Satisficing measures for analysis of risky positions

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Abstract

In this work we consider a class of measures for evaluating the quality of financial positions with uncertain payoffs based on their ability to achieve desired financial goals. In the spirit of Simon [23], we call these measures satisficing measures and show that they are dual to classes of corresponding risk measures. This approach has the advantage that aspiration levels (either competing benchmarks or fixed targets) are often natural for investors to specify, as opposed to the risk-tolerance type parameters, which can be difficult to understand intuitively and hard to appropriately specify, that are necessary for many other approaches (risk measures, utility functions, etc.). Moreover, we explore a class of satisficing measures that have quasi-concavity properties which ensure that they appropriately reward for diversification. This further implies that optimization of these measures can be approached using computationally tractable tools from convex optimization, in contrast to the difficult, combinatorial problems that plague optimization of value-at-risk and related measures. Finally, when our satisficing measures have a particular scale invariance property, we are able to represent them in terms of expected value over an ambiguous probability distribution; additionally, these satisficing measures have a separation property which allows us to compute a single “tangent” portfolio regardless of the investor's desired expected return.

Keywords: satisficing, aspiration levels, targets, risk measures, coherent risk measures, convex risk measures, portfolio optimization.

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1 Introduction

One of the key principles from Simon’s [22] bounded rationality model is that, rather than formulating and solving complicated optimization problems, real-world agents often can choose the first-available actions which ensure that certain aspiration levels will be achieved. In other words, given the computational difficulties in the rational model paradigm, a more sensible (and descriptively accurate) approach may in fact be to view profit not as an objective to be maximized, but rather a constraint relative to some given aspiration level. Simon [23] coined the term satisficing1 to describe this approach.

The focus of this paper is to utilize the concept of aspiration levels from satisficing as a means of quantifying the desirability of investment opportunities with uncertain payoffs. As such, we replace the term “agent” above with “investor,” and we refer to the investment opportunities as “positions.” We restrict ourselves to the scalar case, i.e., the case when each position will realize a payoff representable by a single, real number. Though our primary focus in this paper is on a model useful for financial positions, the framework readily applies to other settings in which risk is an important issue.

In particular, our goal is to provide a framework for measuring the quality of risky positions with respect to their ability to satisfy (i.e., achieve the aspiration level). Aspiration level models based in the spirit of satisficing have been explored before but the connection is usually drawn via probability measures, i.e., the probability of achieving at least an aspiration level. Here we axiomatize a more general class of measures, called satisficing measures, which depend on the investor’s aspiration level. Probability measures are one special case of these.

One important advantage of this approach is that aspiration levels are often very natural for investors to specify, whereas traditional models based on risk measures or utility functions depend critically on tolerance parameters which are often difficult for investors to intuitively grasp and even harder to appropriately assess (certainly, at least, relative to assessing aspiration levels).2

As we have stated, the idea of aspiration levels is not new in the decision theory literature (which often uses the nomenclature “targets”). Some recent work focuses on the probability of achieving an aspiration level as a way of comprehensively and rigorously discussing decision theory without using utility functions; see Castagnoli and LiCalzi [4] and Bordley and LiCalzi [3]. Tsetlin and Winkler [25] consider the case of using aspiration levels along multiple dimensions within utility functions. From the descriptive standpoint, a number of studies have concluded that aspiration levels play an important role in real-world decision making behavior. Lanzillotti’s study [14] interviews executives of 20 large companies and concludes that these managers are primarily concerned about target returns on investment. In another study, Payne et al. [18, 19] illustrate that managers tend to disregard investment possibilities that are likely to under perform against their target. Simon himself [23] also argued that most firms’ goals are not maximizing profit but attaining a target profit. In an empirical study by Mao [15], managers were asked to define what they considered as risk. From their responses, Mao concluded that “risk is primarily considered to be the prospect of not meeting some target rate of

1A blend of the words “satisfy” and “suffice.”
2We concede here that we avoid entirely the issue of ascertaining the “right” aspiration level and assume it is a given primitive; for some recent work which does address this issue, see Bearden and Connolly [2].
return.” The notion of success probability has also been applied in the mathematical finance literature; see, for instance, Müller et al. [17] and Föllmer and Leukert [10].

