

TRADING AGENTS AND LIQUIDITY RISK

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Abstract

A recent area of concern – and analysis – in both financial economics and capital markets has been liquidity. Broadly speaking, liquidity is the ease with which a financial asset can be traded. Liquidity risk, on the other hand, can be defined as the uncertainty associated with the measure of liquidity. Using a simple information-based model of liquidity, we define, develop, and empirically test some measures of liquidity risk, both at the stock- and market-levels. In this model, trading agents are characterized as being driven by superior information, liquidity needs, or hedging requirements. The bid-ask spreads derived from this model have the desired historical properties, and the ability to forecast future liquidity. We also provide empirical evidence that validates the notion that liquidity affects financial market performance.

JEL Classification G 100 (General Financial Markets)

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1. Introduction

A recent area of concern – and analysis – in both financial economics and capital markets has been liquidity. Broadly speaking, liquidity is the ease with which a financial asset can be traded. Liquidity risk, on the other hand, can be defined as the uncertainty associated with the availability of liquidity.

The financial press has been replete with articles on liquidity and its effects, especially during the “quant-driven” crisis of August 2007, the ongoing global financial crisis involving credit markets and financial institutions that commenced in 2007, and the LTCM crisis of August 1998. In May 15, 2008 Federal Reserve Chairman Ben Bernanke remarked that “Another crucial lesson from recent events is that financial institutions must understand their liquidity needs at an enterprise-wide level and be prepared for the possibility that market liquidity may erode quickly and unexpectedly”

In the academic literature, Bernanke and Gertler (1995) were the first to show that financial market liquidity affects the real economy.² They demonstrate that sustained periods of high (low) liquidity can lead to positive (negative) shocks to the real economy. In a nutshell, financial market liquidity affects the availability of credit to corporates and households, which in turn affects investments and consumption, leading to an effect on output.

In their classic paper, “Asset pricing and the bid-ask spread,” Amihud and Mendelson (1986) demonstrate that for a short holding period, transaction costs are important and the liquid asset is the better investment, despite being more expensive than the illiquid asset.³ However, over longer measurement periods, transaction costs are less important and the illiquid asset is the better investment. As a consequence, investors such as pension funds and insurance companies) can potentially use illiquid instruments to fund long-term liabilities and pocket the extra return premium.

In a related paper on illiquidity, Amihud (2002) proposes a measure of price impact that is intuitive and simple to implement. It is based on the λ measure of Kyle (1985), which measures the marginal impact of price with respect to the trading volume^{4,5}. The Amihud measure, *ILLIQ*, more specifically is the average ratio of the absolute return in a stock to

its trading volume in any given period, i.e. $\frac{1}{T} \sum_{t=1}^T \frac{|r_t|}{V_t}$ where r_t is the daily stock’s return, V_t is the trading volume in dollars during that day, and T is the number of trading days during the time period.

² B. Bernanke and Gertler, “Monetary Policy and Asset Price Volatility,” Journal of Economic Perspectives, 1995.

³ Y. Amihud and H. Mendelson, “Asset pricing and the Bid-Ask Spread,” Journal of Financial Economics, 1986.

⁴ Y. Amihud, “Illiquidity and stock returns: cross-section and time series effects,” Journal of Financial Markets, 2002.

⁵ A.S. Kyle, “Continuous auctions and insider trading,” Econometrica, 1985.

The drivers of liquidity include demand / supply pressures and inventory risk, where extreme events can cause order imbalances and inventory overload. Secondly, agents may lack the propensity to trade when natural counterparties are not immediately available. Related to this point, a paper on Latent Liquidity (see Mahanti, Nashikkar, Subrahmanyam, Chacko and Mallik, *Journal of Financial Economics*, 2008),⁶ which is defined as the weighted average turnover of investors who hold a security, can be used to proxy for liquidity in markets with sparse transactions data. Mahanti et al find that in the case of U.S. Corporate Bonds, those held by active (or higher turnover) managers are more liquid than those held by long-term (or lower turnover) buy-and-hold investors. For bonds that trade frequently, their Latent Liquidity measure even has predictive power for both transaction costs and the price impact of trading, over and above those characteristics traditionally thought to be related to liquidity.

The remaining drivers of liquidity include the possession of private information, short sale constraints, and the funding of liquidity (a.k.a. the cost of margin trading). A paper that addresses the funding issue, i.e., the availability of financing (or the lack thereof) over a short period of time as reflected in repo / reverse repo markets, is the one by Cherian, Jacquier and Jarrow (2004).⁷ They model the price difference between otherwise identical on- and off-the-run Treasury securities as the *auction-induced premium* that the former commands due to the riskless profits owners of the on-the-run (liquid) bond can garner by borrowing at a special repo rate while lending out at the prevailing risk-free rate during the Treasury auction cycle.

In an asset pricing equilibrium context with liquidity risk, Acharya and Pedersen (2005) theoretically solve the liquidity-adjusted CAPM problem, where they find that the security's equilibrium rate of return depends on its expected liquidity as well as on the covariance of its own return net of liquidity costs with that of the market's return net of liquidity costs.⁸ In addition, their model yields the result that investors are more attracted to liquid securities when the market return is low, broadly consistent with the empirical findings of Hameed, Kang and Viswanathan (2007) where negative market returns decrease stock liquidity for high volatility stocks and during times of tightness in the funding market.⁹

Lo, Petrov, and Wierzbicki (2003) explicitly model liquidity into the portfolio construction process.¹⁰ Using mean-variance optimized portfolios adjusted for liquidity in three distinct ways, Lo et al find that portfolios close to each other on the traditional mean-variance efficient frontier can differ substantially in their liquidity characteristics. Their analysis also reveals that simple forms of liquidity optimization can yield

⁶ S. Mahanti, A. Nashikkar, M. Subrahmanyam, G. Chacko and G. Mallik, "Latent Liquidity: A New Measure of Liquidity, with an Application to Corporate Bonds," *Journal of Financial Economics*, 2008.

⁷ J. Cherian, E. Jacquier and R. Jarrow, "A Model of the Convenience Yields in On-the-run Treasuries," *Review of Derivatives Research*, 2004.

⁸ V. Acharya and L. Pedersen, "Asset Pricing with Liquidity Risk," *Journal of Financial Economics*, 2005.

⁹ A. Hameed, W. Kang and S. Viswanathan, "Stock Market Declines and Liquidity," NUS Working Paper, 2007.

¹⁰ A. Lo, C. Petrov, and M. Wierzbicki, "It's 11 PM — Do You Know Where Your Liquidity Is? The Mean-Variance Liquidity Frontier," *Journal of Investment Management*, 2003.

significant benefits in reducing a portfolio's liquidity-risk exposure without sacrificing a great deal of expected return per unit of risk.

In our paper, we define, develop, and empirically test some measures of liquidity risk, both at the stock- and market-levels. We then introduce a simple information-based model of liquidity to help fix some key ideas and illustrate the comparative statics when the market explicitly takes into account liquidity. By conjoining the theoretical model with the empirical one, we are able to provide forecasts of future liquidity as well the composition of the types of traders in different market environments. We finally provide empirical evidence that validates the notion that liquidity affects financial market performance.

2. An Empirical Model of Liquidity Risk

A question that immediately comes to mind is how one measures liquidity. The two most common forms used in practice are:

1. The ease or availability of financing over a short term (example, as reflected in repo / reverse repo markets);
2. The ease with which positions can be liquidated.

The second measure provides for the ability to liquidate positions without significantly affecting prices: i.e., being able to buy or sell a security for the value we expect at the time we expect. Measure (2) is the one most commonly used by market participants, and the one we adopt in this paper.

The liquidity modeling approach in the current paper follows a 2-step process. In the first step, we estimate the ease with which positions can be liquidated. This is defined as the magnitude of price movements resulting from order size and is based on Amihud's measure of illiquidity described above. We extend Amihud's measure by using intraday trading data. Each trading day is divided into 10 equal trading time horizons with each time horizon equal to 45 minutes (The normal trading hours for U.S. equities is between 9.30 and 4 P.M.). For each time horizon t , we compute Amihud's *ILLIQ* measure as

$$ILLIQ_t = \frac{|r_t|}{V_t} \quad \text{Equation (1)}$$

Where r_t is the stock return during the time horizon t and V_t is the total dollar trading value during t .

So as to allow for comparisons across time to account for inflation and stock issuance, the measure is appropriately normalized. The normalization process adjusts $ILLIQ_t$ by the factor $\frac{M_0}{M_t}$, where M_0 is the total market capitalization of the largest 3000 U.S. equities as of Jan 1, 1993 and M_t is the total market capitalization as of the prior quarter.

The second step of the process estimates the uncertainty associated with the cost of liquidating the position. We define this uncertainty as the liquidity risk. In order to calculate this, we formulate a time-series model of $ILLIQ_t$ then estimate the liquidity risk as the illiquidity shock, as reflected in the standard error of the “noise” term. More formally, we use the specification proposed by Amihud (2002), where liquidity risk is computed as the standard error in the AR(1) equation:

$$ILLIQ_t = a + b * ILLIQ_{t-1} + \varepsilon_t \quad \text{Equation (2)}$$

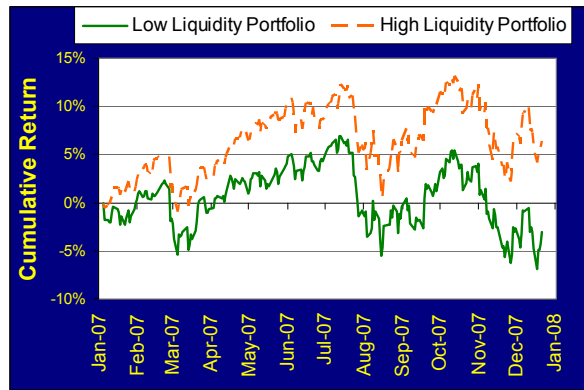
where t represents the trading horizon described above and $t-1$ represents the prior trading horizon. The regression is performed using trailing 1 quarter’s worth of data.

