Implied Migration Rates from Credit Barrier Models*

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Abstract

The risk-neutral credit migration process captures quantitative information which is relevant to the pricing theory and risk management of credit derivatives. In this article, we derive implied migration rates by means of a recently introduced credit barrier model which is calibrated on the basis of aggregate information such as credit migration rates and credit spread curves. The model is characterized by an underlying stochastic process that represents credit quality, and default events are associated to barrier crossings. The stochastic process has state dependent volatility and jumps which are estimated by using empirical migration and default rates. A risk-neutralizing drift and forward liquidity spreads are estimated to consistently match the average spread curves corresponding to all the various ratings. The implied migration rates obtained with our credit barrier model are then compared with those obtained via the Jarrow-Lando-Turnbull model and the Kijima-Komoribayashi model in a detailed example.

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1 Introduction

Credit rating agencies such as Moody's and Standard and Poor's identify the credit worthiness of sovereign borrowers and corporations. The rating agencies also provide migration and default rates based on historical data. The pricing of interest rate derivatives such as basket default swaps and collateralized bond obligations require accurate pricing models which are consistent with empirical credit migration and default rates, in order to capture the relevant components of the price of credit risk and to model correlations between credit spreads and default arrival times in a natural way.

This difficult class of models has attracted a great deal of interest in recent derivatives literature. In this article, we introduce new credit barrier models extending those in Iscoe, Kreinin, and Rosen (1999), Hull and White (2001) and Avellaneda and Zhu (2001). These models are based on a notion of *distance to default* related to credit ratings, with default events represented as the first hitting time of a barrier. One can think of the distance to default as a measure of an obligor's leverage relative to the volatility of its asset values. Unlike the precursors, our model attempts to capture aggregate information including migration rates and average spread curves for all ratings within a unified framework.

Pricing models for credit sensitive assets can be divided into at least four main categories: (i) structural models, (ii) credit barrier models, (iii) reduced-form models, (iv) credit-rating based models.

Structural models for credit risk were first proposed by Black and Scholes (1973) and Merton (1974). In its many variants by Kim, Ramaswamy, and Sundaresan (1993), Nielsen, Saa-Requejo, and Santa-Clara (1993), Longstaff and Schwartz (1995) and others, this approach attempts to directly model the borrower's balance sheet. The firm's fixed liabilities constitute a barrier point for the value of its assets. If assets drop below that barrier, the firm is unable to support its debt and default occurs. Estimating these models is a subtle art as extracting a default barrier from accounting statements requires significant judgment to account for complex liability structures, which is generally information that not even public firms are bound to disclose in full detail.

Structural models also have an interesting version for risk management purposes: according to the CreditMetrics-KMV approach, multi-factor Merton models can be estimated in such a way to reflect equity correlations and so that single name marginals are in agreement with expected default frequencies within a given fixed time horizon.

Pricing applications to basket credit derivatives prompt one to go beyond the single period framework. To reach an agreement with credit spread data within an inter-temporal setting while implementing correlations via equity proxies, Hull and White (2001) price credit default swaps using credit barrier models in which the underlying process is a standard Brownian motion and defaults are triggered by the crossing of a moving barrier adapted to a given spread curve. This was an idea that originated in Iscoe, Kreinin, and Rosen (1999). The model was extended in Avellaneda and Zhu (2001). The literature on credit barrier models also includes working papers by Hyer, Lipton, Pugachevsky, and Qui (1999), Gordy and Heitfield (2001), Douady and Jeanblanc (2002) and Iscoe and Kreinin (2002).

Reduced-form models were considered by Duffie and Singleton (1999) and Duffie (1998). Reduced-form models differ fundamentally from structural-form models in the degree of predictability of the default. In these models, an exogenous random variable drives default and the probability of default over any time interval is nonzero. Empirical evidence concerning reduced-form models is rather limited. Using the Duffie and Singleton (1999) framework, Duffee (1999) finds that these models have difficulty in explaining the observed term structure of credit spreads across firms of different qualities. In particular, such models have difficulty accounting for the inter-temporal transitions from relatively flat yield spreads when firms have low credit risk to steeper yield spreads when the risk of default increases.

Credit rating migration rates represent rich aggregate information about the credit quality process under the real world measure. Jarrow, Lando, and Turnbull (1997) introduce a method to risk-neutralize the historical credit migration rate matrix and adapt it to credit spread curves. Kijima and Komoribayashi (1998) propose an alternative risk-neutralization method that we find to be more robust (see the discussion below and in Appendix E). In order to obtain a continuous-time process from a single-period transition matrix, it is often necessary to regularize the empirically observed matrix by methods given by Israel, Rosenthal, and Wei (2001) and Kreinin and Sidelnikova (2001). Moreover, in Kijima, Komoribayashi, and Suzuki (2002) a method to correlate migrations in this model by means of exogenous factors is also proposed.

We find that these interesting models manifest limitations and unrealistic features when used to build continuous time Markov processes because of the discrete nature of credit classes. For instance, credit spreads are constant and jump whenever the credit class changes. Furthermore, the methods to risk-neutralize historical migration matrices are rather ad-hoc. For instance, in Jarrow, Lando, and Turnbull (1997) the factors used to risk-neutralize real world probabilities for upgrades and downgrades are postulated to be the same. Although intuition would suggest ascribing a higher implied probability to downgrades rather than upgrades, there is no natural criterion in this model to capture this effect.

Finally, correlations for the discrete process are not directly relatable to equity correlations due to the discrete nature of the credit quality process, but rather has to be implied by historical correlations of migration data. This data is rather sparse and not readily available. In addition, the correlation structure supported by discrete models is constrained and not entirely flexible. These shortcomings of discrete rating models can be overcome by adopting a continuous underlying to model credit quality. In this case, spreads are not constant if the credit quality variable is in the same rating category and credit quality variables corresponding to different names can be reasonably correlated using equity proxies.

In our credit barrier models, the driving process is identified with the observable credit rating and is estimated consistently to both the migration rates and pricing information on yield spreads. Credit quality is modelled as a continuous variable f_t varying between 0 and 1 which undergoes a jump process with state dependent volatility in continuous time. This variable captures the firm's fundamental information and is directly related to credit ratings, as credit migrations and default events correspond to barrier crossings. Rich statistical information on the process followed by the credit quality under the real-world measure is given by the historical credit migration and default rates. To explain this data we find that it is necessary to allow our explanatory variable to have a volatility dependent on the distance to default, as higher quality ratings are less volatile than lower quality ones. In the context of a simple diffusion model, as already noticed in Gordy and Heitfield (2001), it is not possible to achieve accurate fits for migration rates of retaining the same rating and of changing two or three ratings. To overcome this difficulty we have the option to either include jumps as in Zhou (1997) or a stochastic volatility component. We opt for the former solution since a stationary process under the real-world measure can be estimated in terms of observable quantities. On the other hand, a stochastic volatility model would require marking to market an hidden volatility variable.

To calibrate under the risk-neutral measure, we allow for a time-dependent risk-neutralizing drift. Since our model attempts to capture average spread curves for all credit ratings at once, the estimation requires disentangling spreads due uniquely to the market price of credit risk from the *liquidity* and the *differential taxation* premia. This difficult problem is studied in a recent article by Elton, Gruber, Agrawal, and Mann (2001) and in a working paper by Huang and Huang (2002). The latter authors make use of a variety of different structural models and estimates on the equity risk premium to disentangle the liquidity from the credit premium. In this article we take a different approach which however yields qualitatively similar results. Namely, we make use of the estimates of the tax premia in Elton, Gruber, Agrawal, and Mann (2001) and adjust the risk-neutralizing drift in credit barrier models in such a way that forward liquidity spreads for investment grade bonds are strictly positive and the sum of their squares across ratings and maturities is minimum.

We observe that our model is capable of reproducing the qualitative features of observed term structures of credit spreads across the full range of credit ratings, with curves for high quality ratings being fairly flat and curves for lower quality ratings being inverted and steep. To achieve a quantitatively accurate match with market spread data, a necessary step for pricing purposes, one can adjust the term structure of *forward liquidity spreads* depending on the initial ratings. This allows one to calibrate derivative models and can also be taken as a basis for relative value analysis.

For technical reasons, we ask that our models be integrable in terms of special functions. This condition turns out to be of crucial technical importance for the estimation and calibration aspects of the problem, as accurate expressions for transition probabilities are required for the numerical fitting procedure. Analytical tractability eases the calculation of other important quantities such as first passage times across the credit barriers. It also simplifies tasks such that the generation of Monte Carlo scenarios for possibly correlated credit histories under both the real-world and risk-neutral measures. These tasks are the elementary building blocks required to assemble pricing algorithms for applications ranging from pricing of basket credit derivative to risk assessment for credit exposures.

The present paper expands upon and extends the results of Albanese, Campolieti, Chen, and Zavidonov (2003). More detailed derivations of the main formulae are presented here that could not be included in Albanese, Campolieti, Chen, and Zavidonov (2003). The main contribution of this paper is the introduction of forward liquidity spreads and the presentation of the implied transition probabilities.

The remainder of this paper is organized as follows. Section 2 describes the proposed credit model, Section 3 gives a numerical example using historical data to calibrate the real-world measure and market data to calibrate the risk-neutral measure. Results from this model are compared with the previous models of Jarrow, Lando, and Turnbull (1997) and Kijima and Komoribayashi (1998). Section 4 concludes. Appendices A through E are included to illuminate some of the more technical points in this article.

2 Model specification

Suppose that for a given credit rating system there are K different credit classes, with the first rating being the lowest quality rating, the second being the second lowest, up to the K^{th} rating being the highest quality rating. Notice that this inverts the notation of Jarrow, Lando, and Turnbull (1997).

In our credit barrier model, we refer to the underlying stochastic process f_t as the credit quality process for a defaultable bond. The value of f_t is restricted to the interval $[0, 1]$ with the lower boundary at zero being absorbing and the upper boundary at one being unattainable. The interval $[0, 1]$ is subdivided into K sub-intervals with equal length. That is, define the credit barriers b_i for $i = 0, \ldots, K$ to be the constant values $b_i = i/K$. Then, for $i = 1, \ldots, K$, the i^{th} subinterval $(b_{i-1}, b_i]$ corresponds to the i^{th} credit class in that if f_t is in the sub-interval $(b_{i-1}, b_i]$, then we say that the defaultable bond has a credit rating of i. If f_t is absorbed at the lower boundary, then we say that the bond has defaulted, with the default time occurring at the hitting time of the barrier.

In the following sub-section, the stochastic process which the credit quality variable follows under the real-world measure is described. The stochastic process is characterized by a state-dependent volatility, and an analytic expression for its propagator is given. With the propagator, it is possible to calculate transition and default probabilities. In sub-section 2.2, jumps are added to the stochastic process, which completes the description of the process under the real-world measure. The transformation to the risk-neutral measure is carried out by the addition of a time-dependent drift, as described in sub-section 2.3.

2.1 Diffusion process for the real-world measure

The credit quality variable f_t satisfies a stochastic differential equation of the form:

$$df_t = \mu(f_t)dt + \sigma(f_t)dW_t, \tag{1}$$

where the functional forms of the drift μ and the volatility σ are restricted so that results in Albanese, Camplolieti, Carr, and Lipton (2001) can be utilized to give an explicit formula for the transition probability density function (propagator) $U_f(f,t;f_0)$ with initial state f_0 at time $t=0$. We present a derivation of the formula for $U_f(f,t;f_0)$ in Appendix A. Notice that the transition probability density function is completely specified by the four parameters q_1, q_2, ρ and θ, and the nonlinear transformation between f_t and ϕ_t, as described in Appendix A.

With a formula for the transition probability density function, it is possible to calculate transition probabilities between credit ratings and also default probabilities. Let $p_{ij}(t)$ be the probability of starting at $t=0$ with initial credit rating i and having a final credit rating of j at some time $t>0$. We make a simplification by restricting the credit quality variable to only take on the values $f_i = \frac{2i-1}{2K}$ for $i=1,\ldots,K$. Notice that f_i is the mid-point of the sub-interval $(b_i, b_{i+1}]$. This is in some sense an average of possible initial credit quality values within the given credit class. Then, the expression for $p_{ij}(t)$ is merely an integral over the final credit rating of the transition probability density function. Explicitly, we have:

$$p_{ij}(t) \quad = \quad \int_{b_{j-1}}^{b_j} U_f(f,t;f_i)df. \tag{2}$$

Also, the probability of defaulting by time t conditioned to starting with an initial credit rating i is given by the survival probability subtracted from unity:

$$p_i^D(t) = 1 - \int_0^1 U_f(f,t;f_i)df. \tag{3}$$

An expression that eases the numerical burden of calculating the default probabilities is given in Appendix B.

