

Figure 8. Estimated drift (two-dimensional)

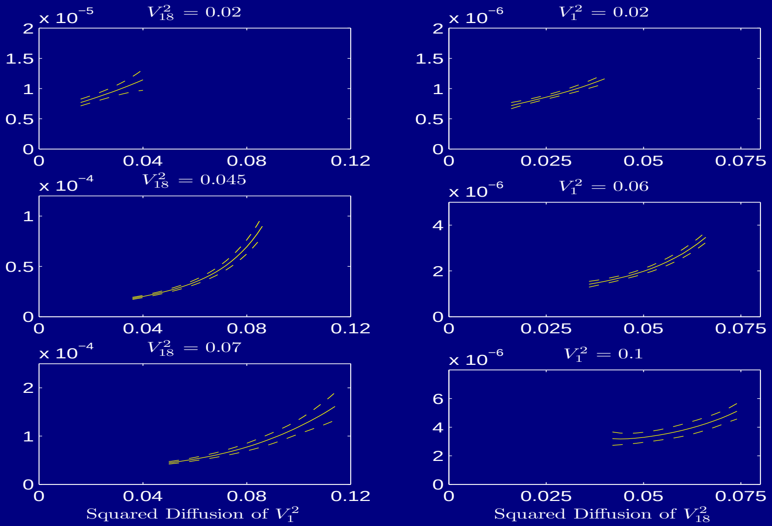

Figure 9. Estimated squared diffusion (two-dimensional)

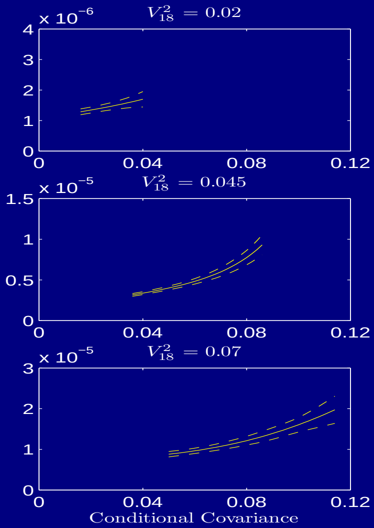

Figure 10. Estimated conditional covariance

Dynamic Features of State Variables (cont'd)

Dynamic Features of State Variables (cont'd)

Nonparametric bootstrap test of the linearity of the drift, squared diffusion, and conditional covariance of V_1^2 and V_{18}^2.

Dynamic Features of State Variables (cont'd)

Nonparametric bootstrap test of the linearity of the drift, squared diffusion, and conditional covariance of V_1^2 and V_{18}^2.

For each of five functions involved, we consider several hypothesized functional forms that are linear in at least one component of (V_1^2, V_{18}^2).

Table 7. Testing the linearity in the drift, squared diffusion, and conditional covariance

	μ_1	μ_{18}	σ_1^2	σ_{18}^2	$\sigma_{1,18}$
$a + bV_1^2$	0.00		0.00		
$a + bV_{18}^2$		0.13		0.00	
$a + b_1 V_1^2 + b_2 V_{18}^2$	0.01	0.41	0.01	0.04	0.32
$g(V_1^2) + bV_{18}^2$	0.43	0.33	0.20	0.02	0.31
$bV_1^2 + g(V_{18}^2)$	0.01	0.42	0.01	0.12	0.32

6. Robustness Checks

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- Alternative factor construction: using average implied volatility rather than variance-swap prices.
- Nonlinear drift: several specific nonlinear models.

7. Conclusion

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The results indicate that two variance-swap prices, with a short maturity and a long maturity, capture the dynamics of the state variables in the options prices. The model with the two state variables is able to eliminate most of the persistent pricing errors.

The drift is nonlinear and the diffusion term is inconsistent with the square root diffusion process. Extending the volatility process to higher dimensions is of little help. Rather, improving upon the specification of the two-factor volatility process is a promising direction in modeling options prices.