Using the above model for Illiquidity (Equation 1) and Liquidity Risk (Equation 2), we formally introduce the following liquidity indicators:

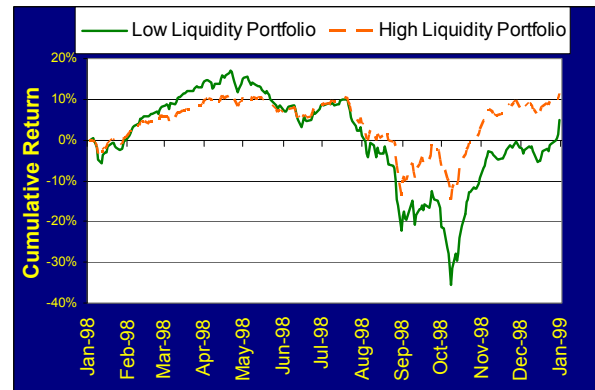
- a) **Market Illiquidity Level (MIL)** is defined as the illiquidity level aggregated for an entire market. We compute the $ILLIQ$ measure for each stock each week, denoted by Stock Illiquidity Level (SIL). The median value of the SILs across all the stocks computed weekly is expressed as an index and denoted **Market Illiquidity Level (MIL)**. MIL is our barometer of liquidity conditions for an Equities Market. An increase in this level indicates deteriorating liquidity conditions. The MIL for U.S. Equities is based on an initial value of 100 registered on Jan 8, 1993.
- b) **Stock Liquidity Rating (SLR)** is defined as stock's liquidity risk (using Equation 2 with trailing one quarter of intraday trading data) categorized into one of ten liquidity risk buckets {AAA, AA, A, BBB, BB, B, CCC, CC, C, D}, with AAA having the least risk and D the greatest.
- c) **Market Illiquidity Factor (MIF)** measures how liquidity risk is priced by market participants. For U.S. equities, it is computed daily as follows. At the beginning of each quarter, the largest 3000 U.S. equities are sorted by SLR. A portfolio is formed by going long the 500 highest liquidity risk stocks and going short the 500 lowest liquidity risk stocks. This portfolio is reconstituted at the beginning of each quarter and the constituents are held constant for the remainder of the quarter. The daily return of this portfolio is the Market Illiquidity Factor (MIF). It measures the cumulative return of illiquid securities relative to liquid securities. The MIF for U.S. equities is based on an initial value of 100 registered on April 1, 1993.
- d) **Stock Liquidity Beta (SLB)** measures the sensitivity of a stock’s return to liquidity risk. It is computed using a multi-factor OLS regression, where the factors are a market factor (defined using the largest 3000 U.S. stocks rebalanced quarterly) and MIL returns. The OLS regression is computed every quarter using trailing 5 years of daily equity closing price data.

Next we provide some casual empirical evidence using the above liquidity indicators. According to Figure 1, illiquid equities, as ranked by the Stock Liquidity Rating (SLR), had underperformed liquid equities by 15.8% between December 28, 2006 (peak) and

May 20, 2008 (trough). This was the biggest margin attained since the LTCM crisis of 1998, where liquid securities, between April 13, 1998 (peak) and October 15, 1998 (trough), outperformed illiquid ones by 18.4%.



Liquidity Crisis of 2007



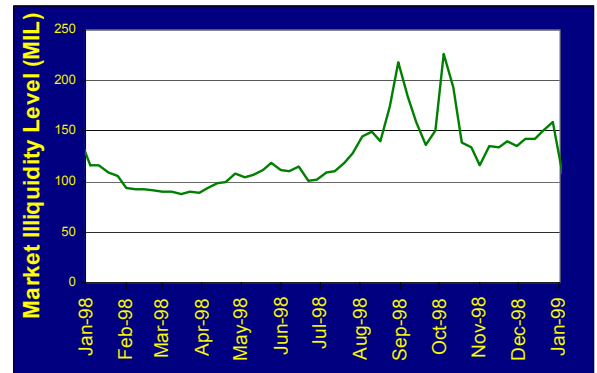
Liquidity Crisis of 1998

Fig 1: Pricing of Liquidity Risk: This figure presents the daily performance of illiquid and liquid stock portfolios during the 1998 and 2007 liquidity crises. The stocks are selected from a universe of 3000 largest public U.S. equities by market capitalization, as determined at the beginning of the quarter. The stocks are ranked on the basis of liquidity risk determined using prior quarters' intra-day trading data. The stock's liquidity rank is denoted as SLR (or Stock Liquidity Rating). The low liquidity portfolio is given as the equal-weighted portfolio of the lowest 500 stocks by liquidity risk. The high liquidity portfolio is given as the equal-weighted portfolio of the highest 500 stocks by liquidity risk.

The Market Illiquidity Level (MIL), which is the median illiquidity level for stocks in the entire market, succinctly captures the time-variation in market liquidity. Indeed, Figure 2 illustrates that liquidity is a significant source of risk in U.S. equity markets, especially during periods of market stress. In fact we will demonstrate that liquidity is statistically and economically significant in explaining asset returns.



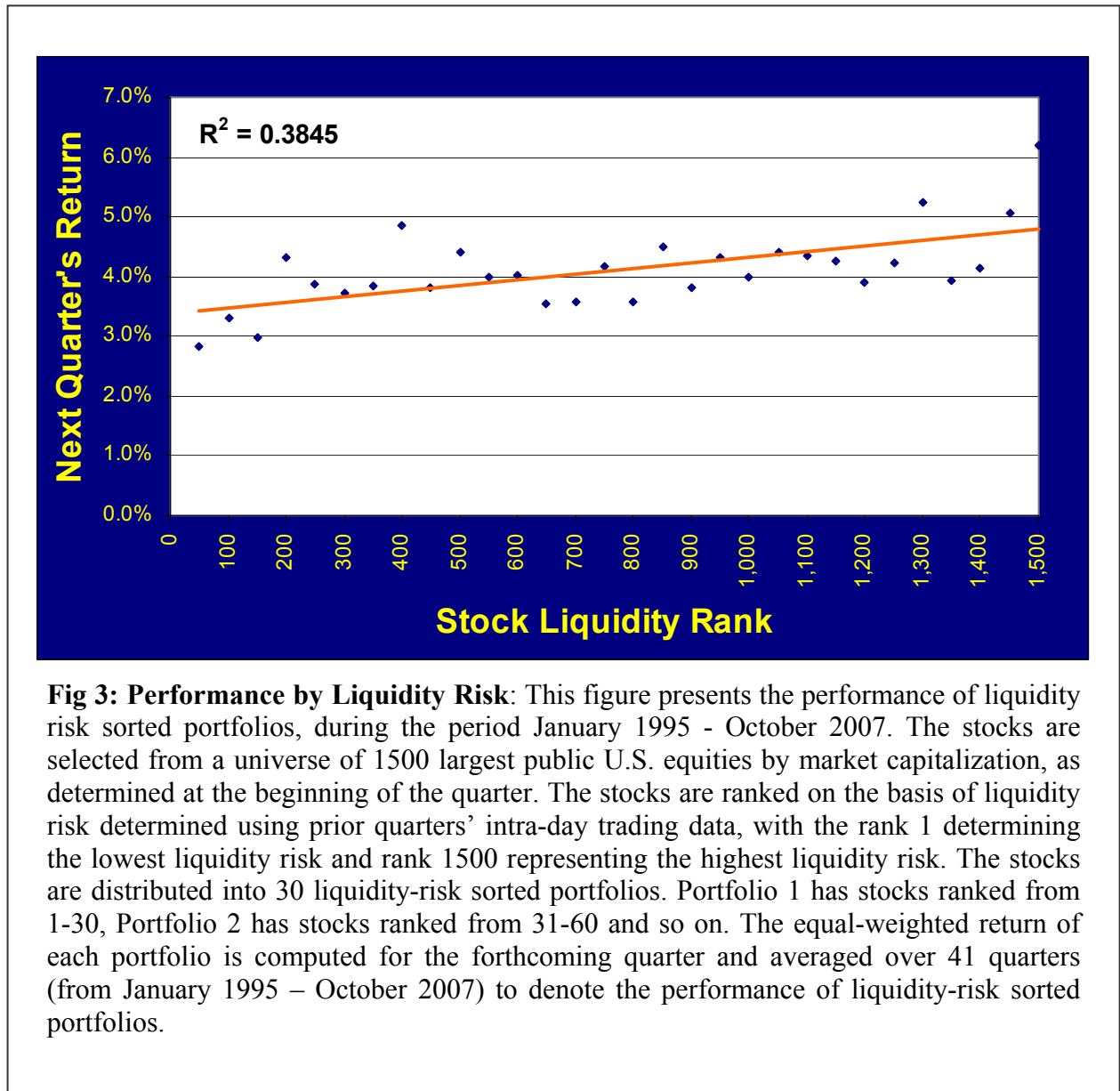
Liquidity Crisis of 2007



Liquidity Crisis of 1998

Fig 2: Level of Illiquidity: This figure presents the levels of illiquidity during the 1998 and 2007 liquidity crises. The stocks are selected from a universe of 3000 largest public U.S. equities by market capitalization, as determined at the beginning of the quarter. The weekly level of illiquidity for each stock is determined using intra-day trading data. This is denoted as SIL (or Stock Illiquidity Level). The median SIL across the 3000 stocks is denoted as MIL (or Market Illiquidity level). A higher value of MIL denotes a higher degree of illiquidity. A lower value of MIL denotes a lower degree of illiquidity.

Finally, evidence that lends additional support to the hypothesis that liquidity affects financial market performance is provided in Figures 3 and 4. An experiment comprising of 30 portfolios formed on the basis of its liquidity risk rank (as provided by our SLR model) from the largest 1500 stocks in the U.S. rebalanced quarterly between January 1995 and October 2007, yields the following results:



A corresponding analysis conducted using the stock's liquidity beta rank yields even more compelling results about the influence of liquidity on asset price returns (see Figure 4).