2.2 Adding jumps

With the credit quality variable f_t specified as a diffusion process as in the previous sub-section, we found that it was not possible to have a good match with empirical transition probabilities. Specifically, for all initial credit ratings, we could not simultaneously match the probability of retaining the same credit rating and the probabilities of changing two or three credit ratings. The low kurtosis of the distribution is the same problem as was encountered by Gordy and Heitfield (2001) with normal distributions.

Under the present model, there are two possible solutions to the problem of low kurtosis. One is by allowing jumps in the stochastic process and the other is in allowing the state-dependent volatility to be stochastic as well. The disadvantage in having a stochastic volatility is that the initial volatility would have to be estimated from the available data, which is a difficult task given the infrequency of rating transitions. Thus, we choose the first alternative and introduce jumps in the stochastic process.

Jumps are introduced via a stochastic time change similar to the variance-gamma model of Madan, Carr, and Chang (1998). While Madan, Carr, and Chang (1998) specialize to the case of geometric Brownian motion, the variance-gamma model generalizes in a natural way. We obtain a process $\tilde{f}_t(\nu)$ with jumps by evaluating the credit quality process f_t at a random time given by a gamma process $\gamma(t,1,\nu)$ with unit mean and variance rate ν. That is:

$$\tilde{f}_t(\nu) \quad = \quad f_{\gamma(t,1,\nu)} \tag{4}$$

There is an intuitive financial interpretation to this stochastic time change. The gamma process $\gamma(t,1,\nu)$ can be thought of as a mapping from calendar time t to financial time $\tau = \gamma(t,1,\nu)$, which measures financial activity up to calendar time t. The gamma process is not continuous, but instead advances in jumps of various sizes as t increases. Thus, jumps have the interpretation of rapid advances in financial activity with respect to calendar time. While this explanation of the stochastic time change is intuitive, we need not directly analyze the rate financial activity in order to estimate the jump parameter ν—only transition probabilities need to be considered.

In our model, default occurs if the process f_τ hits the barrier at zero, possibly during an unobservable excursion during a jump in financial time. Thus, the transition probability density function $\tilde{U}_f(f,t;f_0)$ is obtained by integrating the probability kernel in financial time against the gamma density function. That is:

$$\tilde{U}_f(f,t;f_0) = \int_0^\infty U_f(f,s;f_0)\tilde{\Gamma}(s,t)ds, \tag{5}$$

with the gamma density function defined as:

$$\tilde{\Gamma}(s,t) = \frac{s^{t/\nu-1}e^{-s/\nu}}{\Gamma(t/\nu)\nu^{t/\nu}}. \tag{6}$$

Here, $\Gamma(x)$ is the usual Gamma function. With the expression for the propagator $\tilde{U}_f(f,t;f_0)$, it is possible to obtain the transition probabilities and default probabilities under the stochastic time change with equations analogous to (2) and (3). The transition probabilities under the stochastic time change become:

$$\tilde{p}_{ij}(t) = \int_{b_{j-1}}^{b_j} \tilde{U}_f(f,t;f_0)df, \tag{7}$$

and the default probabilities become:

$$\tilde{p}_{ij}^D(t) = 1 - \int_0^1 \tilde{U}_f(f,t;f_0)df. \tag{8}$$

Again, an expression that makes the calculation for the default probabilities less numerically intensive is given in Appendix B.

Then, with this model it is possible to fit historical data with a high degree of accuracy. Thus, problems in risk management can be dealt with in this real-world framework.

2.3 Risk-neutralizing drift

In order to deal with derivative pricing problems, it is necessary to bring the model into a risk-neutral setting. We do this by the addition of a time-dependent drift on the credit quality process. Note that we add the drift term prior to the stochastic time change, and retain the same jump rate in both the real-world and risk-neutral measures. We opted against having a market price for jump risk so that other data such as derivative prices would not have to be used to calibrate the parameters.

The addition of a drift term can be described in terms of a time-dependent non-linear transformation with the functional form $g_t = G(f_t,t)$. By Itô's Lemma, the stochastic differential equation satisfied by the process g_t is:

$$dg_t = \left[G'(f_t,t)\mu(f_t) + \dot{G}(f_t,t) + \frac{1}{2}G''(f_t,t)\sigma^2(f_t)\right]dt + G'(f_t,t)\sigma(f_t)dW_t. \tag{9}$$

In order to apply the Fundamental Theorem of Finance, it is required that this transformation leave the volatility invariant. This leads to the condition:

$$\sigma(g) = G'(f,t)\sigma(f). \tag{10}$$

Rearranging and integrating, we obtain:

$$\int_0^{G(f,t)} \frac{d\xi}{\sigma(\xi)} = \int_0^f \frac{d\xi}{\sigma(\xi)} + a(t). \tag{11}$$

for some function of time $a(t)$. In Appendix C we show that the the transformation $G(f,t)$ is completely specified by the choice of $a(t)$. In addition, the pricing kernel $U_g(g,t;g_0)$ can be calculated by solving a boundary value problem, where the location of the boundary is dependent on the function $a(t)$. Thus, default occurs whenever the non-zero barrier is hit, and the risk-neutralizing drift is calibrated by adjusting the location of the barrier so that the default probabilities match those implied by the market.

The stochastic time change with the same variance rate as with the real-world measure is used so that the time transformed process $\tilde{g}_t(\nu)$ is given by:

$$\tilde{g}_t \quad = \quad g_{\gamma(t,1,\nu)} \tag{12}$$

As in the real-world measure, we define defaults to occur whenever the barrier is hit, even during an unobservable excursion. Then, the pricing kernel is given by:

$$\tilde{U}_g(g,t;g_0) \quad = \quad \int_0^\infty U_g(g,s;g_0)\tilde{\Gamma}(t,s)ds. \tag{13}$$

Thus, as with the real-world transition probabilities in (7) and (8), the risk-neutral transition probabilities are:

$$\tilde{q}_{ij}(t) \quad = \quad \int_{G(b_{j-1},0)}^{G(b_j,0)} \tilde{U}_g\left(g,t;G(f_i,0)\right)dg, \tag{14}$$

and the risk-neutral default probabilities are:

$$\tilde{q}_i^D(t) \quad = \quad 1 - \int_0^{G(1,0)} \tilde{U}_g\left(g,t;G(f_i,0)\right)dg. \tag{15}$$

3 Numerical example

In this section, we show that our model can be calibrated such that the fit is accurate to both the historical data in the real-world measure and the market data in the risk-neutral measure. Having calibrated the model, we produce risk-neutral transition probabilities and compare them qualitatively with the risk-neutral transition probabilities produced from the methods of Jarrow, Lando, and Turnbull (1997) and Kijima and Komoribayashi (1998).

3.1 Calibrating under the real-world measure

In the real-world measure, the calibration problem is to match the transition probabilities given in Eq. (7) at one year with the historical transition probabilities, and to match the default probabilities given in Eq. (8) at one, three and five years with the historical default probabilities.

For the historical data, we used the full one year transition probability matrix taken from Carty (1997). The matrix as provided contains an additional category labelled “Withdrawn Rating”. This category includes all issues that were rated at the beginning of the given time period but not rated at the end of the observation period, whether it was because the debt was retired, due to a lack of information available to Moody’s or any number of other reasons. If the debt was retired, then a withdrawn rating has no implication on the credit risk. On the other hand, if the rating was withdrawn due to a lack of information, this could be the result of a degradation in credit quality.

Since the withdrawn rating gives no clear indication about the credit risk of the issuer, we adjusted the transition matrix to be conditional upon the issuer not having their rating withdrawn within the observation period. This data is presented in Table 1.

For historical default probabilities, we used one, three and five year time periods. The additional time horizons were used for the default probabilities due to the relative importance of the default probabilities in credit risk models. The three and five year transition matrices were not used because the data was too noisy for the low probability events since the averaging was taken over a shorter time period (over 1983-1994 and 1983-1992 respectively for the three and five year transition matrices, as opposed to 1983-1996 for the one year transition matrix). As with the transition probabilities, the data was taken from Carty (1997) and is withdrawn rating adjusted.

Under the real-world measure, the model was calibrated to minimize the the difference between the components $\tilde{p}_{ij}(t)$ and $\tilde{p}_i^D(t)$ of the model and the corresponding components from historical data. In order to fully specify the model, it is necessary to set the free parameters q_1, q_2, θ, ν, and ρ. In addition, while the f-space where the credit levels and barriers were evenly spaced was convenient for expository reasons, it was found that in this space the model could not be calibrated accurately with the volatility functions that were used. Thus, it was necessary to allow the credit barriers and initial credit quality values to be free parameters. In the space of ϕ, we call the credit barriers $c_0, c_1, \ldots, c_K$ and the initial credit quality values $\phi_1, \ldots, \phi_K$.

Thus, there are a total of $2K + 4$ free parameters (since c_0 and c_K are fixed). While this seems a formidable numerical task, significant simplifications can be made. As an overview, given a set of parameters q_1, q_2, θ, ν, and ρ, one can obtain estimates on the initial credit quality values for lower rated firms using the default probabilities. From these, one can obtain estimates on the credit barriers for the lower rated firms using the diagonal elements of the transition probability matrix. Estimates for the credit barriers and credit quality values of higher rated firms are given in turn by using the probabilities of retaining the same rating and the probabilities of downgrades, respectively. Thus, given q_1, q_2, θ, ν, and ρ we can estimate all of the credit barriers and initial credit quality values. We now describe this algorithm in detail.

Given a set of parameters q_1, q_2, θ, ν, and ρ, default probabilities can be efficiently calculated by use of Eq. (B.6). So, for the lower credit ratings where the probability of default is significant, approximations for the initial credit levels ϕ_i can be made by matching the calculated model default probabilities with the historical default probabilities.

Suppose that the lowest i_0 credit levels can be determined in this way. With an approximation of ϕ_1 made for this parameter set, it is possible to make an approximation on c_1, since we only need c_1 and ϕ_1 in order to determine $\tilde{p}_{1,1}$. Likewise, for $i \leq i_0$, we need only c_{i-1} and ϕ_i to determine $\tilde{p}_{i,i}$. The diagonal elements (corresponding to the probability that the credit ratings do not change) are used because they tend to be the largest elements of the transition matrix.

To this point, we have approximations on $\phi_1, \ldots, \phi_{i_0}$ and $c_1, \ldots, c_{i_0}$. Given ϕ_{i_0+1}, we can calculate $\tilde{p}_{i_0+1,j}$ for $j \leq i_0$. Thus, an approximation of ϕ_{i_0+1} can be made by matching these model transition probabilities with the historical probabilities. In turn, c_{i_0+1} can be approximated as before, by matching the diagonal element $\tilde{p}_{i_0+1,i_0+1}$. Thus, we can iterate this process until all of the credit levels and barriers are estimated for this set of parameters.

Thus, given a set of parameters q_1, q_2, θ, ν, and ρ, we can obtain a first approximation on the credit levels and barriers. After determining a reasonable set of parameters, the approximations on the credit levels and barriers can be further refined to better fit the historical data.

The results of the fit with the transition probabilities for a selection of initial credit ratings is given in Fig. 1. The fit with the default probabilities for all initial credit ratings and for one, three and five years is given in Fig. 2. It is seen that the match is quite good across all ratings and for all time horizons in the case of the default probabilities. The state-dependent volatility needed to achieve this fit is shown in Fig. 3. The local volatility pictured is a function f, the space in which the credit ratings are evenly spaced. The main feature to notice about the volatility function is that it is greater for lower quality credit ratings, as we would expect.

It is interesting to note that the variance rate ν that is estimated from the historical data has a value of 8.1 years. Relative to the time horizons involved, this is a large value for the variance rate. This implies that large jump amplitudes are required to model the observed transition probabilities. And since the jump amplitudes are affected by the state-dependent volatility we see that larger jumps generally occur in the lower quality ratings.

3.2 Calibrating the drift in the risk-neutral measure

After having calibrated the real-world model, it is necessary to calibrate the risk-neutralizing drift so as to match market prices of bonds. We use aggregate data for bond prices as opposed to individual issue prices. The Bridge Evaluator corporate spreads for Industrials are available from http://www.bondsonline.com. Here, yield spreads above the U.S. treasury rate are given for each alpha-numeric credit rating. The data used in this numerical example was for February 10, 2003. The corporate yield spreads are given in Table 3 and the Treasury rates are given in Table 4.

As explained in Appendix C, the risk-neutralizing drift is determined by the specification of a default boundary in x-space. We parameterized the default boundaries by arctangent functions of the form:

$$b(t) \quad = \quad A\left[\arctan\left(\frac{t-B}{C}\right) - \arctan\left(-\frac{B}{C}\right)\right] \tag{16}$$

This form was used in order that $b(t) > 0$ for $t > 0$ and $b(0) = 0$. A positive barrier for positive time was required to increase the probabilities of default. With this parameterization, there is great flexibility in the effect of the boundary on default probabilities, and thus credit spreads.