Of course, one of the drawbacks of maximizing the success probability alone is that it tacitly assumes that the modeler is indifferent to the level of losses and gains. It does not address how catastrophic losses can be (or how exceptional gains can be) when “extreme” (i.e., low probability) events occur. A number of studies have in fact suggested that subjects are not completely insensitive to such magnitude variations, particularly with respect to losses; see, for instance, Payne et al. [18]. More recently, Diecidue and van de Ven [8] have argued that a model which solely maximizes the success probability is “too crude to be normatively or descriptively relevant.”

Our primary contributions in this paper are as follows:

1. We define axiomatically the concept of a satisficing measure, which depends on a position’s performance relative to a given aspiration level. These satisficing measures can be thought of as a generalization of the success probability approach. We also show that satisficing measures have a rather deep, dual connection to risk measures; in particular, we prove a representation theorem which shows that every satisficing measure can be written in terms of a parametric family of risk measures.

2. In general, a satisficing measure need not reward diversification. When we impose an additional property of quasi-concavity on satisficing measures, however, diversification is rewarded. Moreover, the representation theorem in this case holds over a parametric family of convex risk measures introduced by Föllmer and Schied [11]. Quasi-concavity of satisficing measures further ensures that they are generally not insensitive to the magnitude of losses below (or gains above) aspiration levels, which is in stark contrast to probability measures. In addition, the quasi-concavity property is important computationally, as it readily admits the use of efficient algorithms for optimization over satisficing measures.

3. Finally, when we impose an additional property of scale invariance on a satisficing measure, our representation theorem then ties into to the well-known class of coherent risk measures popularized by Artzner et al. [1]. This class of satisficing measures has an alternate interpretation in terms of expected values over ambiguous probability distributions, which, interestingly, connects us back to Simon’s contention that probabilities often not known exactly. Furthermore, we show a “separation theorem” holds for portfolio optimization problems over this class of satisficing measures.

Before we proceed, we comment briefly on our choice of nomenclature. The idea of defining a satisficing measure as something over which we can optimize may seem to be somewhat of an oxymoron to purists. Indeed, Simon’s original presentation of satisficing was as a computationally feasible alternative to the potentially onerous approach of maximizing. On the other hand, we are not the first to use the aspiration level idea from satisficing within decision theoretic approaches based on optimization (e.g., [3], and Charnes and Cooper, [5]). Moreover, the rapid proliferation of massive computational capabilities has radically changed the way decisions are made in many industries, particularly finance (Simon
himself [24] was obviously quite aware of and interested in the impact of high-speed computing). Our choice of the word satisficing largely reflects a connection to the concept of aspiration levels as a means of quantifying satisfaction with a risky position.

The outline of the paper is as follows. In Section 2, we motivate and define the axioms of satisficing measures and state our most general representation theorem relating these to risk measures; in Section 3, we consider imposing additional properties on the satisficing measures and then connect to convex and coherent risk measures. Finally, Section 4 considers portfolio optimization with satisficing measures as well as a computational example.

2 Defining satisficing measures

Let \((\Omega, \mathcal{F}, P)\) be a measure space and let \(\mathcal{X}\) be a set of random variables on \(\Omega\), i.e., a set of functions \(X : \Omega \rightarrow \mathbb{R}\). Each \(X \in \mathcal{X}\) represents the payoff (or return) of a different, risky position. Throughout the paper, we will use the notation \(X \geq Y\) for \(X, Y \in \mathcal{X}\) to represent state-wise dominance, i.e., \(X(\omega) \geq Y(\omega)\) for all \(\omega \in \Omega\). Similarly, \(X > Y\) denotes strict state-wise dominance (\(X(\omega) > Y(\omega)\) for all \(\omega \in \Omega\)).

We consider the situation in which the investor has an aspiration level \(\tau\) which she hopes to achieve via these positions. We assume \(\tau\) is a random variable on \(\Omega\) as well, which, of course, includes the case of \(\tau\) being a constant. This encompasses the cases of both individual investors, who have their own financial goals (usually constant targets) to meet, as well as the case of professional managers, who often compete against alternative, risky positions (benchmarks).