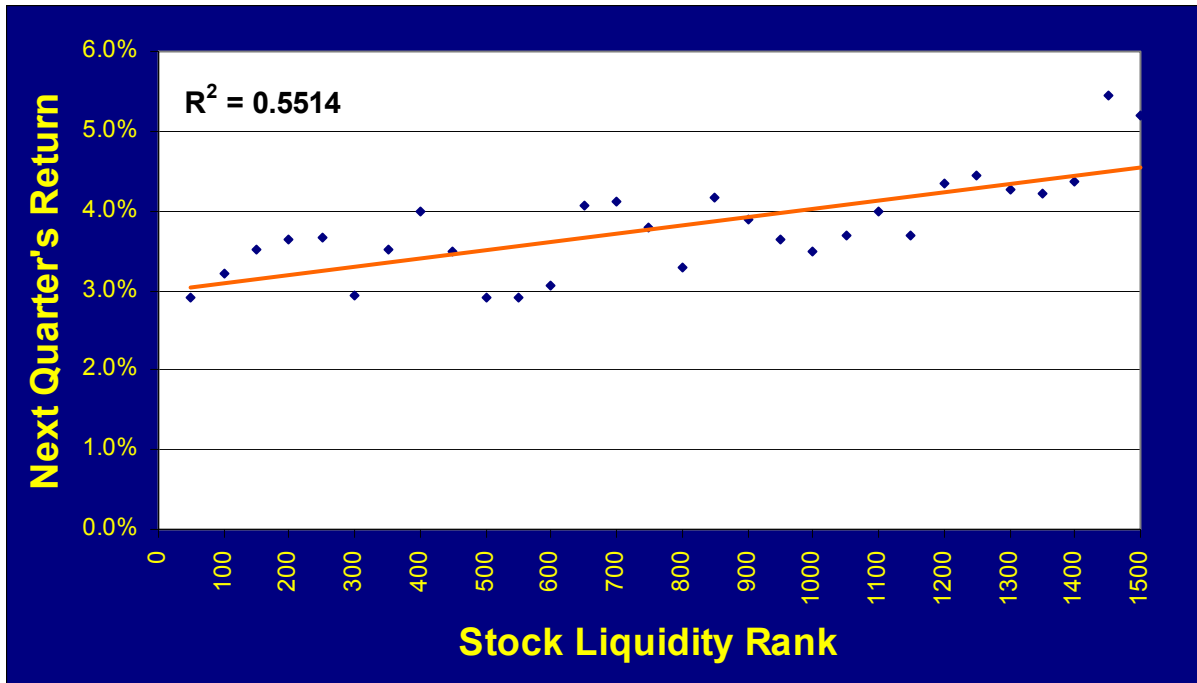


Fig 4: Performance by Stock Liquidity Beta: This figure presents the performance of liquidity beta sorted portfolios, during the period January 1995 - October 2007. The stocks are selected from a universe of 1500 largest public U.S. equities by market capitalization, as determined at the beginning of the quarter. The liquidity risk beta for each stock is computed using a two-factor model and trailing 5 years of daily stock returns. The factors used are Liquidity Factor (MIF) and Market Factor. The Liquidity Factor is given as the daily excess return of higher liquidity risk stocks relative to the lower liquidity risk stock (using a universe of 1500 largest public U.S. equities by market capitalization). The Market Factor is given as the daily return of the S&P 1500 index (which represents the largest public U.S. equities by market capitalization). The liquidity risk beta is given as the sensitivity of the stock's return to the Liquidity Factor. The stocks are ranked on the basis of liquidity risk beta determined using prior daily stock return data, with rank 1 determining the lowest liquidity risk beta and rank 1500 representing the highest liquidity risk beta. The stocks are distributed into 30 liquidity risk beta-sorted portfolios. Portfolio 1 has stocks ranked from 1-30 by liquidity risk beta, Portfolio 2 has stocks ranked from 31-60 by liquidity risk beta and so on. The equal-weighted return of each portfolio is computed for the forthcoming quarter and averaged over 41 quarters (from January 1995 – October 2007) to denote the performance of liquidity-risk beta sorted portfolios.

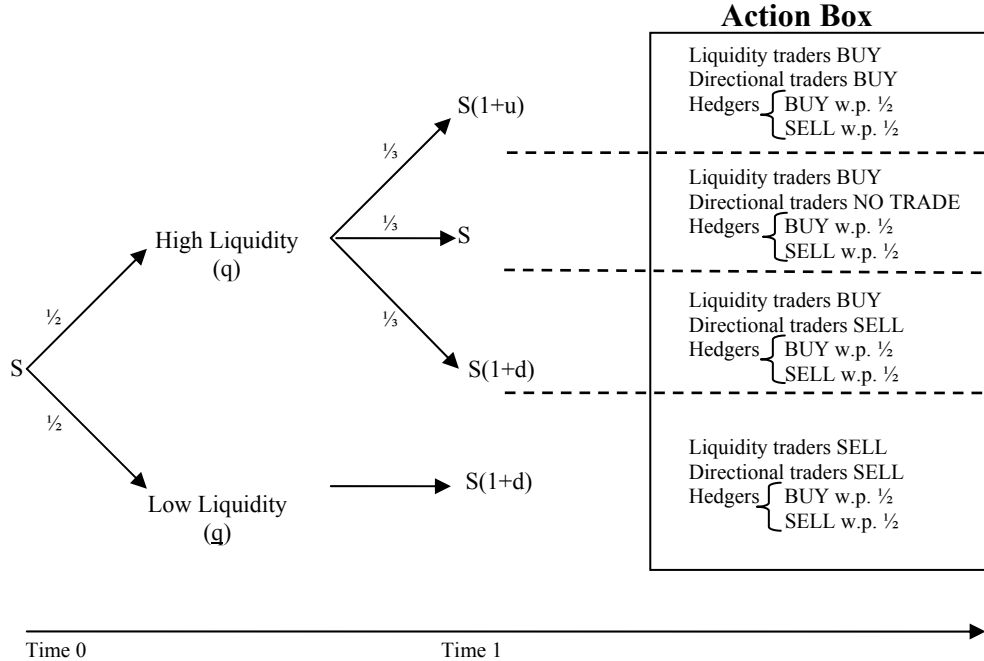
A few broad observations about liquidity would be appropriate at this juncture. Liquidity is neither a fixed property nor concept, it can suddenly dry up (as in the current severe global financial crisis involving credit markets, which commenced in 2007), influence asset price returns, and serve as a significant source of risk. Casual empirical evidence also reveals that size and trading volume, while important, are insufficient proxies of liquidity.

3. A Simple Model of (Informed) Liquidity Trading

It is well-accepted that there are various motivations, both based on superior information or otherwise, that enters the price formation process. These include informed agents who trade on better models of momentum, valuation, and private information derived from proprietary fundamental research and analysis. Current market events have demonstrated that there exists a different class of agents who trade strategically based on market liquidity conditions, somewhat independently of the informed trading based on fundamentals. To that end our measure of liquidity risk heretofore derives from the Kyle (1985) model, which analyses strategic informed trading behavior in the presence of a market maker and uninformed noise (liquidity) traders. In this section we enhance the Kyle information trading structure by introducing informed *liquidity traders* whose trading behavior is influenced by market liquidity conditions.

Our model structure follows the Bayesian Nash equilibrium construct utilized in much of the market microstructure literature.¹¹ The market is one with incomplete information where a fraction $0 < \alpha < 1$ of the traders are more informed about the stock's market direction (henceforth called *directional traders*), while another segment $0 < \beta < 1$ of traders are informed about market liquidity affecting the stock's return (henceforth called *liquidity traders*). So as to ensure equilibrium with no market breakdowns, we include the fraction $0 < 1 - \alpha - \beta < 1$ of *hedgers* who are uninformed about both liquidity and direction. All traders are risk neutral. For notational convenience we assume the risk-free rate is zero. The 1-period stock price process, probabilities, and trade structure can be described by the following tree diagram:

¹¹ See L. Glosten and P. Milgrom, "Bid, Ask and Transaction Prices in Specialist Market with Heterogeneously Informed Traders," *Journal of Financial Economics*, 1985; D. Easley and M. O'Hara, "Price, Trade Size, and Information in Securities Markets," *Journal of Financial Economics*, 1987; and J. Cherian and R. Jarrow, "Options Markets, Self-fulfilling Prophecies, and Implied Volatilities, *Review of Derivatives Research*, 1998.



There are 4 possible stock outcomes at time 1, which is common knowledge and fully revealed to all market participants at that time: $\{(S(1+u)(1+q)); (S(1+q)); (S(1+d)(1+q)); (S(1+d)(1+q))\}$, which we shall represent by $\{S^1; S^2; S^3; S^4\}$. Since at time 0 the **directional traders** are already informed about whether the stock price is **up** ($u > 0$) or **down** ($d < 0$), they trade according to the trade structure depicted in the diagram. Similarly, the **liquidity traders** at time 0 are already informed about whether markets are going to be liquid ($q > 0$) or illiquid ($q < 0$) in the next period. Since q is positive, liquid markets are viewed as a favorable outcome for stocks. Since $(u, q) > 0$ and $(d, q) < 0$,

$$S(1+u)(1+q) > S(1+q) > S(1+d)(1+q) > S(1+d)(1+q).$$

The prior commonly-held beliefs for the likelihoods of these four states are given along the branches of the tree diagram. The probability of a liquid market occurring is $\frac{1}{2}$. Given that a liquid market occurs, the up, flat, and down stock price outcomes are equally likely outcomes. In illiquid markets, the stock price will only go down, given the lack of market confidence. **Liquidity traders** BUY if markets are liquid, and SELL otherwise; **directional traders** BUY if markets are up, SELL if markets are down, and do nothing otherwise; while **hedgers**, being uninformed, BUY or SELL with equal probability. The actions of the various traders are denoted in the “Action Box” of the above tree diagram.