Given a boundary, the drift and the transformation $G(f, t)$ is determined. In turn, the risk-neutral default probabilities $\tilde{q}_i^D(t)$ can be calculated by Eq. (15). From this, if we take the simple case of constant interest rates, then we can derive the term structure of credit spread rates $s_i(t)$ for i-rated bonds by:

$$s_i(t) \quad = \quad -\frac{\ln[1 - \tilde{q}_i^D(t)(1 - R_i)]}{t}. \tag{17}$$

Here, R_i is the recovery rate of bonds starting with an initial rating of i. For the numerical example, we use historical recovery rates as given in Altman and Kishore (1998) and displayed here in Table 5. As an alternative, one could also use implied recovery rates obtained by comparing swap rates for single name default swaps with upfront prices of out-of-equilibrium default swaps: this venue is reviewed in a forthcoming paper by the authors.

The credit spread is the portion of the quoted yield spread that is due solely to credit risk, with the remaining portion of the yield due to other factors such as differential taxation between government and corporate bonds, and liquidity effects. It is found in Huang and Huang (2002) that the credit spread tends to be a relatively small fraction of the yield spread for investment grade bonds, while it is a larger fraction for low quality bonds. As such, our risk-neutralization was calibrated in order that this empirical observation holds true in the model. More specifically, we asked that the forward liquidity spread be strictly positive for investment grade bonds and that the sum of their squares for all maturities be minimum. Forward liquidity spreads are plotted against credit quality in Fig. 4 while their term structures are given in Fig. 5 The comparison between yield and credit spreads for the calibration are shown in Fig. 6. These are calculated with a default boundary that is shown in Fig. 7.

In order to be able to directly compare the spreads quoted in Table 3 with the spreads derived from the model, it is necessary to adjust the quoted spreads for taxation effects. For the US corporate bond market, Elton, Gruber, Agrawal, and Mann (2001) find that the mid-point of effective state tax rates is 4.875%. Assuming zero-coupon bonds, the tax-adjusted spread rate is then:

$$s_i(t) - 4.875\%[s_i(t) + r(t)] \tag{18}$$

where $r(t)$ is the treasury yield given in Table 4.

Our goodness of fit criterion is formulated in terms of forward liquidity spreads. The forward liquidity spread $c(T, T+\tau, i)$ for the period $[T, T+\tau]$ and the average spread curve of rating i, computed with simple compounding over the period τ, is defined so that

$$f^{\text{mkt}}(T, T+\tau, i)) = (f^{\text{mdl}}(T, T+\tau, i) + c(T, T+\tau, i))) \tag{19}$$

where $f^{\text{mkt}}(T, T+\tau, i)$ and $f^{\text{mdl}}(T, T+\tau, i)$ are the model and market forward rates, respectively. The estimates are gauged in such a way that the sum of the squares of $c(T, T+\tau, i)$ over all investment grade ratings and over maturities of 0, 1, 2, 3, 5 and 7 years, is minimum. The values for the period τ are 1, 1, 1, 2, 2 and 3 years, respectively.

Forward liquidity spreads are plotted against credit quality in Fig. 4 while their term structures are given in Fig. 5. One can notice that investment grade bonds have consistently low forward liquidity yield spreads, at most 40 to 60 basis points in our example. High yield bonds instead have much higher forward liquidity spreads. Notice in particular the BB yield, which is positive and worth more than 250 basis points for short maturities. This large liquidity spread can be justified by the trading restriction that most investors have which forces them to sell off (at a discount) investment grade bonds once they are downgraded to a BB rating. Lower quality high yield bonds instead have instead a negative liquidity spread, signaling the fact that these asset are sought after because of their speculatively high returns. Longer maturity CCC bonds instead sell at a heavy discount.

All these features could not realistically be accounted for by a different choice of risk-neutralizing drift. Although we considered only a restricted class of drift functions, we can hardly imagine a specification for the drift which is large and negative just below the investment grade barrier and then becomes large and positive for higher yield credit qualities: far from reproducing the observed liquidity effect, these conflicting properties would decrease accuracy. We are led to conclude that no specification of the risk-neutral measure is consistent with observed yield spreads. A consistent pricing framework based on the pricing measures here constructed will have to take into account liquidity spreads separately through an ad-hoc mechanism such as the following: one could price a derivative such as a basket credit default swap neglecting liquidity spreads, find hedge ratios and then find liquidity corrections to the derivative price by pricing the actual cost of the hedging strategy. We plan to discuss such pricing frameworks in a forthcoming paper.

3.3 Risk-neutral transition probabilities

Having calibrated the model for the risk-neutral measure, it is possible to calculate risk-neutral transition probabilities by use of Eq. (14). The ratio of the model risk-neutral transition probabilities to the model real-world transition probabilities are plotted in Figs. 10 and 11 for one and five year horizons, respectively.

We now compare these results to the risk-neutral transition probabilities of Jarrow, Lando, and Turnbull (1997) and Kijima and Komoribayashi (1998) (hereafter JLT and KK, respectively). The procedure for calibrating both of these models is given in Appendix E, and the data used was the same as was used to calibrate the credit barrier model—namely, Tables 1, 2, 3 and 4. Only the one year default probabilities were used from Table 2 and a constant recovery rate of 40% was used. The ratio of the model risk-neutral transition probabilities to the model real-world transition probabilities for the JLT model are plotted in Figs. 12 and 13, for time-horizons of one and five years respectively, and the corresponding graphs for the KK model are shown in Figs. 14 and 15. Note that the ratios were plotted only for transitions where the real-world transition probability was greater than 0.5%. None of the ratios are actually zero, with the exception of the Ba2 transition to itself (marked with an asterisk) in Fig. 12 for the JLT model.

It was found that the JLT scheme was very unstable. Even for the first period the risk premia could not be specified so that the bonds were priced exactly. With the minimization scheme, errors magnified with each iteration until 5 years, when the Caa bonds were mispriced by about 10% while the other lower grade bonds had about 3-5% error. The higher grade bonds continued to be priced correctly.

The KK scheme was much more stable. the $\ell_i(t)$ could be specified so that bonds were exactly priced up until a maturity of 4 years. However, in order to calculate $\ell_j(4)$ (to get $\tilde{\mathbf{Q}}(0,5)$), we had to use the minimization routine. This resulted in a 4.3% error in the Caa bond with a 5 year maturity, and $< .5\%$ for the other grades.

Both the JLT and KK schemes suffer the drawback that the restriction of the risk premia to be independent of the final state is not economically intuitive. Since risk-neutral default probabilities are greater than real-world default probabilities, we would expect risk-neutral downgrade probabilities to be greater than real-world downgrade probabilities (ie. $\tilde{q}_{i,j} \geq q_{i,j},\ j < i$), and risk-neutral upgrade probabilities to be less than real-world upgrade probabilities (ie. $\tilde{q}_{i,j} \leq q_{i,j},\ j > i$). In the credit barrier model, this is what occurs. This is in contrast to both the JLT and KK models. For the one year horizon, the JLT and KK model yield constant adjustments across ratings (except for staying in the same rating for JLT, and defaulting for KK). The JLT model has the further defect that the risk-neutral probability of Ba2 bonds staying in the same rating is zero.

At a maturity of 5 years, the JLT model displays disorderly behavior, with the ratio of risk-neutral to real-world transition probabilities not even monotonic for some of the ratings. The KK model still yields relatively constant ratios, with the exception of Caa bonds.

4 Conclusion

A credit model is specified in which for a given defaultable bond, a stochastic credit quality process is an indicator of credit migrations and defaults in a real-world measure. The process has a state-dependent volatility and jumps are introduced by a stochastic time change. The event of hitting the absorbing lower boundary corresponds to a default event. A time-dependent drift is applied to the credit quality process in order to change to a risk-neutral measure.

Empirical credit migration probabilities and default probabilities are reproduced with good accuracy in the real-world measure. The risk-neutralizing drift so that the model also agrees with market spread curve data.

The analytic integrability of the model is essential in order that it can be calibrated with precision. Knowing the probability kernels, Monte Carlo algorithms can be made efficient, with possible application to correlated credit processes. In addition to uses in credit risk management, instruments such as basket default swaps can be priced.

Appendix A. Diffusion process

This appendix describes in detail the diffusion process followed by the credit quality variable f_t in financial time.

The problem of calibration was found to have greater tractability if we removed the restriction that the credit barrier levels be evenly spaced. After calibrating the model in which the barrier levels and initial ratings are free parameters, we can return to the model in which the barrier levels and initial ratings are evenly spaced by applying a (non-unique) monotonic nonlinear transformation.

Following Albanese and Campolieti (2001), we restrict ourselves to diffusion process which are integrable by reduction by changes of variable and of measure to an underlying x_t which follows a special case of a CIR process:

$$dx_t = (2\theta + 2)dt + 2\sqrt{x_t}dW_t \tag{A.1}$$

Under the transformation

$$\Phi(x) \quad = \quad \frac{2\sigma_0 I_\theta(\sqrt{2\rho x})}{q_2 \hat{u}(\sqrt{2\rho x})} \tag{A.2}$$

where $\hat{u}(x) = q_1 I_\theta(x) + q_2 K_\theta(x)$, I_θ and K_θ the modified Bessel functions, and with numeraire:

$$\frac{e^{\rho t} x^{\theta/2}}{\hat{u}(\sqrt{2\rho x})}, \tag{A.3}$$

the process $\phi_t = \Phi(x_t)$ is driftless and satisfies the stochastic differential equation:

$$d\phi_t \quad = \quad \sigma(\phi_t) dW_t. \tag{A.4}$$

Here, the state-dependent volatility function is given by:

$$\sigma(\Phi(x)) \quad = \quad \frac{2\sigma_0}{\sqrt{x}\hat{u}^2(\sqrt{2\rho x})}. \tag{A.5}$$

From Eq. (A.4), we see that the transition probability density function, $U_\phi(\phi, t; \phi_0)$, in the space of the variable ϕ solves the forward-time Kolmogorov equation:

$$\frac{\partial U_\phi}{\partial t} \quad = \quad \frac{1}{2}\frac{\partial^2}{\partial \phi^2}(\sigma(\phi)^2 U_\phi). \tag{A.6}$$

From Eq. (A.2) notice that the function $\Phi(x)$ is monotonic with positive derivative given by:

$$\frac{d\Phi(x)}{dx} \quad = \quad \frac{\sigma_0}{x\hat{u}^2(\sqrt{2\rho x})} \tag{A.7}$$

so that we can define a function X to be the inverse transformation of Φ. That is,

$$x = X(\phi) = \Phi^{-1}(\phi) \tag{A.8}$$

It is shown in Appendix D that the pricing kernel for ϕ_t is related to the pricing kernel $U_x(x, t; x_0)$ for x_t (with initial value x_0) by:

$$U_\phi(\Phi(x), t; \Phi(x_0)) \quad = \quad e^{-\rho t}\frac{x}{\sigma_0}\left(\frac{x_0}{x}\right)^{\theta/2}\frac{\hat{u}^3(\sqrt{2\rho x})}{\hat{u}(\sqrt{2\rho x_0})} U_x(x, t; x_0) \tag{A.9}$$

This is a general formula that holds for generic boundary conditions that specify $U_x(x, t; x_0)$. In the case of our model, in the real-world measure there is an absorbing boundary at $x = 0$ corresponding to defaults. In this case we can solve Eq. (A.1) in analytically closed form to obtain:

$$U_x(x, t; x_0) \quad = \quad \frac{e^{-\frac{x+x_0}{2t}}}{2t}\left(\frac{x}{x_0}\right)^{\theta/2} I_\theta\left(\frac{\sqrt{x x_0}}{t}\right) \tag{A.10}$$

Thus, in ϕ-space the transition probability distribution is given by:

$$U_\phi(\Phi(x), t; \Phi(x_0)) \quad = \quad e^{-\rho t - \frac{x+x_0}{2t}}\frac{x}{2\sigma_0 t}\frac{\hat{u}^3(\sqrt{2\rho x})}{\hat{u}(\sqrt{2\rho x_0})} I_\theta\left(\frac{\sqrt{x x_0}}{t}\right). \tag{A.11}$$

Appendix B. Default probabilities

The default probabilities $p_i^D(t)$ are a crucial component of the transition matrix, and in this appendix we give an alternate means to calculating them without needing to numerically integrate the propagator as in (3) and (8). Also, the effect on the default probabilities of changing the various parameters is investigated. The equivalent of (3) in ϕ-space is:

$$p_i^D(t) \quad = \quad 1 - \int_0^{\phi_{max}} U_\phi(\phi, t; \phi_i) d\phi, \tag{B.1}$$

where ϕ_{max} is the unattainable upper boundary (corresponding to $x \to \infty$ and $f = 1$). Differentiating this respect to t and substituting (A.6), we see that

$$\frac{dp_i^D(t)}{dt} \quad = \quad \frac{1}{2}\frac{\partial}{\partial \phi}\left(\sigma(\phi)^2 U_\phi\right)\Big|_{\phi=0} \tag{B.2}$$

since the upper boundary in unattainable, giving a zero density boundary condition. Substituting the expression for $U_\phi(\phi, t; \phi_i)$ given in (A.11), using an integral expression for K_θ and a series expansion for I_θ (see 8.432 9 and 8.445, respectively, of Gradshteyn and Ryzhik (2000)), (B.2) is equal to:

$$\frac{q_2 e^{-\rho t - x_i/2t}}{2\hat{u}(\sqrt{2\rho x_i}) t^{\theta+1}} \left(\frac{x_i}{2\rho}\right)^{\theta/2}. \tag{B.3}$$

By integrating this expression we find a simple integral representation for $p_i^D(t)$:

$$p_i^D(t) = \frac{q_2}{2\hat{u}(\sqrt{2\rho x_i})} \left(\frac{x_i}{2\rho}\right)^{\theta/2} \int_0^t \frac{e^{-\rho s - \frac{x_i}{2s}}}{s^{1+\theta}} ds. \tag{B.4}$$

From this result one can see that if $q_2 = 0$ the probability of default will be zero and the probability density function U_ϕ integrates to unity for any time $t \geq 0$.