Given an uncertain payoff \(X \in \mathcal{X}\), we define the target premium \(V\) to denote the excess payoff above the aspiration level, i.e., \(V = X - \tau \in \mathcal{V}\), where \(\mathcal{V}\) is also set of random variables on \(\Omega\). Without loss of generality and for notational convenience, we assume \(\mathcal{V} = \mathcal{X}\). In other words, we will assume each of the payoffs \(X \in \mathcal{X}\) already has the aspiration level embedded within it, and therefore we will suppress the notation \(\tau\) in everything that follows. Thus, a position achieves the aspiration level if and only if the realized target premium \(X(\omega)\) satisfies \(X(\omega) \geq 0\).

In the spirit of satisficing, one measure of satisfaction with respect to a target premium \(X \in \mathcal{X}\) is

\[
\rho(X) \triangleq P\{X \geq 0\}.
\tag{1}
\]

Obviously, such a measure is equivalent to the expected value of a \(\{0, 1\}\)-utility function \(u(x)\) which is 1 if and only if \(x \geq 0\).

We now define a more general notion of satisficing measures, of which probability measures are one example. We are attempting to get the best of both worlds: on the one hand, we are exploiting the fact that aspiration levels are simple to understand and (often) natural to specify; on the other, by imposing additional properties in the following section, we will circumvent some of the main difficulties with probability measures (e.g., its inability to reward diversification or distinguish magnitudes).

**Definition 1.** A function \(\rho : \mathcal{X} \rightarrow [0, \bar{\rho}]\), where \(\bar{\rho} \in \{1, \infty\}\), is a satisficing measure defined on the target premium if it satisfies the following axioms for all \(X, Y \in \mathcal{X}\):

...
1. Attainment content: If $X \geq 0$, then $\rho(X) = \bar{\rho}$.

2. Non-attainment apathy: If $X < 0$, then $\rho(X) = 0$.

3. Monotonicity: If $X \geq Y$, then $\rho(X) \geq \rho(Y)$.

4. Gain continuity: $\lim_{a \downarrow 0} \rho(X + a) = \rho(X)$.

These axioms are rather straightforward to motivate in light of satisficing. Attainment content reflects the fact that satisficing is primarily concerned with achieving the aspiration level; if a position always achieves this, then we are always satisfied with the outcome, and therefore fully “content.” On the flip side, a position which never reaches the aspiration level does not satisfy us in the slightest (non-attainment apathy). Monotonicity is quite clear as well: if one position never under performs another, then we must be at least as satisfied with it. Finally, gain continuity is simply right continuity of $\rho$: if we augment our position with an infinitesimally small but positive amount, in the limit we cannot improve our satisfaction.

In other words, we are indifferent to small gains in the payoff, but not necessarily small losses (in particular, if we are right at the aspiration level).

Remark 1. An immediate criticism of this framework is that, if the target premia are all above (below) zero, then any satisficing measure cannot distinguish among any of the positions. On the one hand, if aspiration levels are truly preference-dependent and in no way influenced by the available opportunities, then it is true that the satisficing measures will not be very useful in such circumstances. On the other hand, it is not unreasonable to assume that aspiration levels are often affected by the opportunities available to the investor. In such cases, one could argue that the investor is too pessimistic (optimistic) and should be reassessing her aspiration level. In fact, Simon [22]’s paper provides an example of selling a house and the agent’s aspiration level is determined, in part, via an “exploration” phase in which she learns about the climate of her housing market. In short, the satisficing measures we define are most useful when the aspiration levels are set such that positions can fall both above and below this level.

Clearly, $\mathbb{P}\{X \geq 0\}$ is one example of a satisficing measure; $\mathbb{P}\{X > 0\}$, however, violates gain continuity, and is therefore not a satisficing measure.

We also point out that our definition allows for satisficing measures to map to either $[0, 1]$ or all of $\mathbb{R}_+$. We will give examples of both cases in Section 2.2.