We now proceed to solve for the equilibrium solution. The market maker determines her equilibrium bid and ask prices as follows:

$$\begin{aligned} \text{ASK Price} &= \text{Exp} [\text{Stock Price at Time 1} \mid \text{BUY}] \\ &= S(1+u)(1+q)\text{Pr} [S^1 \mid \text{BUY}] + S(1+q)\text{Pr} [S^2 \mid \text{BUY}] + \\ &\quad S(1+d)(1+q)\text{Pr} [S^3 \mid \text{BUY}] + S(1+d)(1+q)\text{Pr} [S^4 \mid \text{BUY}] \end{aligned}$$

$$\begin{aligned} \text{BID Price} &= \text{Exp [Stock Price at Time 1 | SELL]} \\ &= S(1+u)(1+q)\text{Pr [S}^1 \text{ | SELL]} + S(1+q)\text{Pr [S}^2 \text{ | SELL]} + \\ &\quad S(1+d)(1+q)\text{Pr [S}^3 \text{ | SELL]} + S(1+d)(1+q)\text{Pr [S}^4 \text{ | SELL]} \end{aligned}$$

Furthermore, to ensure trader's actions are optimal, the liquidity trader:

$$\begin{aligned} \text{BUYS if and only if ASK} &< \text{Exp [Stock Price at Time 1 | q]} \\ &\Leftrightarrow \text{ASK} < \frac{1}{3} [S(1+u)(1+q) + S(1+q) + S(1+d)(1+q)] \\ \text{SELLS if and only if BID} &> \text{Exp [Stock Price at Time 1 | q]} \\ &\Leftrightarrow \text{BID} > S(1+d)(1+q) \end{aligned}$$

The directional trader on the other hand:

$$\begin{aligned} \text{BUYS if and only if ASK} &< \text{Exp [Stock Price at Time 1 | (S(1+u)]} \\ &\Leftrightarrow \text{ASK} < S(1+u)(1+q) \\ \text{SELLS if and only if BID} &> \text{Exp [Stock Price at Time 1 | S(1+d)]} \\ &\Leftrightarrow \text{BID} > [\frac{1}{4}S(1+d)(1+q) + \frac{3}{4}S(1+d)(1+q)] \end{aligned}$$

Result 1 (BID / ASK Prices)

a) The equilibrium ASK Price is given by:

$$\begin{aligned} &S(1+u)(1+q)(\alpha + \beta + 1)/(6 - 4\alpha) + S(1+q)(1 + \beta - \alpha)/(6 - 4\alpha) \\ &+ S(1+d)(1+q)(1 + \beta - \alpha)/(6 - 4\alpha) + S(1+d)(1+q)(3(1 - \alpha - \beta))/(6 - 4\alpha) \end{aligned}$$

b) The equilibrium BID Price is given by:

$$\begin{aligned} &S(1+u)(1+q)(1 - \alpha - \beta)/(6 + 2\alpha) + S(1+q)(1 - \alpha - \beta)/(6 + 2\alpha) \\ &+ S(1+d)(1+q)(1 + \alpha - \beta)/(6 + 2\alpha) + S(1+d)(1+q)(3 + 3\alpha + 3\beta)/(6 + 2\alpha) \end{aligned}$$

Result 2 (BID / ASK Spread)

Using Result 1 and some algebra, the bid / ask spread (ASK – BID) is given by:

$$\frac{1}{2} \left\{ S(1+u)(1+q) [(6-\alpha)(\alpha+\beta) + 3\alpha] + S(1+q) [(6-\alpha)(3\alpha+\beta) - 15\alpha] \right. \\ \left. + S(1+d)(1+q) [(6-\alpha)(\beta-\alpha) + 3\alpha] + 3*S(1+d)(1+q) [-(6-\alpha)(\alpha+\beta) + 3\alpha] \right\} / \\ [(3+\alpha)(3-2\alpha)]$$

The proofs of the 2 results can be found in the Appendix.

To ensure that an equilibrium in bid and ask prices indeed exists, the informed traders' optimality conditions should obtain. Using the optimality conditions, Result 1, and simple algebra, we get the next result.

Result 3 (Optimality Conditions for Equilibrium)

a) The liquidity trader buys if and only if:

$$9(\alpha+\beta-1)(1+d)(1+q) > (1+u)(1+q)(7\alpha + 3\beta-3)+(1+q)(\alpha+3\beta-3)+(1+d)(1+q)(\alpha+3\beta-3)$$

She sells if and only if:

$$(1+u)(1+q)(1-\alpha-\beta)+(1+q)(1-\alpha-\beta)+(1+d)(1+q)(1+\alpha-\beta) > (1+d)(1+q)(3-\alpha-3\beta)$$

b) On the other hand, the directional trader buys if and only if:

$$(1+u)(1+q)(5\alpha+\beta-5) < (\alpha-\beta-1)(1+q)(2+d)+(1+d)(1+q)(3\alpha+3\beta-3)$$

She sells if and only if:

$$(2+u)(1+q)(1-\alpha-\beta) + (1+d)(1+q)(2\alpha+3\beta) > (1+d)(1+q)(2+\beta)$$

4. Applications – Forecasting Liquidity and Determining Trader Types

The natural question to ask at this stage is whether by conjoining the empirical model of Section 2 with the information-based model of liquidity introduced in Section 3, one can forecast future period liquidity. Additionally, we would like to construct and analyze the composition of the different traders at different points in time using the same models and market information. This is what we set out to do in this section.

In order to determine next period's liquidity we proceed as follows:

Step 1

For each security i and week t we decompose the total return into its liquidity and market components using the following OLS regression equation:

$$\text{Total_return}_{it} = c_{it} + (\text{beta}_{i,t,\text{liquidity}} * \text{MIF}_t) + (\text{beta}_{i,t,\text{market}} * \text{Market_return}_t) + \varepsilon_{it}$$

where,

- MIF_t is the weekly liquidity risk factor given by the cumulative return of illiquid securities relative to liquid securities between period $t-1$ and t as defined in Section 2.
- Market_return_t is the weekly Russell 3000 index return between period $t-1$ and t .

Step 2

Each week we determine market liquidity conditions by using the Market Illiquidity Level (MIL) indicator from Section 2. When MIL is increasing, it indicates deteriorating liquidity. Conversely, when MIL is decreasing, it indicates improving liquidity.

Similarly we determine rising market conditions by positive weekly performance of the Russell 3000 index. Conversely, we determine declining market conditions by negative weekly performance of the Russell 3000 index.

For each security i and week t , we define:

q_{it} : $\text{Beta}_{i,t,\text{liquidity}} * \text{MIF_return}_t$, during improving liquidity (MIL decreasing)

\underline{q}_{it} : $\text{Beta}_{i,t,\text{liquidity}} * \text{MIF_return}_t$, during deteriorating liquidity (MIL increasing)

u_{it} : $\text{Beta}_{i,t,\text{market}} * \text{Market_return}_t$, during rising market (positive Russell 3000 return)

d_{it} : $\text{Beta}_{i,t,\text{market}} * \text{Market_return}_t$, during declining market (negative Russell 3000 return)

Step 3

For each security i and week t , we calculate the rolling 1 year average (i.e., prior 52 weeks) of the above parameters as:

$$q_{it_avg}: \frac{1}{52} \sum_{t=1}^{52} 1_{\{\text{improving liquidity}\}} q_{it}$$

$$\underline{q}_{it_avg}: \frac{1}{52} \sum_{t=1}^{52} 1_{\{\text{deteriorating liquidity}\}} \underline{q}_{it}$$

$$u_{it_avg}: \frac{1}{52} \sum_{t=1}^{52} 1_{\{\text{rising market}\}} u_{it}$$

$$d_{it_avg}: \frac{1}{52} \sum_{t=1}^{52} 1_{\{\text{falling market}\}} d_{it}$$

where $1_{\{X\}}$ is the indicator function with respect to event X occurring.

Step 4

By utilizing the average parameter estimates from Step 3, we compute Bid-Ask spreads for the following time period, i.e. week $t+1$, using Result 2 from Section 3. The Bid-Ask spread is computed over a range of 100 values for α and 100 values for β , each of which is uniformly distributed between 0 and 1 for a total combination of 10^4 potential $[\alpha, \beta]$ values, which have to satisfy the Optimality Conditions set out in Result 3.

In Figure 5, we report the time series distribution of the computed Bid-Ask spread values. First we compute the average of Bid-Ask spread for each stock each week, where the average is computed across the 10^4 combinations of the $[\alpha, \beta]$ values described above. For each week, we report the median value of this Bid-Ask spread which is computed across the largest 3000 U.S. equities. It is obvious from the distribution that the Bid-Ask spread are quite sensitive to improving and declining liquidity periods.

In Figure 6, we report similar results for high liquidity risk and low liquidity risk sub-portfolios (drawn from a universe of 3000 largest U.S. equities). It is evident from the figure that there is a substantial difference in Bid-Ask spread values across the two sub-portfolios that vary through time.



Fig 5: Bid-Ask Spread: This figure presents the median bid ask spread drawn from a distribution generated by explicitly modeling trading agents as informed traders, liquidity driven traders and hedgers. The model parameters are estimated using historical data on the largest 3000 U.S. equities (as of the beginning of the quarter). The Bid-Ask spread is estimated each week for each combination of trading agents from a total of 10^4 combinations. The median value of the Bid-Ask spread, across the largest 3000 U.S. equities is reported each week.