Let us investigate the long-time limit $t \to \infty$ for $p_i^D(t)$. In this limit, the integral in (B.4) tends to

$$p_i^D(\infty) \quad = \quad \frac{q_2 K_\theta(x_i)}{q_1 I_\theta(x_i) + q_2 K_\theta(x_i)} \tag{B.5}$$

by use of 8.432 6 in Gradshteyn and Ryzhik (2000). We see that unless $q_1 = 0$, the credit quality variable will not attain the default state with probability one, even at arbitrarily large time. On the other hand, if $q_1 = 0$, by using Eqs. (A.7) and (A.11) in the integral expressions for the various probabilities the transition and default probabilities are independent of q_2. Thus, for $q_1 = 0$ the transition and default probabilities depend only on the parameters ρ and θ.

Incorporating the stochastic time change introduced in sub-section 2.2, the default probabilities are calculated by integrating the default probabilities $p_i^D(t)$ in financial time against the distribution $\tilde{\Gamma}(s, t)$ defined in (6) to yield

$$\tilde{p}_i^D(t) \quad = \quad \frac{q_2}{2\hat{u}(\sqrt{2\rho x_i})} \left(\frac{x_i}{2\rho}\right)^{\theta/2} \int_0^\infty \Gamma\left(\frac{z}{\nu}, \frac{t}{\nu}\right) \frac{e^{-z-\frac{x_i}{2z}}}{z^{1+\theta}} dz, \tag{B.6}$$

where $\Gamma(x, a)$ is the incomplete Gamma function:

$$\Gamma(x, a) \quad = \quad \frac{1}{\Gamma(a)} \int_x^\infty y^{a-1} e^{-y} dy. \tag{B.7}$$

If the variance rate ν approaches zero so that there are no jumps, the incomplete Gamma function $\Gamma(z/\rho\nu, t/\nu)$ approaches the Heavyside step function $\theta(\rho t - z)$ and we get Eq. (B.4), as expected.

Appendix C. Barriers and Drifts

Here, we describe the pricing kernel under the risk-neutral measure.

Let $b(t)$ be a positive function in x-space where we impose a Dirichlet boundary condition. That is, the pricing kernel $U_x(x,t;x_0)$ in x-space satisfies the boundary conditions

$$\begin{aligned} U_x(x,0;x_0) &= \delta(x-x_0) \\ U_x(b(t),t;x_0) &= 0 \\ \lim_{x\to\infty} U_x(x,t;x_0) &= 0 \end{aligned} \tag{C.1}$$

and within this domain is a solution to the forward-time Kolmogorov equation:

$$\frac{1}{2}\frac{\partial U_x}{\partial t} = x\frac{\partial^2 U_x}{\partial x^2} + (1-\theta)\frac{\partial U_x}{\partial x} \tag{C.2}$$

If $b(t)$ is uniformly zero, the solution formula for U_x is given in Eq. (A.10). For non-zero boundaries $b(t)$, the solution must be found numerically.

In order to obtain an explicit formula for the transformation G, it is convenient to apply the iso-volatility transformation in Eq. (11) to the process ϕ_t. In this coordinate, using Eqs. (A.5) and (A.7) we have:

$$\begin{aligned} \int_0^c \frac{d\xi}{\sigma(\xi)} &= \int_0^c \frac{\sqrt{X(\xi)}\ \hat{u}^2\left(\sqrt{2\rho X(\xi)}\right)}{2\sigma_0} d\xi \\ &= \int_0^{X(c)} \frac{d\Phi(x)}{dx}\frac{\sqrt{x}\ \hat{u}^2\left(\sqrt{2\rho x}\right)}{2\sigma_0} dx \\ &= \int_0^{X(c)} \frac{1}{2\sqrt{x}} dx \\ &= \sqrt{X(c)}. \end{aligned} \tag{C.3}$$

With (C.3) we can derive from Eq. (11) that

$$G(\phi,t) = \Phi\left[\left(\sqrt{X(\phi)} + a(t)\right)^2\right]. \tag{C.4}$$

It is possible to change variables to map the time-dependent boundary $b(t)$ in x-space to 0 after the transformation G is applied. By Eq. (C.4), this is done by setting

$$a(t) = -\sqrt{b(t)} \tag{C.5}$$

Let $\Phi_g(g,t)$ be the function such that $G(\Phi_g(g,t),t) = g$ for all times t. Its derivative is found by Eq. (10) and we can obtain an expression for the pricing kernel $U_g(g,t;g_0)$ simply by a change of variables so that

$$U_g(G(\Phi(x),t),t;G(\Phi(x_0),t)) = U_\phi(\Phi(x),t;\Phi(x_0))\frac{\sigma(\Phi(x))}{\sigma\left(G(\Phi(x),t)\right)} \tag{C.6}$$

We make the assumption that we can expand U_x in powers of $\bar{x} = x - b(t)$ at the boundary $b(t)$:

$$U_x(x,t;x_0) = \sum_{m=1} \alpha_m(x_0,t;[b(\cdot)])\bar{x}^m, \tag{C.7}$$

where $\{\alpha_m\}_{m=1}$ is a sequence of functions dependent on x_0 and t and functionally dependent on $b(\cdot)$. Notice that the constant term is zero since U_x vanishes at the boundary. Then it can be shown that the probability of default is given by:

$$p^D(t) \quad = \quad \rho \int_0^t \alpha_1(x_0, s; [b(\cdot)])ds \tag{C.8}$$

Here, the $\alpha_1(x_0, s; [b(\cdot)])$ term is just the gradient of U_x at the boundary. In the case $b(t) = 0$, this is seen to reduce to the formula for the default probability given in (B.4), as expected.

Appendix D. Exact solutions

In this appendix we derive the formula in (A.9) in Appendix A. An intuitive financial derivation of this formula follows from the Fundamental Theorem of Finance upon properly interpreting the processes.

Consider a multi-currency financial model where domestic interest rates are zero, the process x_t in (A.1) is interpreted as a price process for an asset denominated in a foreign currency and $\phi_t = \Phi(x_t)$ is the price process for a contingent claim (a quanto option) in the domestic currency. Assume that under the risk-neutral measure where ϕ_t is driftless, the underlying foreign price process x_t obeys the equation

$$dx_t \quad = \quad \mu(x_t)dt + \nu(x_t)dW_t \tag{D.1}$$

for some drift function $\mu(x)$. Assume also that the volatility $\nu(x_t)$ is such that, for some choice of the drift function $\lambda(x)$, one can solve the stochastic differential equation

$$dx_t \quad = \quad \lambda(x_t)dt + \nu(x_t)dW_t \tag{D.2}$$

in analytically closed form. By solving, we mean that it is possible to find the *pricing kernel* $u(x, t; x_0)$, which can be interpreted as the price of a butterfly spread option of maturity t for which the area under the payoff function is one and in the limit as the spread goes to zero around the point x, where x_0 is the spot price of the underlying foreign asset.

Our objective is to show that if the volatility function for the quanto option $\phi_t = \Phi(x_t)$ is defined as:

$$\sigma(\Phi(x)) \quad = \quad \frac{\sigma_0 \nu(x) \exp\left(-2\int^x \frac{\lambda(y)}{\nu(y)^2} dy\right)}{\hat{v}(x,\rho)^2}, \tag{D.3}$$

then it is possible to find the pricing kernel for the quanto option ϕ_t in analytically closed form. Here, ρ is a real valued parameter and the function $\hat{v}(x)$ is defined as the solution of the equation:

$$\frac{\nu^2(x)}{2}\hat{v}_{xx}(x,\rho) \quad = \quad \rho\hat{v}(x,\rho) - \lambda\hat{v}_x(x,\rho). \tag{D.4}$$

Notice that this equation can be interpreted as the Laplace transform of the Black-Scholes equation with drift $\lambda(x)$. Finally, the function $X(\phi)$ in (D.3) and its inverse $\Phi(x)$ are defined as the solutions of the equation

$$\frac{dX(\phi)}{d\phi} \quad = \quad \frac{\nu(x)}{\sigma(\phi)}. \tag{D.5}$$

The key mathematical derivation involves a change of numeraire asset given by a process γ_t defined as follows:

$$\gamma_t \quad = \quad \frac{e^{\rho t}}{\hat{v}(x_t, \rho)} \tag{D.6}$$

and satisfies the equation

$$d\gamma \quad = \quad \left(\rho - \mu\frac{\hat{v}_x}{\hat{v}} + \nu^2\left[\left(\frac{\hat{v}_x}{\hat{v}}\right)^2 - \frac{1}{2}\frac{\hat{v}_{xx}}{\hat{v}}\right]\right)\gamma dt + \sigma^\gamma\gamma dW, \tag{D.7}$$

where the log-normal volatility of γ_t is given by

$$\sigma^\gamma \quad = \quad -\frac{\hat{v}_x\nu}{\hat{v}}. \tag{D.8}$$

Substituting Eq. (D.4), we find that

$$\frac{d\gamma}{\gamma} \quad = \quad \left(\frac{\mu-\lambda}{\nu}\sigma^\gamma + (\sigma^\gamma)^2\right)dt + \sigma^\gamma dW. \tag{D.9}$$

To demonstrate that γ_t defines a domestic asset price process, consider this equation in the original risk-neutral measure where the domestic quanto option price process ϕ_t is driftless. In this case, using Ito's Lemma on the inverse mapping $x_t = X(\phi_t)$, we arrive at a stochastic differential equation of the form in Eq. (D.1) with drift given by

$$\mu(x) \quad = \quad \frac{\sigma(\Phi)^2}{2}\frac{d}{d\Phi}\frac{dX(\Phi)}{d\Phi} \quad = \quad \frac{\sigma(\Phi)^2}{2}\frac{d}{d\Phi}\frac{\nu(x)}{\sigma(\Phi)}. \tag{D.10}$$

where $\Phi = \Phi(x)$. Using the chain rule for differentiation, and expressing all functions in terms of x, we then have

$$\mu(x) \quad = \quad \frac{\sigma\nu}{2}\frac{d}{dx}\left(\frac{\nu}{\sigma}\right) \quad = \quad \frac{\nu}{2}\left[\nu_x - \frac{\nu}{\sigma}\sigma_x\right], \tag{D.11}$$

where $\sigma \equiv \sigma(\Phi(x))$ is the volatility function for the quanto option of price ϕ_t. Hence, by substituting into the expression for the risk-neutral drift of γ_t in (D.9) we find

$$\frac{\mu-\lambda}{\nu}\sigma^g + (\sigma^g)^2 \quad = \quad \left[\lambda + \frac{\nu^2}{2}\frac{\sigma_x}{\sigma} - \frac{1}{2}\nu\nu_x\right]\frac{\hat{v}_x}{\hat{v}} + \nu^2\left(\frac{\hat{v}_x}{\hat{v}}\right)^2. \tag{D.12}$$

Using the expression (D.3) for the volatility of the quanto option ϕ_t, we find that

$$\frac{\sigma_x}{\sigma} \quad = \quad \frac{\nu_x}{\nu} - \frac{2\lambda}{\nu^2} - \frac{2\hat{v}_x}{\hat{v}}. \tag{D.13}$$

Substituting into (D.12), we find that the drift of γ_t under the risk-neutral measure vanishes, as it ought to be for a domestic asset. Hence, γ_t can be interpreted as the process for a numeraire asset.

Next, consider Eq. (D.9) again, but this time under the measure having γ_t as numeraire. Under this pricing measure the price of risk is σ^γ and

$$d\gamma_t \quad = \quad (\sigma^\gamma)^2\gamma_t dt + \sigma^\gamma\gamma_t dW. \tag{D.14}$$

Comparison with Eq. (D.9) shows that under this measure the drift of the underlying process x_t is λ, as stated. This implies that the risk-neutral pricing kernel for the quanto option of volatility (D.3) is given by

$$U(\phi, t; \phi_0) \quad = \quad \frac{\nu(X(\phi))}{\sigma(\phi)}\frac{\hat{v}(X(\phi),\rho)}{\hat{v}(X(\phi_0),\rho)}e^{-\rho t}u(X(\phi), t; X(\phi_0)). \tag{D.15}$$

Appendix E. Minimization schemes for JLT and KK

Here, we give a brief overview of the JLT and KK calibration schemes and explicitly state the minimization scheme that is required if an exact match to market prices cannot be made.