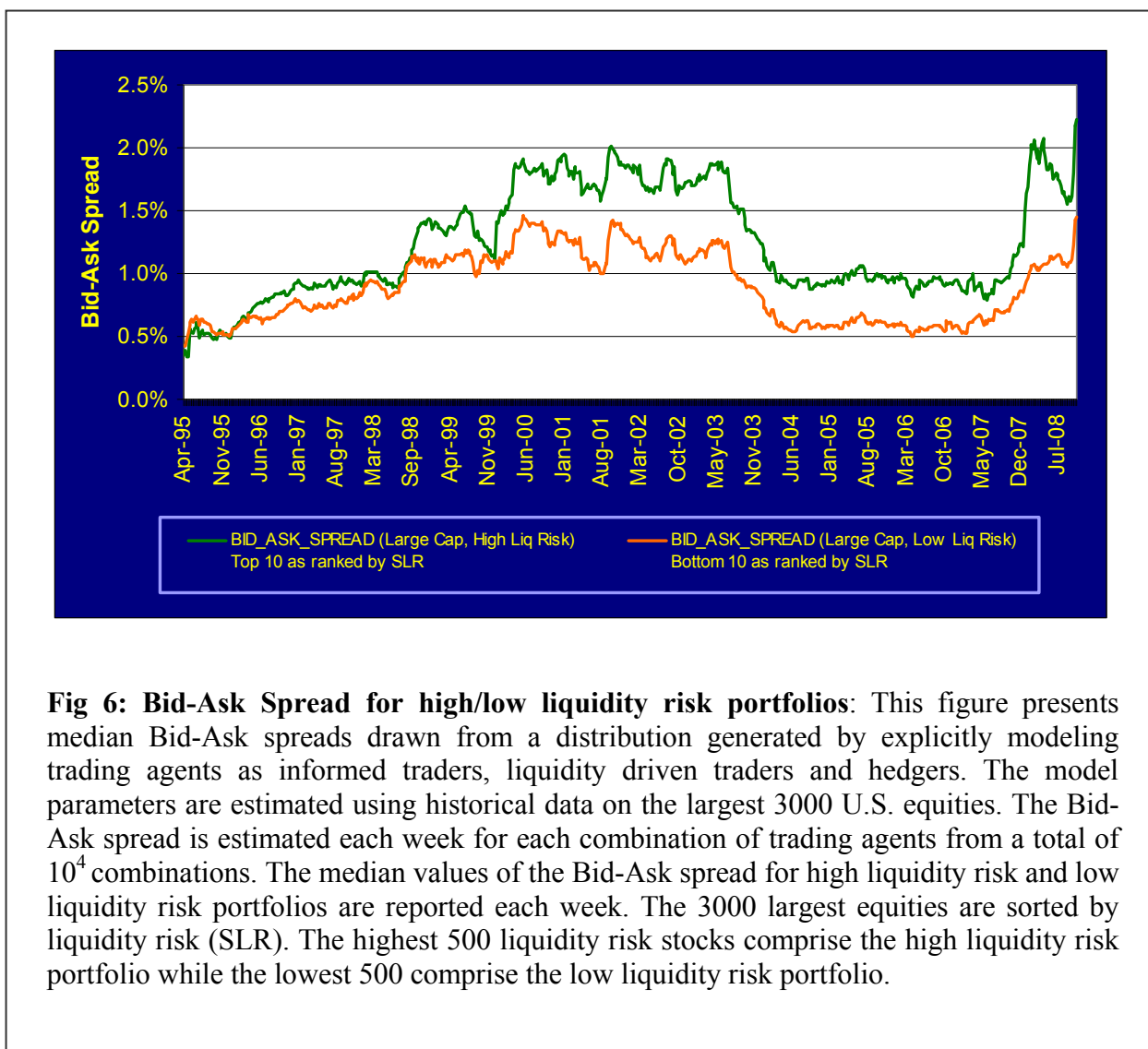


Fig 6: Bid-Ask Spread for high/low liquidity risk portfolios: This figure presents median Bid-Ask spreads drawn from a distribution generated by explicitly modeling trading agents as informed traders, liquidity driven traders and hedgers. The model parameters are estimated using historical data on the largest 3000 U.S. equities. The Bid-Ask spread is estimated each week for each combination of trading agents from a total of 10^4 combinations. The median values of the Bid-Ask spread for high liquidity risk and low liquidity risk portfolios are reported each week. The 3000 largest equities are sorted by liquidity risk (SLR). The highest 500 liquidity risk stocks comprise the high liquidity risk portfolio while the lowest 500 comprise the low liquidity risk portfolio.

It is natural to consider how the composition of trading agents impacts the Bid-Ask spread. In order to study this relationship, we calibrate the model defined in Section 2, with historical data as described above and study how Bid-Ask spread varies with changes in composition of trading agents as defined by the levels of *directional traders* (α), *liquidity traders* (β) and *hedgers* (γ). Figure 7 demonstrates that the Bid-Ask Spread decreases as the proportion of *hedgers* increase. This is to be expected as the *hedgers* are uninformed; as a consequence, they represent both sides of the (no-information) trade equally. Conversely, as the proportion of informed traders increases, the Bid-Ask spread widens due to the presence of informational asymmetry in the market.

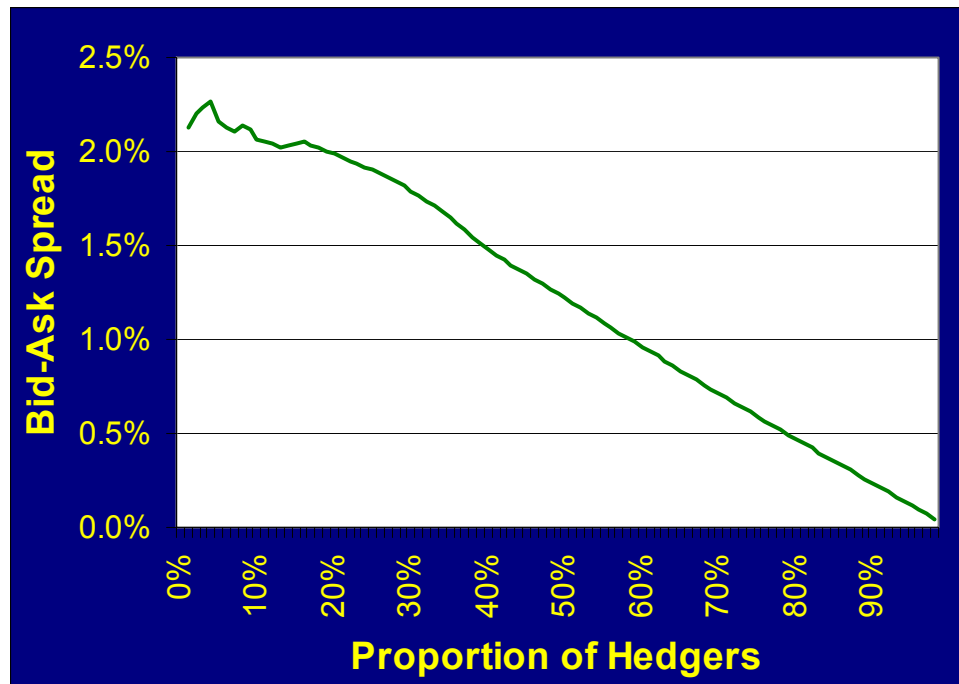


Fig 7: Bid-Ask Spread versus Proportion of Hedgers: This figure presents Bid-Ask spread variation against the proportion of hedgers generated by explicitly modeling trading agents as informed traders, liquidity driven traders and hedgers. The model parameters are estimated using historical data on the largest 3000 U.S. equities. The bid ask spread is estimated each week for each combination of trading agents from a total of 10^4 combinations. The average Bid-Ask spread for different levels of hedging is represented above.

We next study the effect of the Bid-Ask spread on the relative probability distribution of *directional traders* (α) and *liquidity traders* (β). We measure the relative distribution by the ratio β/α . For a given range of *hedgers*, Table 1 indicates that the Bid-Ask spread is inversely proportional to the β/α ratio. In other words, for a fixed level of hedging, the Bid-Ask spread decreases as the proportion of *liquidity traders* increases with respect to *directional traders*.

Level of Hedging	Correlation (β/α , Bid-Ask)
0 - 20%	-0.43
20 - 40%	-0.46
40 - 60%	-0.55
60 - 80%	-0.64
80 - 100%	-0.81

Table 1: Correlation of Bid-Ask spread with (β/α): This figure presents Bid-Ask spread variation against the ratio of liquidity traders (β) to directional traders (α) for a given level of hedging. The model parameters are estimated using historical data on the largest 3000 U.S. equities. The Bid-Ask spread is estimated each week for each combination of trading agents from a total of 10^4 combinations. The average Bid-Ask spread for different levels of β/α (for a given level of hedging) is computed, and the correlation is presented above.

Finally, in order to imply the probability distribution of agents (Φ_t) using the information-based model, we consider the functional model described above that takes as its input parameters the distribution of trading agents as well as the return attributable to various states of the model (i.e., increasing liquidity, decreasing liquidity, improving markets and deteriorating markets).

For each week t , define:

- α_t : Probability distribution of *directional traders*
- β_t : Probability distribution of *liquidity traders*
- γ_t : Probability distribution of *hedgers*
- Φ_t : Probability distribution of agents defined by the set $\{\alpha_t, \beta_t, \gamma_t\}$

Define the market average¹² of the stocks movement during the various states of the model (improving liquidity, deteriorating liquidity, up market, down market), calibrated from historical data as:

- q_t : Average(q_{it}), across each stock i
- \underline{q}_t : Average(\underline{q}_{it}), across each stock i
- u_t : Average(u_{it}), across each stock i
- d_t : Average(d_{it}), across each stock i

Define

- Bid-Ask $_{\Phi,t}$: Bid-Ask spread (computed using Result 3 from Section 3) by using the parameters: $q_t, \underline{q}_t, u_t, d_t, \alpha, \beta, \gamma$
- MIL $_t$: Market Illiquidity Level at time t , as defined in Section 2

¹² The market averages are computed across the largest 3000 stocks. The universe of stocks is rebalanced quarterly.

Define the time series vectors (denoted in bold)

Bid-Ask $_{\Phi,t}$: {Bid-Ask $_{\Phi,0}$ Bid-Ask $_{\Phi,1}$ Bid-Ask $_{\Phi,t}$ }

MIL $_t$: {MIL $_0$ MIL $_1$ MIL $_t$ }

The “implied” Φ generates the Bid-Ask series **Bid-Ask** $_{\Phi,t}$ that has the strongest long-run relationship with a Liquidity indicator given by **MIL** $_t$. Long run relationship between variables is typically determined using a co-integration analysis. The “implied” Φ (that gives the distribution of agents implied from historical observations) is the Φ that generates Bid-Ask values that has the highest degree of co-integration with the Liquidity indicator. In order to determine the “implied” Φ , we proceed as follows:

Step 1

We construct the co-integration equation for each Φ_t by using 5 years of trailing weekly data. The OLS equation is given as:

$$\mathbf{Bid-Ask}_{\Phi,t} = \mathbf{c}_t + \mathbf{MIL}_t + \boldsymbol{\varepsilon}_t$$

Step 2

The series **Bid-Ask** $_{\Phi,t}$ and **MIL** $_t$ will be co-integrated if the residuals $\boldsymbol{\varepsilon}_t$ is stationary.

We test for stationarity by using the unit root equation of the residuals (with drift) as given by:

$$\Delta \boldsymbol{\varepsilon}_t = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 \boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\xi}_t$$

Thus, $\boldsymbol{\varepsilon}_t$ is stationary if we can reject $H_0: \boldsymbol{\alpha}_1 = 0$. The null hypothesis is rejected i.e., $\boldsymbol{\varepsilon}_t$ is stationary if the t-stat of $\boldsymbol{\alpha}_1 \leq -2.86$ (the 5% Dickie-Fuller critical value). If the null hypothesis is rejected, we can conclude that **Bid-Ask** $_{\Phi,t}$ and **MIL** $_t$ are co-integrated. On the other hand, if the null hypothesis is accepted, we can conclude that **Bid-Ask** $_{\Phi,t}$ and **MIL** $_t$ are not co-integrated (i.e. no long-run relationship can be established)

Step 3

We assemble the set of Φ for which we can establish a co-integration relationship between **Bid-Ask** $_{\Phi,t}$ and **MIL** $_t$ as defined in Step 2. The “implied” Φ is given for the value that generates the most negative t-stat.

Figure 8 below gives this implied distribution which indicates that strategic trading varies with the market environment. For example, during periods of market stress, strategic trading is dominated by liquidity needs. This is evidenced from the heightened level of *liquidity traders* during the 2007-2008 liquidity crises.

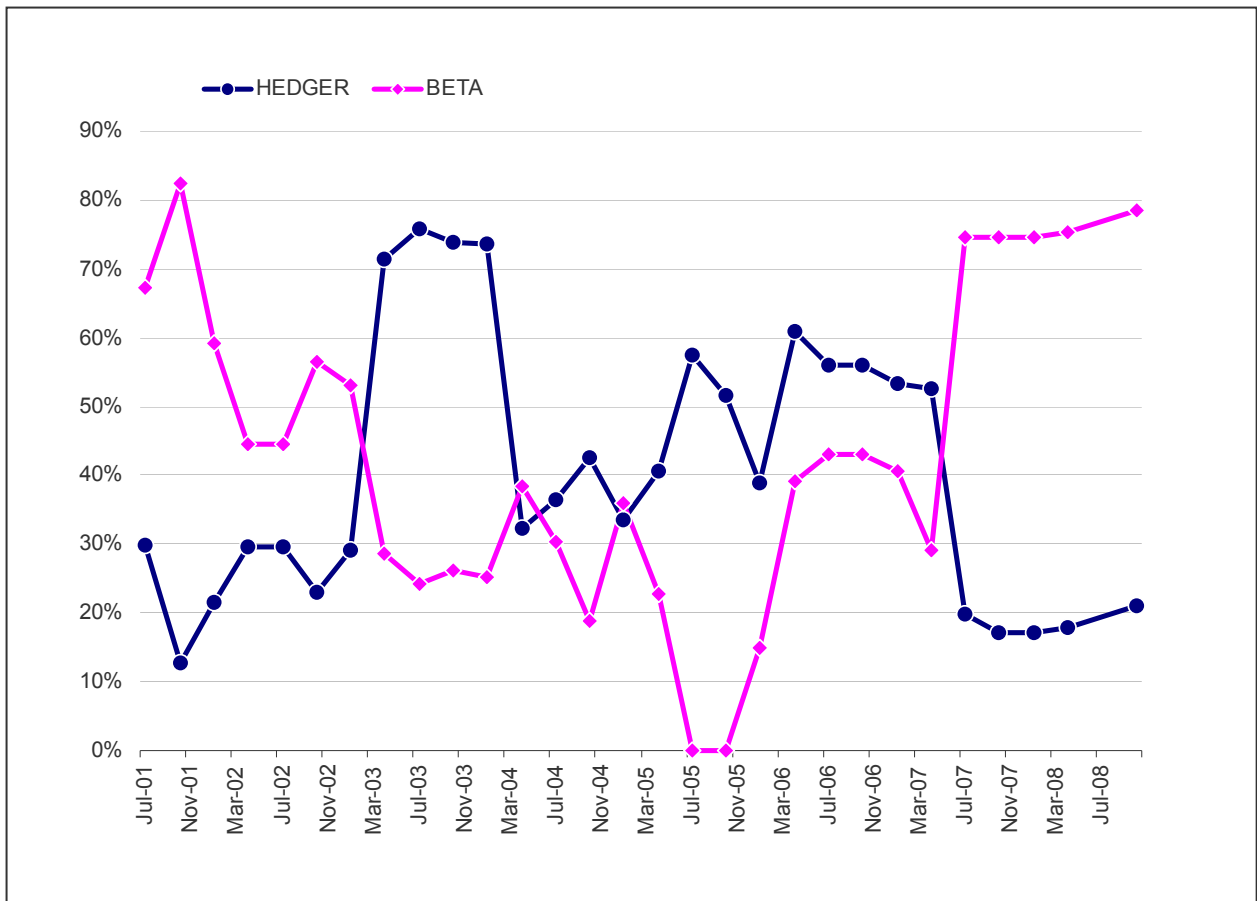


Fig 8: Agents’ Implied Distribution: This figure shows the implied probability distribution of agents using the information-based model calibrated to the historical data. The input parameters are the distribution of trading agents as well as the return attributable to various historical states of the model (i.e. increasing liquidity, decreasing liquidity, improving markets and deteriorating markets). In the first step, the returns attributable to the various states are computed by using historical behavior of stocks. In the second step, we imply the “optimal” probability distribution of agents that provides a Bid-Ask spread estimate that is most co-integrated with Market Illiquidity Level (MIL).

5. Impact of Liquidity on Equity Pricing

Traditional asset pricing models are based on standard, perfect competition Walrasian equilibrium markets that are frictionless. However markets are plagued by various forms of illiquidity and transaction costs. Hence, prices are not always at fundamental value, rather are affected by trading activity. As a consequence, asset pricing models should incorporate liquidity as an endogenous parameter.

First we study the asset pricing distribution when the trailing liquidity is improving versus when the trailing liquidity is deteriorating. Table 2 demonstrates that equities exhibit superior performance (higher expected return and lower volatility) when the prior

liquidity, as measured using trailing 12 weeks of observations of Market Illiquidity Level, improves. This is in contrast to the case when the prior liquidity deteriorates. Further, Table 2 shows that this dichotomy in asset price behavior is more marked for illiquid stocks (as proxied by Russell 2000 Index) than liquid stocks (as proxied by Dow Jones Industrial Average Index). The empirical evidence hence demonstrates that illiquid stocks outperform liquid stocks when liquidity improves and vice versa.

Russell 2000	DJI	Russell 2000	DJI
Return	Return	Std Deviation	Std Deviation
-5.3%	6.8%	24.0%	19.4%
18.4%	9.0%	15.4%	14.0%

Table 2: Prior Liquidity Influences U.S. Equity Returns: This table shows the performance of the Dow Jones Industrial Average (DJI) and Russell 2000 returns when the prior liquidity is improving and when the prior liquidity is deteriorating. Liquidity is improving when the Market Illiquidity Level decreases over the prior one quarter. Liquidity is deteriorating when the Market Illiquidity Level increases over the prior one quarter. The returns and standard deviation of returns are based on weekly data for the period April 1993 - October 2008, and are expressed as annualized numbers.

This property is further evidenced by a naive trading strategy that takes a long position in the Russell 2000 and a short position in the Dow Jones Industrial Average (DJI) when prior liquidity is improving. Conversely, the strategy takes a short position in the Russell 2000 and a long position in the DJI when prior liquidity is deteriorating. Liquidity is improving when the Market Illiquidity Level decreases over the prior one quarter. Liquidity is deteriorating when the Market Illiquidity Level increases over the prior one quarter. (The strategy does not take into account transaction costs.) Figure 9 demonstrates that the strategy strongly outperforms a naïve benchmark based loosely on the Fama/French Small-Minus-Big (SMB)¹³ factor: in our case, we long the Russell 2000 (Small Cap) and short the DJI (Large Cap) to proxy for the SMB factor. The trading strategy performance is also quite robust across various liquidity cycles.

Finally, Figure 10 shows that the trading signal, when using various lagged 12-week horizons for the Market Illiquidity Level (e.g. excluding the current week, the last 2 weeks and so on), is also quite persistent.

¹³ E. Fama and K. French, "Common Risk Factors in the Returns on Stocks and Bonds," Journal of Financial Economics, 1993.

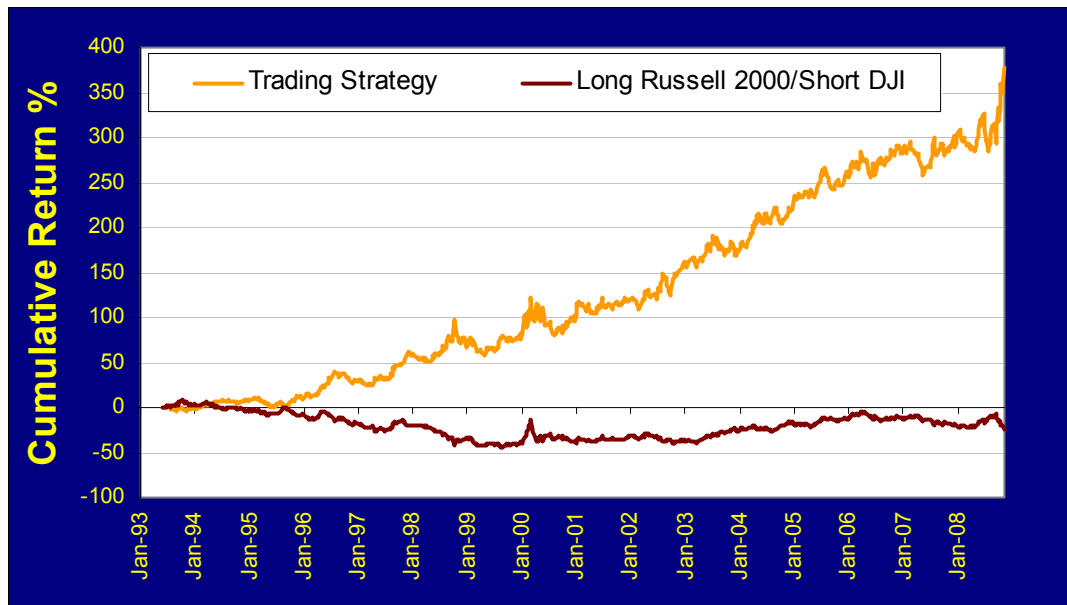


Fig 9: Liquidity Based Index Trading Strategy Performance: This table shows the cumulative return of a long/short trading strategy based on the Dow Jones Industrial Average (DJI) and Russell 2000 index returns. The strategy takes a long position in Russell 2000 and a short position in DJI when prior liquidity is improving. Conversely, the strategy takes a short position in Russell 2000 and a long position in DJI when prior liquidity is deteriorating. Liquidity is improving when the Market Illiquidity Level (MIL) decreases over the prior one quarter. Liquidity is deteriorating when the MIL increases over the prior one quarter. The returns of the strategy are based on weekly rebalancing for the period April 1993 - October 2008. The strategy does not take into account transaction costs.