Let X_t be a time-homogenous Markov chain taking on values in a finite state space $S = \{1, \ldots, K\}$. State 1 represents the highest credit rating, down to state $K-1$ which represents the lowest credit rating. State K is associated with default; as such it is an absorbing state.

For the discrete time model, we can designate a one-period transition matrix:

$$\mathbf{Q} = \begin{pmatrix} q_{1,1} & \cdots & q_{1,K-1} & q_{1,K} \\ q_{2,1} & \cdots & q_{2,K-1} & q_{2,K} \\ \vdots & \ddots & \vdots & \vdots \\ q_{K-1,1} & \cdots & q_{K-1,K-1} & q_{K-1,K} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{E.1}$$

Here, $q_{i,j}$ is the real-world transition probability of starting in credit rating class i and being in rating j in the next period.

Let the corresponding process $\tilde{X}_t$ be the credit rating process under the risk-neutral probability measure $\tilde{P}$. It is not necessary that $\tilde{X}_t$ be time-homogenous or Markovian. Thus, the time t one-period transition matrix can be designated as:

$$\tilde{\mathbf{Q}}(t, t+1) = \begin{pmatrix} \tilde{q}_{1,1}(t,t+1) & \cdots & \tilde{q}_{1,K-1}(t,t+1) & \tilde{q}_{1,K}(t,t+1) \\ \tilde{q}_{2,1}(t,t+1) & \cdots & \tilde{q}_{2,K-1}(t,t+1) & \tilde{q}_{2,K}(t,t+1) \\ \vdots & \ddots & \vdots & \vdots \\ \tilde{q}_{K-1,1}(t,t+1) & \cdots & \tilde{q}_{K-1,K-1}(t,t+1) & \tilde{q}_{K-1,K}(t,t+1) \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{E.2}$$

Here, $\tilde{q}_{i,j}(t, t+1)$ is the risk-neutral transition probability of starting in credit rating class i at time t and being in rating j in the next period $t+1$.

Let $v_i(t, T)$ denote the time t price of a bond with credit rating i and maturity T, with $i = 0$ denoting the default-free bond. If the spot rate process for the default-free bond is independent of the credit rating process $\tilde{X}_t$, then we have:

$$v_i(t,T) = v_0(t,T)[1 - \tilde{q}_{i,K}(t,T) + R\tilde{q}_{i,K}(t,T)] \quad 1 \le i \le K-1, \tag{E.3}$$

where R is the recovery rate. We can calculate the $\tilde{\mathbf{Q}}(t, T)$ matrix in order to obtain the default probabilities $\tilde{q}_{i,K}(t, T)$ from the one-step matrices via the formula:

$$\tilde{\mathbf{Q}}(t,T) = \tilde{\mathbf{Q}}(t,t+1)\tilde{\mathbf{Q}}(t+1,t+2)\ldots\tilde{\mathbf{Q}}(T-1,T). \tag{E.4}$$

In general, the implied transition probabilities are obtained from the real-world transition probabilities by:

$$\tilde{q}_{i,j}(t,t+1) = \pi_{i,j}(t)q_{i,j} \quad 1 \le i, j \le K, \tag{E.5}$$

where the $\pi_{i,j}(t)$ are called the *risk premium adjustments*. These risk premium adjustments are chosen such that the bond prices given by Eq. (E.3) match market prices and must be such that $\tilde{\mathbf{Q}}(t, t+1)$ is a valid transition probability matrix. That is, we require:

$$\tilde{q}_{i,j}(t,t+1) \ge 0 \ \forall i,j \quad \text{and} \quad \sum_{j=1}^{K} \tilde{q}_{i,j}(t,t+1) = 1 \ \forall i \tag{E.6}$$

The difference between the JLT model and the KK model is in the specification of the risk-neutral adjustments.

JLT

JLT assume that for $i \neq j$ the risk-neutral adjustments are independent of the final state. That is, $\pi_{i,j}(t) = \mu_i(t)$ for $i \neq j$. The freedom for the $i = j$ case is required in order to normalize the sum of possible transitions from the state i. So, the risk-neutral probabilities are given by:

$$\tilde{q}_{i,j}(t, t+1) = \begin{cases} \mu_i(t) q_{i,j}; & i \neq j \\ 1 - \mu_i(t)(1 - q_{i,i}); & i = j \end{cases} \tag{E.7}$$

Notice that positivity is retained only if $0 < \mu_i(t) < \frac{1}{1-q_{i,i}}$. JLT describe an iterative procedure that determines the $\mu_i(t)$ so that the market prices are exactly matched. However, the algorithm is valid only if the one-period default probability is non-zero for every credit class. Since this is rarely the case, JLT remedy this by specifying the zero $q_{i,K}$'s to be a minimum value of 0.0001, and then subtracting this from the main diagonal. Furthermore, it is frequently the case that the algorithm gives values of $\mu_i(t)$ that yield negative risk-neutral probabilities. In this event, it is necessary to perform a minimization of the difference between model and market prices rather that obtaining an exact match. We give an explicit algorithm for such a minimization here.

At the t^{th} time step, given a vector $\bar{\mu}(t-1) = (\mu_1(t-1), \ldots, \mu_{K-1}(t-1), 1)'$ the risk-neutral transition matrix $\tilde{\mathbf{Q}}(0,t)$ can be determined by Eqs. (E.4) and (E.7). With this, the bond prices can be calculated using Eq. (E.3). Thus, we can choose the vector $\bar{\mu}(t-1)$ such that

$$\sum_{j=1}^{K-1} [v_i(0,t) - v_i(0,t)^{\text{market}}]^2 \tag{E.8}$$

is minimized, subject to the positivity constraint $0 < \mu_i(t-1) < \frac{1}{1-q_{i,i}}$. In fact, this is a linearly constrained, linear least-square minimization problem. By Eq. (E.3), the square-root of each term in Eq. (E.8) is:

$$v_i(0,t) - v_i(0,t)^{\text{market}} = v_0(0,t)(R-1)\tilde{q}_{i,K}(0,t) - \left[v_i(0,t)^{\text{market}} - v_0(0,t)\right] \tag{E.9}$$

Now, $\tilde{q}_{i,K}(0,t)$ is the (i,K) element of the matrix $\tilde{\mathbf{Q}}(0,t) = \tilde{\mathbf{Q}}(0,t-1)\tilde{\mathbf{Q}}(t-1.t)$. That is,

$$\begin{aligned} \tilde{q}_{i,K}(0,t) &= \sum_{j=1}^{K} \tilde{q}_{i,j}(0,t-1)\tilde{q}_{j,K}(t-1,t) \\ &= \sum_{j=1}^{K-1} \tilde{q}_{i,j}(0,t-1)\mu_j(t) q_{j,K} + \tilde{q}_{i,K}(0,t-1) \end{aligned} \tag{E.10}$$

So, we see that by Eq. (E.9), if $\mathbf{A}$ is a $(K-1) \times K$-matrix and $\mathbf{b}$ is a $K-1$-vector with components:

$$A_{i,j} = v_0(0,t)(R-1)\tilde{q}_{i,j}(0,t-1)q_{j,K} \quad b_i = v_i(0,t)^{\text{market}} - v_0(0,t) \tag{E.11}$$

then the K vector $\bar{\mu}(t)$ that minimizes $\|A\mu - b\|$ subject to the constraints $0 \leq \mu \leq \mathbf{w}$ (where the inequalities are component-wise) minimizes (E.8). Here, $\mathbf{w}$ is the K vector with components $\frac{1}{1-q_{i,i}}$. Notice that if exact prices can be obtained, this minimization algorithm will obtain the same values for $\mu_i(t)$ as the exact, iterative scheme.

For the numerical example given in Section 3, this scheme yielded errors in the bond prices that were consistently less than 4% except for Caa rated bonds with a maturity of 5 years, where the error was 10%.

KK

Rather than the condition that leads to (E.7), KK use the condition $\pi_{i,j}(t) = \ell_i(t)$ for $j \neq K$. Then, the risk-neutral probabilities are given by:

$$\tilde{q}_{i,j}(t, t+1) = \begin{cases} \ell_i(t) q_{i,j}; & i \neq j \\ 1 - \ell_i(t)(1 - q_{i,K}); & j = K \end{cases} \tag{E.12}$$

KK give an explicit iterative scheme that specifies the $\ell_i(t)$ so that market prices are exactly matched by the model. One advantage over the JLT formulation is that the default probabilities $q_{i,K}$ are not required to be non-zero, so that the adjustment that JLT requires is not needed. The positivity constraint in this case is $0 < \ell_i(t) < \frac{1}{1-q_{i,K}}$, and if this is violated, then the same minimization scheme can be carried out as in the JLT case.

While KK find that their scheme is more stable than JLT (that is, the exact solution produces valid values for the risk premia more often), it was found in our numerical example of Section 3 that the exact solution produced a negative value for $\ell_{\mathrm{CCC}}(4)$. Thus, the minimization scheme must be used. The method is similar to the JLT case. Eq. (E.10) is replaced by:

$$\tilde{q}_{i,K}(0,t) = \sum_{j=1}^{K-1} \tilde{q}_{i,j}(0, t-1)[1 - \ell_j(t)(1 - q_{j,K})] + \tilde{q}_{i,K}(0, t-1). \tag{E.13}$$

The components for the vector $\mathbf{b}$ remain the same, while the components of $\mathbf{A}$ in this case are:

$$A_{i,j} = v_0(0,t)(R-1)\tilde{q}_{i,j}(0, t-1) \tag{E.14}$$

Then, the K-vector $\mathbf{y}$ that minimizes $\|\mathbf{A}\mathbf{y} - \mathbf{b}\|$ subject to the constraints $0 \leq \mathbf{x} \leq 1$ is the vector $(1-\ell_1(t)(1-q_{1,K}), \ldots, 1-\ell_{K-1}(t)(1-q_{K-1,K}), 1)'$ that minimizes (E.8) for the KK case. Again, this minimization yields the same results as the exact, iterative scheme if the $\ell_i(t)$ given by the latter are allowable.

For the numerical example given in Section 3, this scheme yielded errors in the bond prices that were consistently less than 0.5% except for Caa rated bonds with a maturity of 5 years, where the error was 6%.

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References

Albanese, C., Camplolieti, G., Carr, P., and Lipton, A., 2001. Black-Scholes goes hypergeometric, Risk 14(12), 99–103.

Albanese, C., and Campolieti, G., 2001. New families of integrable diffusions, Working paper, http://www.mathpoint.ca/.

Albanese, C., Campolieti, J., Chen, O., and Zavidonov, A., 2003. Credit barrier models, Forthcoming in Risk 16(6).

Altman, E. I., and Kishore, V. M., 1998. Defaults and returns on high yield bonds: analysis through 1997, NYU Salomon Center Working Paper Series S-98-1.

Avellaneda, M., and Zhu, J., 2001. Distance to default, Risk 14(12), 125–129.

Black, F., and Scholes, M., 1973. The pricing of options and corporate liabilities, Journal of Political Economy 81, 637–59.

Carty, L. V., 1997. Moodys rating migration and credit quality correlation, 1920-1996, Moody's Investors Service July.

Douady, R., and Jeanblanc, M., 2002. A rating based model for credit derivatives, working paper.

Duffee, G. R., 1999. Estimating the price of default risk, Review of Financial Studies 12(1), 197–226.

Duffie, D., 1998. Defaultable term structure models with fractional recovery of par, Graduate School of Business, Stanford University.

——— , and Singleton, K. J., 1999. Modeling the term structures of defaultable bonds, Review of Financial Studies 12(4), 687–720.

Elton, E. J., Gruber, M. J., Agrawal, D., and Mann, C., 2001. Explaining the rate spread on corporate bonds, Journal of Finance LVI(1), 247–277.

Gordy, M., and Heitfield, E., 2001. Of Moody's and Merton: a structural model of bond rating transitions, working paper.

Gradshteyn, I.S., and Ryzhik, I.M., 2000. Table of Integrals, Series, and Products, 6^{th} Edition. (Academic Press).

Huang, J.-Z., and Huang, M., 2002. How much of the corporate-treasury yield spread is due to credit risk?, working paper.

Hull, J., and White, A., 2001. Valuing credit default swaps II: modeling default correlations, Journal of Derivatives 8(3), 12–21.

Hyer, T, Lipton, A., Pugachevsky, D., and Qui, S., 1999. A hidden-variable model for risky bonds, working paper.

Iscoe, I., and Kreinin, A., 2002. Default boundary problem, Working paper.