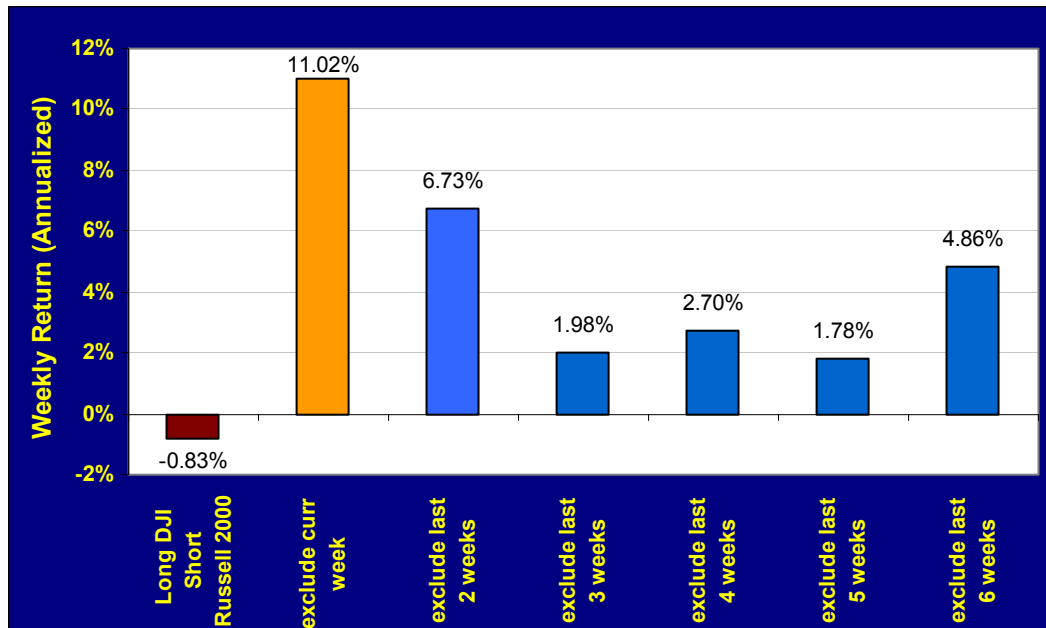


Figure 10: Persistence in Liquidity Based Index Trading Signal: This table shows the performance of a long/short trading strategy based on the Dow Jones Industrial Average (DJI) and Russell 2000 index returns, using various lags of the liquidity indicator. This strategy takes a long position in the Russell 2000 and a short position in the DJI when prior liquidity is improving. Conversely the strategy takes a short position in the Russell 2000 and a long position in the DJI when prior liquidity is deteriorating. Liquidity conditions are determined using various lagged 12-week horizons for the Market Illiquidity Level (e.g., excluding the current week, the last 2 weeks and so on). Liquidity is improving when the Market Illiquidity Level decreases. Conversely, liquidity is deteriorating when the Market Illiquidity Level increases, both measured over the appropriately-lagged horizon. The returns of the strategy are based on weekly rebalancing for the period April 1993 – October 2008. The strategy does not take into account transaction costs.

Table 3 performs a contemporaneous regression of weekly returns of market indices on several risk factors commonly understood to influence asset pricing. The regression is carried out using data from January 1994 through December 2007 (except for the Lehman High Yield index, which starts in August 1998). Table 4 presents correlation coefficients across the risk factors over the same period. It is evident from the results that Market Illiquidity Level (MIL) is significant over and beyond commonly-accepted risk variables used in asset pricing. This further illustrates the importance of liquidity in explaining asset price returns.

	Intercept	Size	Value	T-Bill LIBOR			MIL	Adjusted R2
				T-Bill	OAS	VIX		
Russell 2000	0.84 (3.86)	0.56 (10.34)	-0.71 (-12.74)	-0.34 (-0.5)	-0.02 (-0.95)	-0.03 (-3.14)	-0.07 (-10.23)	0.47
S&P 500	0.89 (4.71)	-0.52 (-11.06)	-0.86 (-17.64)	-0.79 (-1.32)	-0.01 (-0.47)	-0.03 (-3.69)	-0.07 (-11.49)	0.41
DJIA	0.84 (4.12)	-0.58 (-11.46)	-0.62 (-11.83)	-0.94 (-1.45)	-0.01 (-0.66)	-0.03 (-3.07)	-0.08 (-11.71)	0.32
Hang Seng	0.76 (2.03)	-0.14 (-1.43)	-0.52 (-5.36)	-0.93 (-0.77)	0.04 (1.12)	-0.03 (-1.52)	-0.08 (-6.78)	0.11
DAX	1.25 (4.12)	-0.29 (-3.9)	-0.69 (-8.79)	-0.16 (-0.17)	-0.02 (-0.5)	-0.05 (-3.69)	-0.09 (-9.7)	0.22
FTSE	0.79 (3.64)	-0.28 (-5.23)	-0.45 (-8.01)	0.19 (0.27)	-0.01 (-0.28)	-0.04 (-3.42)	-0.06 (-8.55)	0.18
Lehman High Yield	0.33 (3.88)	0.09 (4.85)	0.06 (3.18)	-0.08 (-0.3)	0.01 (0.94)	-0.01 (-2.71)	-0.01 (-5.82)	0.14

Table 3: Contemporaneous regression of market indices with various risk factors:

This table shows the results of OLS regression of weekly returns of market indices with the explanatory variables representing risk factors commonly used in asset pricing. The explanatory variables are formulated using U.S. Equities data. They include:

- Size and Value: % change of Fama/French SMB and HML factors respectively
- T-Bill: Change in yield of 13-week on the run T-Bill
- T-Bill LIBOR OAS: Weekly average of T-Bill LIBOR OAS (T-Bill yield - LIBOR).

This is a commonly used proxy for market liquidity.

- VIX: Weekly average of CBOE Volatility Index
- MIL: % change of Market Illiquidity Level

The regression is carried out using data from January 1994 through December 2007 (except for Lehman High Yield index which starts in August 1998). The t-stats are shown in parenthesis. Coefficients in bold indicate significance level greater than 99%

	Excess Market Ret	Size	Value	T-Bill	T-Bill LIBOR OAS	VIX	MIL
Excess Market Ret	1.00						
Size	0.10	1.00					
Value	-0.56	-0.36	1.00				
T-Bill	0.03	0.04	-0.08	1.00			
T-Bill LIBOR OAS	0.03	0.01	-0.08	0.54	1.00		
VIX	-0.11	-0.08	-0.02	-0.12	-0.01	1.00	
MIL	-0.42	-0.23	0.22	-0.08	-0.07	0.09	1.00

Table 4: Correlation coefficient for risk factors: This table shows the correlation coefficients for various risk factors using weekly data from Jan 1994 through Dec 2007. The factors include

- Excess Market Return, Size and Value: % change of Fama/French R_m minus R_f , SMB and HML factors respectively
- T-Bill: Change in yield of 13-week on the run T-Bill
- T-Bill LIBOR OAS: Weekly average of T-Bill LIBOR OAS (T-Bill yield - LIBOR). This is a commonly used proxy for market liquidity.
- VIX: Weekly average of CBOE Volatility Index
- MIL: % change of Market Illiquidity Level

Finally, we extend our analysis to look at the influence of liquidity in affecting future security returns in the presence of other security-specific variables. We perform a panel regression with fixed effects across publicly-traded U.S. equities with quarterly time series data. The results are reported in Table 5 and confirm that liquidity analytics present opportunities for alpha generation. Moreover, the analysis indicates that in the presence of our liquidity variables, commonly used proxies of liquidity such as size and turnover are rendered insignificant in explaining asset returns.

Explanatory variable (previous quarter value)	Coefficient
previous_quarter_return	-0.04 (-18.25)
illiquidity	-0.23 (-43.93)
liquidity_risk	-0.01 (-2.51)
market_cap	-0.04 (-0.95)
turnover	0.00 (-0.09)
bookvalue_by_price	-0.08 (-20.69)
eps_by_price	-0.01 (-1.53)

Table 5: Panel regression of future return: This Table shows results of a panel regression of securities' future quarter return against various explanatory variables. The regression is performed with fixed effects using quarterly time series data across public U.S. equities from 1993-2007. The explanatory variables include the current quarter's Security Illiquidity level (SIL), Security Liquidity Rank (SLR), market capitalization, turnover, book value/price (using trailing 1 year data) and eps/price (using trailing 1 year data). The various variables enter the regression as a percentile number (0-1), where the percentile value is recomputed every quarter. The t-stats are shown in parenthesis.

6. Conclusion

This paper defined, developed, and empirically tested some measures of liquidity risk, both at the stock- and market-levels. For example, illiquid equities ranked by our Stock Liquidity Rating indicator had underperformed liquid equities by 15.8% during the 2007-08 liquidity crisis, and by 18.4% during the 1998 LTCM crisis.

We next introduced a simple information-based model of liquidity to help fix some key ideas and illustrate the comparative statics when the market explicitly takes into account liquidity. By conjoining this theoretical information-based model with the empirical one, we were able to provide both forecasts of future liquidity as well as the composition of the types of traders in different market environments. For example, during periods of market liquidity stress (such as the 2007-2008 credit crisis), the conjoined model shows that trading is predominantly driven by liquidity needs.