——— , and Rosen, D., 1999. An integrated market and credit risk portfolio model, Algo Research Quarterly 2, 21–37.

Israel, R. B., Rosenthal, J. S., and Wei, J. Z., 2001. Finding generators for Markov chains via empirical transition matrice, with applications to credit ratings, Mathematical Finance 11(2), 245–265.

Jarrow, R. A., Lando, D., and Turnbull, S. M., 1997. A Markov model for the term structure of credit risk spreads, Review of Financial Studies 10(2), 481–523.

Kijima, M., and Komoribayashi, K., 1998. A Markov chain model for valuing credit risk derivatives, Journal of Derivatives 6(1), 97–108.

——— , and Suzuki, E., 2002. A multivariate Markov model for simulating correlated defaults, Journal of Risk.

Kim, I.J., Ramaswamy, K., and Sundaresan, S., 1993. Does default risk in coupons affect the valuation of corporate bonds?, Financial Management 22(3), 117–131.

Kreinin, A, and Sidelnikova, M., 2001. Regularization algorithms for transition matrices, Algo Research Quarterly 4(1/2), 25–40.

Longstaff, F., and Schwartz, E., 1995. A simple approach to valuing risky fixed and floating rate debt, Journal of Finance L(3), 789–819.

Madan, D., Carr, P., and Chang, E., 1998. The variance gamma process and option pricing, European Finance Review 2(1), 79–105.

Merton, R. C., 1974. On the pricing of corporate debt: the risk structure of interest rates, Journal of Finance 29(2), 449–470.

Nielsen, L., Saa-Requejo, J., and Santa-Clara, P., 1993. Default risk and interest rate risk: the term structure of default spreads, working paper, INSEAD.

Zhou, C., 1997. A jump-diffusion approach to modeling credit risk and valuing defaultable securities, Federal Reserve Board, Washington.

(%)	Aaa	Aa1	Aa2	Aa3	A1	A2	A3	Baa1	Baa2	Baa3	Ba1	Ba2	Ba3	B1	B2	B3	Caa-C
Aaa	88.82	5.80	3.28	0.70	0.81	0.38	0.15	0.00	0.00	0.00	0.05	0.00	0.00	0.00	0.00	0.00	0.00
Aa1	2.63	76.92	8.97	8.18	2.71	0.24	0.00	0.25	0.00	0.00	0.12	0.00	0.00	0.00	0.00	0.00	0.00
Aa2	0.64	2.65	80.34	9.55	4.49	1.20	0.76	0.22	0.00	0.00	0.00	0.00	0.07	0.07	0.00	0.00	0.00
Aa3	0.10	0.45	3.02	80.10	10.68	3.99	0.89	0.10	0.26	0.23	0.00	0.05	0.13	0.00	0.00	0.00	0.00
A1	0.04	0.10	0.68	4.62	81.83	7.53	3.04	0.73	0.23	0.19	0.38	0.38	0.07	0.18	0.00	0.00	0.00
A2	0.03	0.04	0.21	0.66	5.71	80.90	7.56	3.18	0.80	0.29	0.21	0.14	0.14	0.03	0.07	0.00	0.03
A3	0.03	0.12	0.00	0.21	1.61	9.02	75.41	6.81	3.88	1.51	0.47	0.19	0.27	0.43	0.04	0.00	0.00
Baa1	0.06	0.00	0.09	0.12	0.16	3.28	8.94	73.44	8.01	3.19	1.00	0.40	0.47	0.65	0.13	0.00	0.00
Baa2	0.00	0.13	0.17	0.16	0.15	0.99	3.82	7.99	74.25	7.90	2.01	0.43	0.72	0.43	0.50	0.31	0.00
Baa3	0.04	0.00	0.00	0.06	0.26	0.61	0.47	4.38	10.58	69.39	6.99	3.15	2.03	0.95	0.33	0.07	0.14
Ba1	0.12	0.00	0.00	0.00	0.18	0.11	0.68	0.92	3.03	6.70	75.05	4.91	4.10	0.83	1.39	1.00	0.12
Ba2	0.00	0.00	0.00	0.00	0.00	0.19	0.13	0.33	0.52	2.46	8.00	73.77	6.18	1.38	4.26	1.77	0.27
Ba3	0.00	0.03	0.03	0.00	0.00	0.23	0.14	0.14	0.22	0.84	2.53	5.02	75.78	2.68	6.23	2.64	0.54
B1	0.03	0.00	0.03	0.00	0.07	0.05	0.20	0.09	0.34	0.43	0.36	2.64	6.42	76.98	1.62	5.33	0.98
B2	0.00	0.00	0.09	0.00	0.16	0.00	0.07	0.16	0.12	0.00	0.25	2.12	3.72	5.81	67.41	7.93	2.75
B3	0.00	0.00	0.07	0.00	0.00	0.00	0.00	0.11	0.18	0.20	0.23	0.32	1.51	4.97	2.51	71.00	4.22
Caa-C	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.67	0.67	0.88	0.00	2.42	2.39	1.50	2.84	57.02

Table 1: Average one year alpha-numeric rating transition matrix. Transition probabilities are averaged over the period 1983-1996. Each row consists of transition probabilities from the same rating, and each column consists of transition probabilities to the same rating. All of the matrix elements are withdrawn rating adjusted. From Carty (1997).

(%)	1 Yr.	3 Yrs.	5 Yrs.
Aaa	0.00	0.00	0.39
Aa1	0.00	0.00	0.50
Aa2	0.00	0.11	0.62
Aa3	0.00	0.11	0.50
A1	0.00	0.45	1.10
A2	0.00	0.23	0.76
A3	0.00	0.47	0.64
Baa1	0.06	0.82	1.83
Baa2	0.06	0.37	1.50
Baa3	0.56	2.21	3.95
Ba1	0.89	4.91	9.66
Ba2	0.75	7.32	14.56
Ba3	2.94	13.90	24.44
B1	4.41	18.46	31.02
B2	9.42	26.84	36.78
B3	14.69	35.50	48.59
Caa-C	31.62	50.14	60.12

Table 2: Average default probabilities by alpha-numeric rating. One year probabilities are averaged over 1983-1996, three year probabilities are averaged over 1983-1994 and five year probabilities are averaged over 1983-1992. All default probabilities are rating-withdrawn adjusted from the full transition matrix. From Carty (1997).

Basis Points	1 YR	2 YR	3 YR	5 YR	7 YR	10 YR
Aaa/AAA	16	21	25	29	40	50
Aa1/AA+	21	26	35	39	50	60
Aa2/AA	26	36	40	44	58	70
Aa3/AA-	31	41	45	52	68	80
A1/A+	41	51	60	71	86	101
A2/A	51	61	74	87	100	120
A3/A-	61	76	90	97	119	135
Baa1/BBB+	83	98	113	125	139	162
Baa2/BBB	103	118	128	141	155	175
Baa3/BBB-	123	131	142	154	167	187
Ba1/BB+	475	450	400	400	375	350
Ba2/BB	625	600	575	525	475	450
Ba3/BB-	750	675	650	600	550	500
B1/B+	900	825	825	675	625	575
B2/B	950	900	850	725	675	625
B3/B-	1025	1000	900	900	850	850
Caa/CCC	1900	1800	1700	1550	1450	1550

Table 3: Yield spreads are Bridge Evaluator Corporate Spreads for Industrials taken from www.bondsonline.com on February 10, 2003

(Basis Points)	1 Yr.	2 Yr.	3 Yr.	5 Yr.	7 Yr.	10 Yr.
Treasury Yield	125	164	206	295	345	398

Table 4: Treasury yields for February 10, 2003.

Original Rating	Recovery Rate(%)
Aaa	68.34
Aa2	59.59
A2	60.63
Baa2	49.24
Ba2	39.05
B2	37.54
Caa	38.02

Table 5: Recovery rate for each credit rating. From Altman and Kishore (1998).

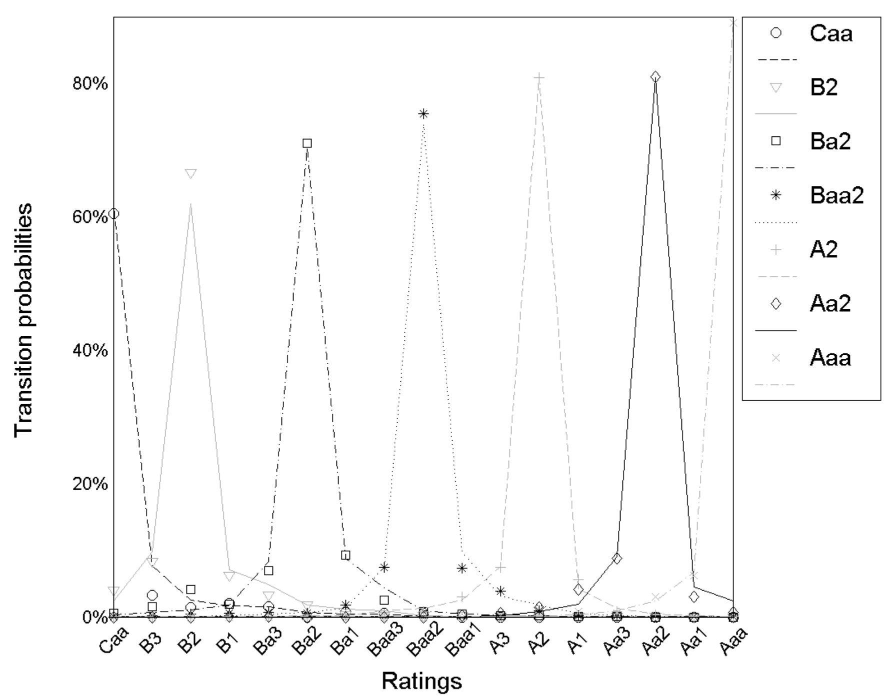

Figure 1: Comparison of model (lines) and historical (dots) one year transition probabilities. Historical transition probabilities are taken from Carty (1997) and are "Withdrawn Rating" adjusted. The parameters used for the model are: $q_2/q_1 = 30$, $\rho = 0.00125$, $\theta = 0.3$ and $\nu = 8.1$.

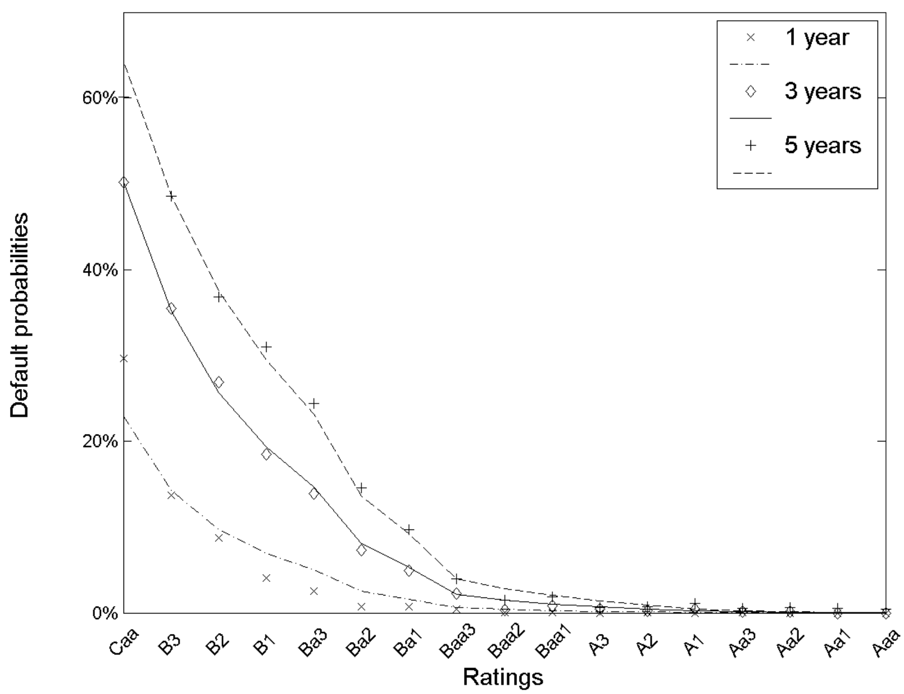

Figure 2: Comparison of model (lines) and historical (dots) default probabilities. Historical default probabilities are taken from Carty (1997) and are "Withdrawn Rating" adjusted.