We provided empirical evidence that validates the notion that liquidity affects financial market performance. As an illustration of the efficacy of our approach to understanding and exploiting liquidity changes, an index trading strategy was developed using the Market Illiquidity Level indicator to generate a long/short trading strategy. It calls for a long (short) position in the Russell 2000 and a short (position) position in the Dow Jones Industrial Average when prior liquidity is improving (deteriorating). The Russell 2000 index serves as a proxy for illiquid equities, while the Dow Jones Industrial Average index serves as a proxy for liquid equities. This liquidity-based index strategy yielded an annualized return of 11% for the period April 1993 – October 2008.

Finally, we demonstrate that in the presence of the liquidity variables introduced in this paper, commonly used proxies for liquidity, such as size and turnover, are rendered insignificant in explaining cross-sectional asset returns.

APPENDIX

We provide the proofs of Results 1 and 2 in this section.

The Bayesian market maker sets the ASK Price as follows:

$$\begin{aligned} \text{ASK Price} &= \text{Exp [Stock Price at Time 1 | BUY]} \\ &= S(1+u)(1+q)\text{Pr} [S^1 | \text{BUY}] + S(1+q)\text{Pr} [S^2 | \text{BUY}] + \\ &\quad S(1+d)(1+q)\text{Pr} [S^3 | \text{BUY}] + S(1+d)(1+q)\text{Pr} [S^4 | \text{BUY}] \end{aligned}$$

We first obtain the posterior probabilities of the market maker using Bayesian updating.

$$\begin{aligned} \text{Pr}[S^1|\text{BUY}] &= \text{Pr}[S^1]*\text{Pr}[\text{BUY}|S^1] / \{ \text{Pr}[S^1]*\text{Pr}[\text{BUY}|S^1]+ \text{Pr}[S^2]*\text{Pr}[\text{BUY}|S^2] + \\ &\quad \text{Pr}[S^3]*\text{Pr}[\text{BUY}|S^3] + \text{Pr}[S^4]*\text{Pr}[\text{BUY}|S^4] \}; \end{aligned}$$

where

$$\begin{aligned} \text{Pr}[S^1]*\text{Pr}[\text{BUY}|S^1] &= \frac{1}{2} \cdot \frac{1}{3} [\alpha + \beta + \frac{1}{2}\gamma] \\ &= \frac{1}{6} \cdot \frac{1}{2} [\alpha+\beta+1] \end{aligned}$$

$$\begin{aligned} \text{Pr}[S^2]*\text{Pr}[\text{BUY}|S^2] &= \frac{1}{2} \cdot \frac{1}{3} [\beta+\frac{1}{2}\gamma] \\ &= \frac{1}{6} \cdot \frac{1}{2} [\beta-\alpha+1] \end{aligned}$$

$$\begin{aligned} \text{Pr}[S^3]*\text{Pr}[\text{BUY}|S^3] &= \frac{1}{2} \cdot \frac{1}{3} [\beta+\frac{1}{2}\gamma] \\ &= \frac{1}{6} \cdot \frac{1}{2} [\beta-\alpha+1] \end{aligned}$$

$$\begin{aligned} \text{Pr}[S^4]*\text{Pr}[\text{BUY}|S^4] &= \frac{1}{2} [\frac{1}{2}\gamma] \\ &= \frac{1}{6} \cdot \frac{1}{2} * 3[1-\alpha-\beta] \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Pr}[S^1]*\text{Pr}[\text{BUY}|S^1] + \text{Pr}[S^2]*\text{Pr}[\text{BUY}|S^2] + \text{Pr}[S^3]*\text{Pr}[\text{BUY}|S^3]+ \text{Pr}[S^4]*\text{Pr}[\text{BUY}|S^4] &= \\ &= \frac{1}{6} \cdot \frac{1}{2} [6-4\alpha] \end{aligned}$$

Substituting for these values into $\text{Pr}[S^1|\text{BUY}]$, we obtain

$$\text{Pr}[S^1|\text{BUY}] = \frac{1}{2} [\alpha+\beta+1] / [3-2\alpha]$$

Similarly,

$$\Pr[S^2|\text{BUY}] = \frac{1}{2} [1+\beta-\alpha] / [3-2\alpha]$$

$$\Pr[S^3|\text{BUY}] = \frac{1}{2} [1+\beta-\alpha] / [3-2\alpha]$$

$$\Pr[S^4|\text{BUY}] = \frac{1}{2} * 3[1-\alpha-\beta] / [3-2\alpha]$$

Simple substitution and algebra yields the ASK Price as

ASK Price =

$$S(1+u)(1+q)*([\alpha+\beta+1] / [6-4\alpha]) + S(1+q) * ([1+\beta-\alpha] / [6-4\alpha]) \\ + S(1+d)(1+q)* ([1+\beta-\alpha] / [6-4\alpha]) + S(1+d)(1+q)* *(3[1-\alpha-\beta] / [6-4\alpha])$$

The market maker sets the BID Price as follows:

$$\text{BID Price} = \text{Exp} [\text{Stock Price at Time 1} | \text{SELL}] \\ = S(1+u)(1+q)\Pr[S^1 | \text{SELL}] + S(1+q)\Pr[S^2 | \text{SELL}] + \\ S(1+d)(1+q)\Pr[S^3 | \text{SELL}] + S(1+d)(1+q)\Pr[S^4 | \text{SELL}]$$

We next obtain the market maker's posterior probabilities.

$$\Pr[S^1|\text{SELL}] = \Pr[S^1]*\Pr[\text{SELL}|S^1] / \{ \Pr[S^1]*\Pr[\text{SELL}|S^1] + \Pr[S^2]*\Pr[\text{SELL}|S^2] + \\ \Pr[S^3]*\Pr[\text{SELL}|S^3] + \Pr[S^4]*\Pr[\text{SELL}|S^4] \};$$

where

$$\Pr[S^1]*\Pr[\text{SELL}|S^1] = 1/6 \cdot \frac{1}{2}[1-\alpha-\beta]$$

$$\Pr[S^2]*\Pr[\text{SELL}|S^2] = 1/6 \cdot \frac{1}{2}[1-\alpha-\beta]$$

$$\Pr[S^3]*\Pr[\text{SELL}|S^3] = 1/6 \cdot \frac{1}{2} [1+\alpha-\beta]$$

$$\Pr[S^4]*\Pr[\text{SELL}|S^4] = 1/6 \cdot \frac{1}{2} [3\alpha + 3\beta + 3]$$

Therefore,

$$\Pr[S^1]*\Pr[\text{SELL}|S^1] + \Pr[S^2]*\Pr[\text{SELL}|S^2] + \Pr[S^3]*\Pr[\text{SELL}|S^3] + \Pr[S^4]*\Pr[\text{SELL}|S^4] = \\ = 1/6 \cdot \frac{1}{2} * [6 + 2\alpha]$$

Substituting for these values into $\Pr[S^1|\text{SELL}]$, we obtain

$$\Pr[S^1|\text{SELL}] = 1/6 \cdot \frac{1}{2}[1-\alpha-\beta] / 1/6 \cdot \frac{1}{2} * [6 + 2\alpha] = [1-\alpha-\beta] / [6 + 2\alpha]$$

Similarly,

$$\Pr[S^2|\text{SELL}] = 1/6 \cdot 1/2[1-\alpha-\beta] / 1/6 \cdot 1/2 * [6 + 2\alpha] = [1-\alpha-\beta] / [6 + 2\alpha]$$

$$\Pr[S^3|\text{SELL}] = 1/6 \cdot 1/2[1+\alpha-\beta] / 1/6 \cdot 1/2 * [6 + 2\alpha] = [1+\alpha-\beta] / [6 + 2\alpha]$$

$$\Pr[S^4|\text{SELL}] = 1/6 \cdot 1/2 [3\alpha + 3\beta + 3] / 1/6 \cdot 1/2 * [6 + 2\alpha] = [3\alpha + 3\beta + 3] / [6 + 2\alpha]$$

Simple substitution and algebra yields the BID Price as

$$\begin{aligned} \text{BID Price} &= \text{Exp [Stock Price at Time 1 | SELL]} \\ &= S(1+u)(1+q)\Pr [S^1 | \text{SELL}] + S(1+q)\Pr [S^2 | \text{SELL}] + \\ &\quad S(1+d)(1+q)\Pr [S^3 | \text{SELL}] + S(1+d)(1+q)\Pr [S^4 | \text{SELL}] \\ &= S(1+u)(1+q)(1 - \alpha - \beta)/(6 + 2\alpha) + S(1+q)(1 - \alpha - \beta)/(6 + 2\alpha) + \\ &\quad S(1+d)(1+q)(1 + \alpha - \beta)/(6 + 2\alpha) + S(1+d)(1+q)(3 + 3\alpha + 3\beta)/(6 + 2\alpha) \end{aligned}$$

The Bid-Ask Spread is given by subtracting the 2 quantities (i.e., ASK Price – BID Price):

$$\begin{aligned} &S(1+u)(1+q)*([\alpha+\beta+1] / [6-4\alpha]) + S(1+q) * ([1+\beta-\alpha] / [6-4\alpha]) \\ &+ S(1+d)(1+q)* ([1+\beta-\alpha] / [6-4\alpha]) + S(1+d)(1+q) * (3[1-\alpha-\beta]/[6-4\alpha]) \\ &- \{ S(1+u)(1+q)(1 - \alpha - \beta)/(6 + 2\alpha) + S(1+q)(1 - \alpha - \beta)/(6 + 2\alpha) + \\ &S(1+d)(1+q)(1 + \alpha - \beta)/(6 + 2\alpha) + S(1+d)(1+q)(3 + 3\alpha + 3\beta)/(6 + 2\alpha) \} \end{aligned}$$

Algebra and simple factorization gives the spread as:

$$\begin{aligned} &1/2 \{ S(1+u)(1+q) [(6-\alpha)(\alpha+\beta) + 3\alpha] + S(1+q) [(6-\alpha)(3\alpha+\beta) - 15\alpha] \\ &+ S(1+d)(1+q) [(6-\alpha)(\beta-\alpha) + 3\alpha] + 3*S(1+d)(1+q) [-(6-\alpha)(\alpha+\beta) + 3\alpha] \} / [(3+\alpha)(3-2\alpha)] \end{aligned}$$