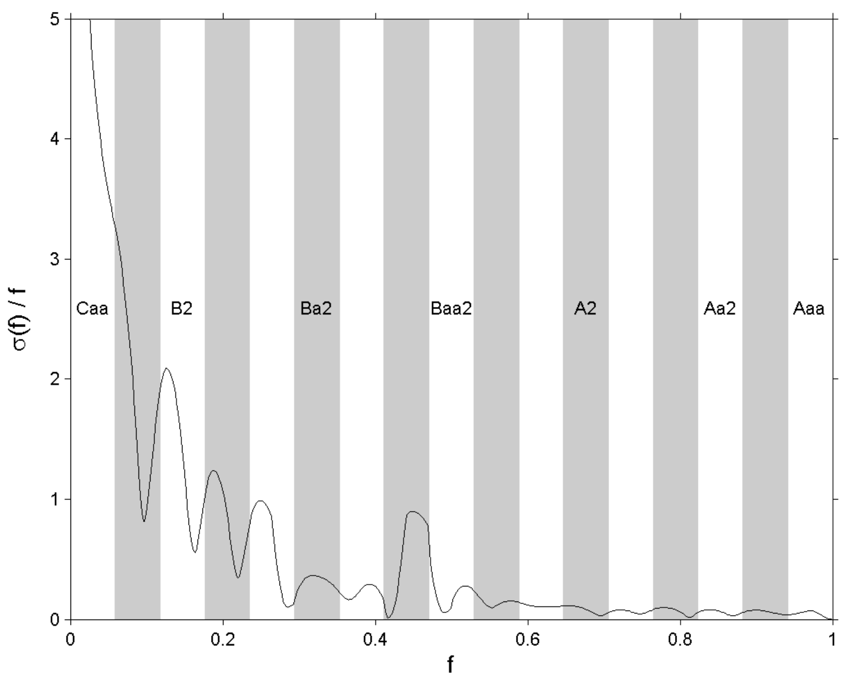

Figure 3: The local volatility as a function of f.

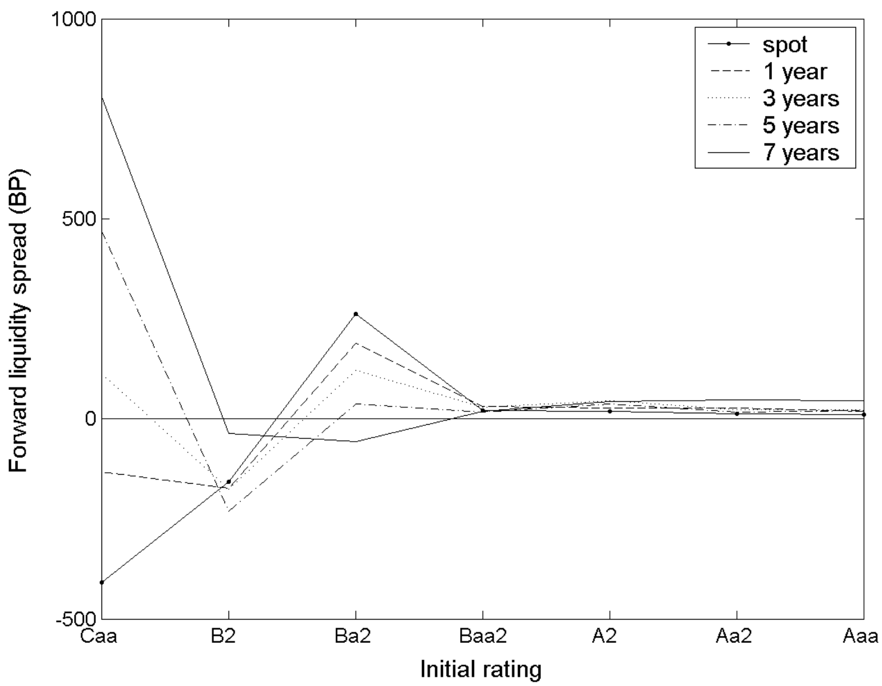

Figure 4: Forward liquidity spreads as a function of credit quality for various maturities.

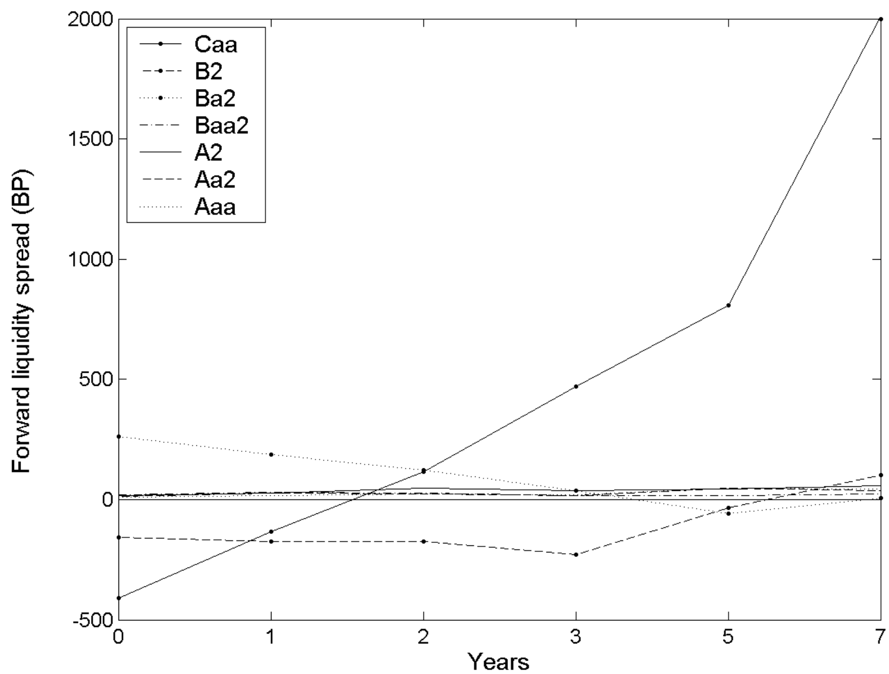

Figure 5: Term structure of forward liquidity spreads for various credit classes

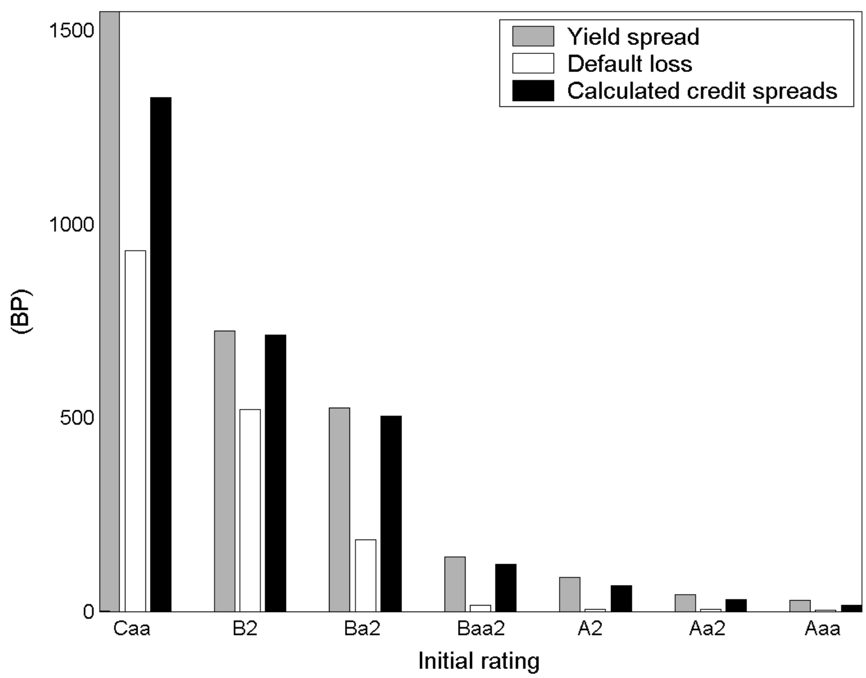

Figure 6: Yield spreads, default loss rates and calculated credit spreads for 5-Year bonds. Yield spreads are Bridge Evaluator Corporate Spreads for Industrials taken from http://www.bondsonline.com on Feb. 10, 2003.

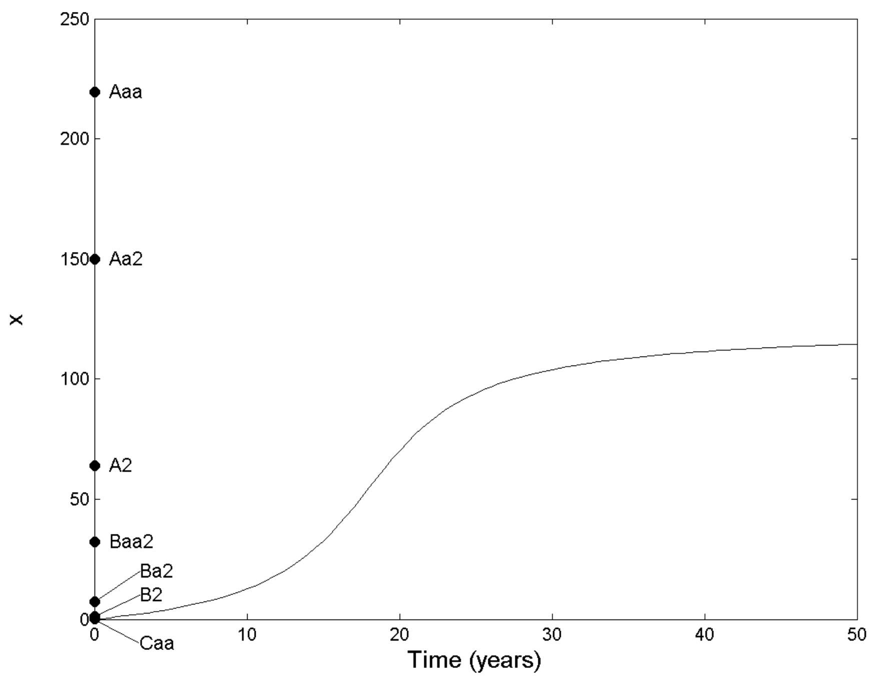

Figure 7: Initial rating levels (dots) and boundary (line) in x-space.

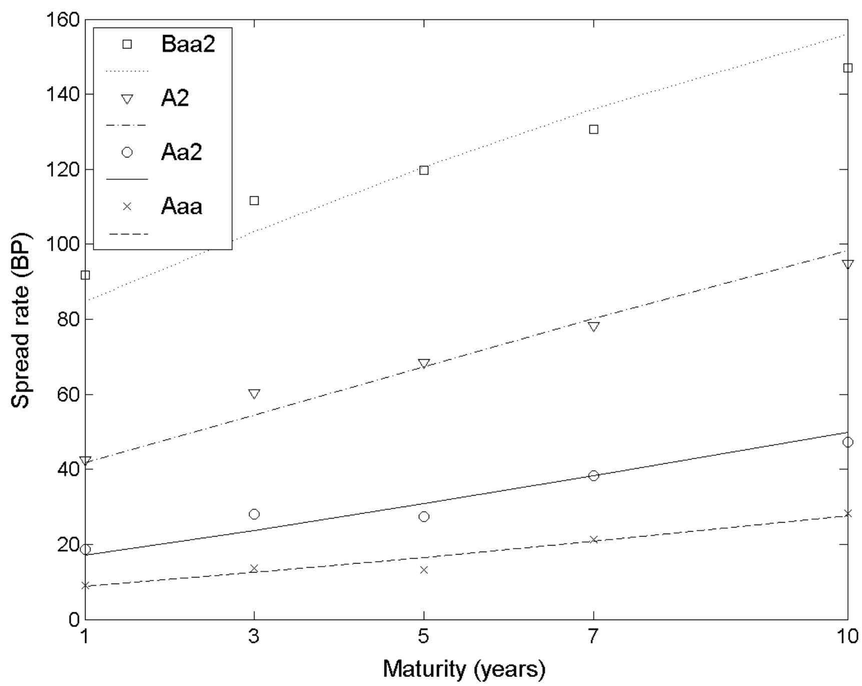

Figure 8: Comparison of the term structures for theoretical and tax-adjusted credit spreads for investment grade rated bonds.

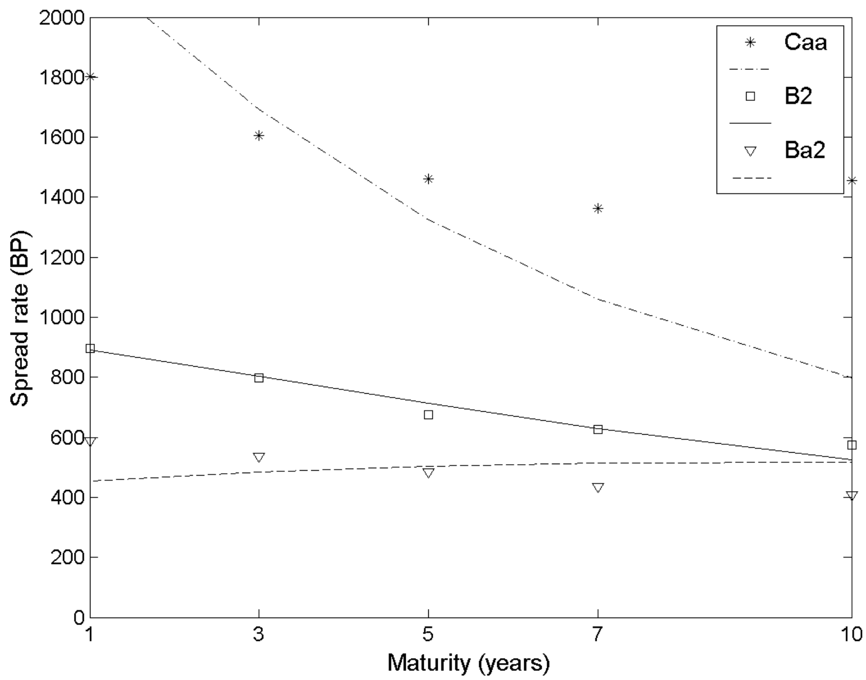

Figure 9: Comparison of the term structures for theoretical and tax-adjusted credit spreads for speculative grade rated bonds.

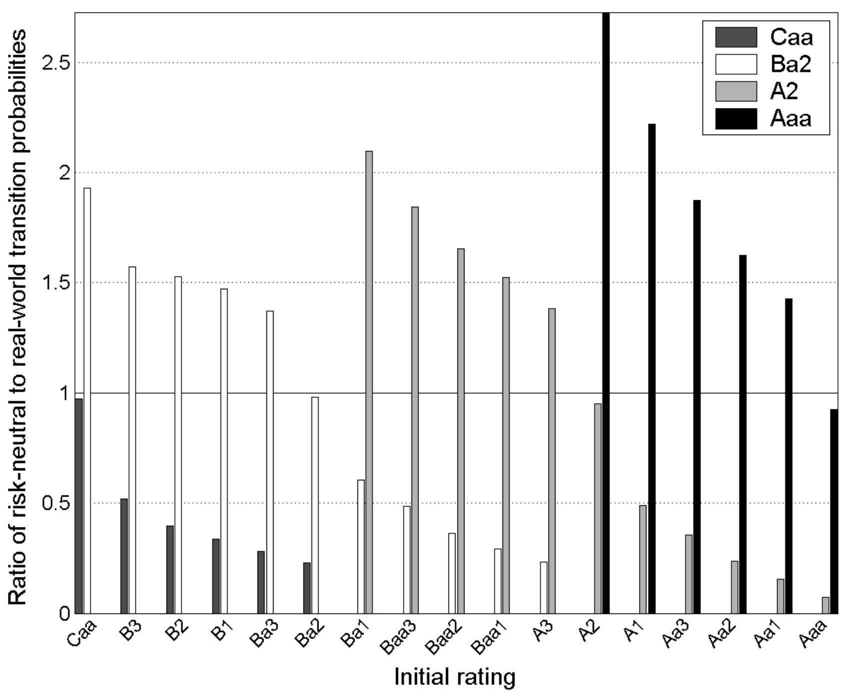

Figure 10: Ratio of risk-neutral probabilities to real-world probabilities for the credit barrier model with a one-year horizon.

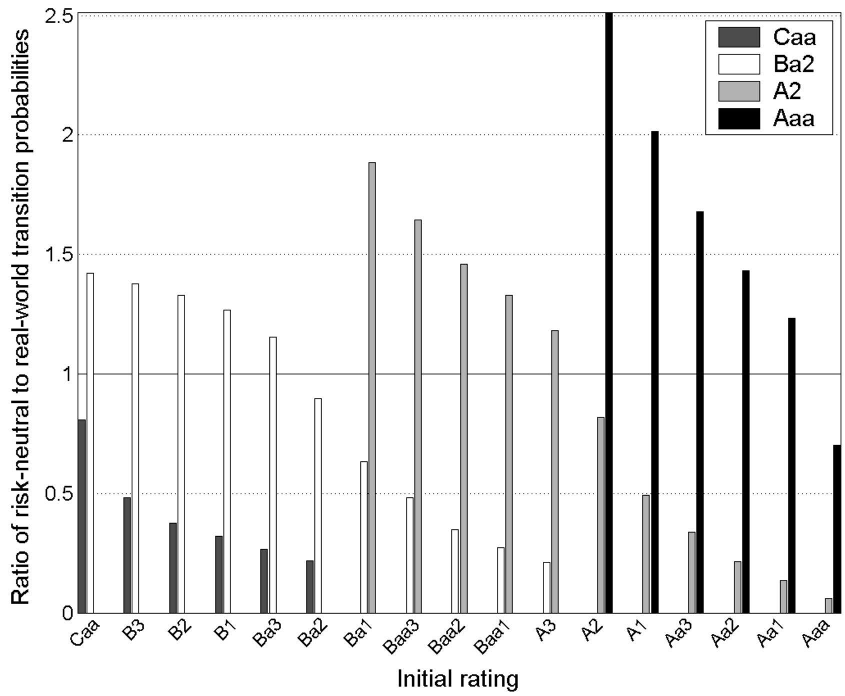

Figure 11: Ratio of risk-neutral probabilities to real-world probabilities for the credit barrier model with a five-year horizon.

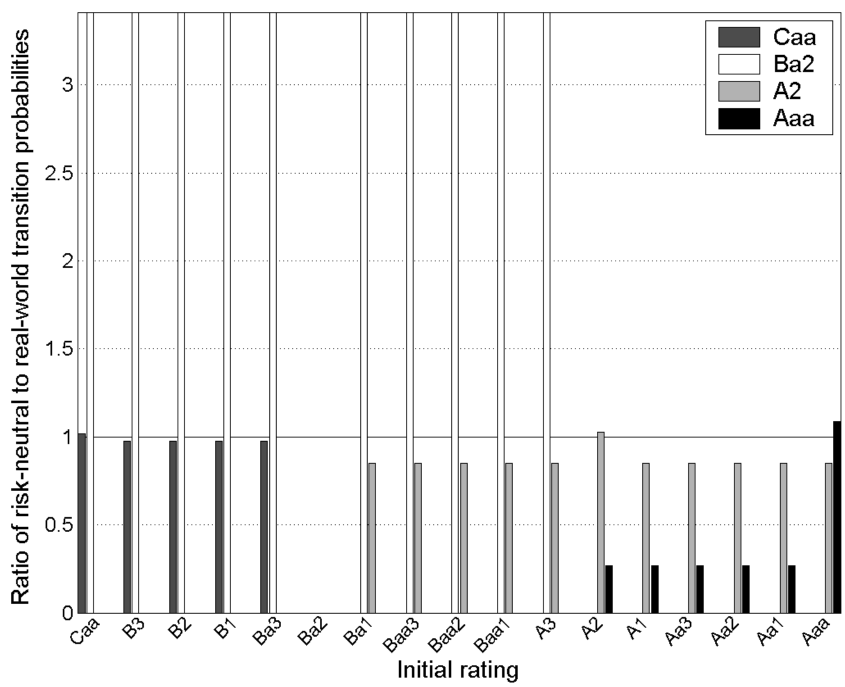

Figure 12: Ratio of risk-neutral probabilities to real-world probabilities for the JLT model with a one-year horizon.

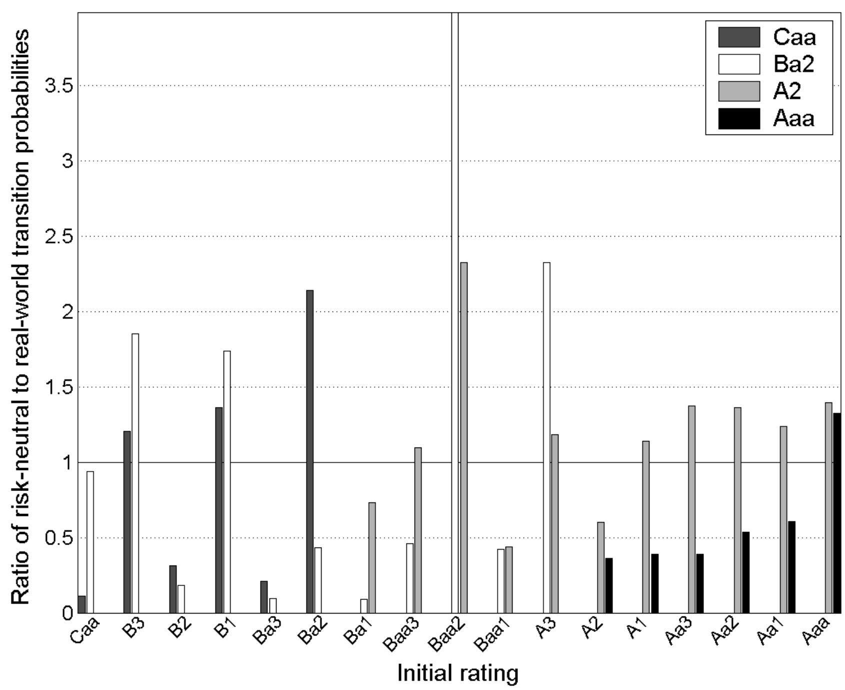

Figure 13: Ratio of risk-neutral probabilities to real-world probabilities for the JLT model with a five-year horizon.

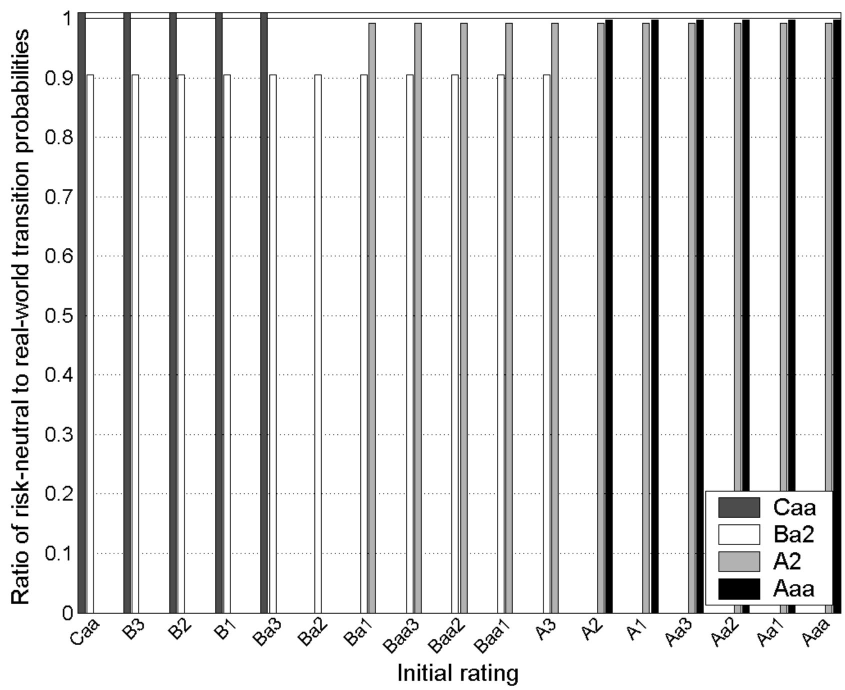

Figure 14: Ratio of risk-neutral probabilities to real-world probabilities for the KK model with a one-year horizon.

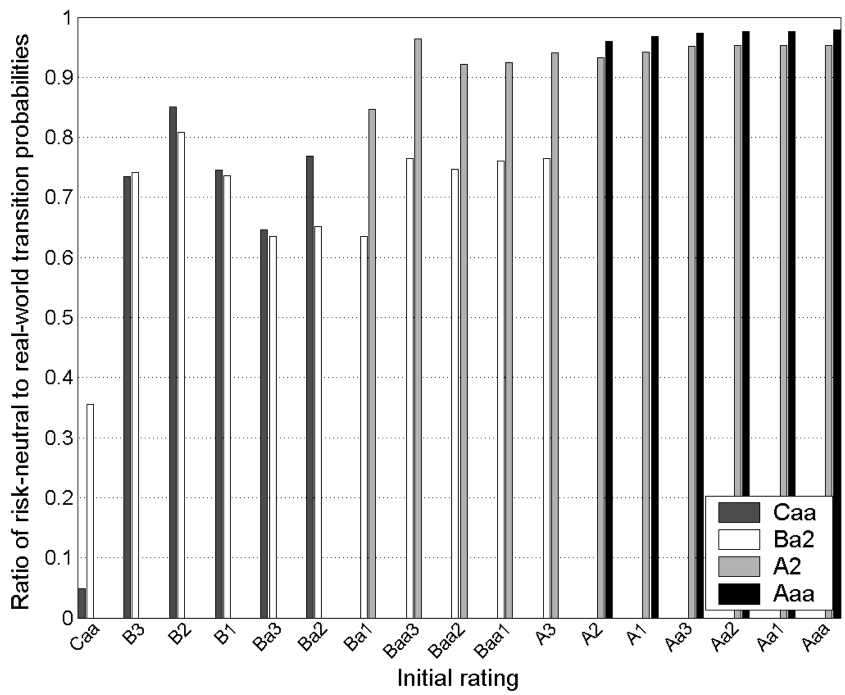

Figure 15: Ratio of risk-neutral probabilities to real-world probabilities for the KK model with a five-year horizon.