2.1 Quasi-concave and coherent satisficing measures

The probability measure $\mathbb{P}\{X \geq 0\}$ is the most obvious example of a satisficing measure. On the one hand, it embodies the concept of satisficing in that it reflects very directly the performance of a position relative to the aspiration level. On the other hand, unfortunately, it suffers a rather critical flaw from an economic perspective in that it does not reflect convex preferences (i.e., does not value diversification).

\[^3\]Note that left continuity is critically different and need not hold. In particular, clearly probability measure $\mathbb{P}\{X \geq 0\}$ is a satisficing measure. Now consider the case when $X$ is a constant; right-continuity, but not left-continuity, holds at $X = 0$. 

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Clearly, any satisficing measure $\rho$ induces a preference relation $\succeq$ with, for any $X, Y \in \mathcal{X}$, $X \succeq Y$ if and only if $\rho(X) \geq \rho(Y)$. Recall that $\succeq$ is a convex preference if, for all $X, Y, Z \in \mathcal{X}$, the implication

$$Y \succeq X, Z \succeq X \Rightarrow \lambda Y + (1 - \lambda)Z \succeq X \ \forall \ \lambda \in [0, 1]$$

holds. It is clear that the induced relation $\succeq$ is convex if and only if the function $\rho$ is quasi-concave, i.e., for all $X, Y \in \mathcal{X}$, $\lambda \in [0, 1]$,

$$\rho(\lambda X + (1 - \lambda)Y) \geq \min\{\rho(X), \rho(Y)\}.$$ 

For perhaps the simplest example of probability’s failure to satisfy quasi-concavity (and hence failure to induce convex preferences), consider a target premium $X$ distributed symmetrically about zero, so $\mathbb{P}\{X \geq 0\} = 1/2$. Now consider an alternative position $Y = -X - \epsilon$, where $\epsilon > 0$. If $\epsilon$ is small enough and the distribution of $X$ is continuous, then $\mathbb{P}\{Y \geq 0\}$ will be close to $1/2$ as well. The position $Z = 1/2(X + Y)$, however, is $-\epsilon/2$ with probability one, so $\mathbb{P}\{Z \geq 0\} = 0$. Therefore, this satisficing measure says we are worse off by diversifying between these positions as opposed to investing into either individually.

One could object that this is a rather silly example; indeed, if aspiration levels are really all we care about, then the probability measure is “doing its job” just fine. From an economic perspective, however, there are a myriad of compelling arguments as to why risk management tools should imply convex preferences (reward diversification). In short, investors should somehow prefer “bundled” positions over “extreme” ones; we refer to the classical references (e.g., [16]) on microeconomic theory for more detail on the justification of convex preferences.

On a related note, we would ultimately like to exploit the simplicity of satisficing measures and use them as a method for selection of positions, and their failure to satisfy quasi-concavity is tantamount to having to select over potentially non-convex spaces of positions. From a computational standpoint, this is in general very difficult when the number of opportunities (or available asset classes in a portfolio setting) is large.

In short, the desire to ensure that satisficing measures recognize diversification appropriately, both for economic and computational reasons, motivates the following.

**Definition 2.** A function $\rho : \mathcal{X} \rightarrow [0, \bar{\rho}]$, where $\bar{\rho} \in \{1, \infty\}$, is a quasi-concave satisficing measure (QSM) defined on the target premium if, in addition to Definition 1, it satisfies the following for all $X, Y \in \mathcal{X}$:

1. Quasi-concavity: If $\lambda \in [0, 1]$, then $\rho(\lambda X + (1 - \lambda)Y) \geq \min\{\rho(X), \rho(Y)\}$.

If, in addition, $\rho$ satisfies

2. Scale invariance: If $k > 0$, then $\rho(kX) = \rho(X)$,

we say $\rho$ is a coherent satisficing measure (CSM).
Notice that, in the simple example we gave, a QSM would have to satisfy 
\[
\min\{\rho(X), \rho(Y)\} \leq 0,
\]
i.e., either \(\rho(X) = 0\) or \(\rho(Y) = 0\), which is not the case for probability measure.

The scale invariance axiom arises in situations in which, like with probability measures, we are indifferent to symmetric scalings of the payoffs with respect to our aspiration level. Note that quasi-concavity alone implies, for any \(\lambda \in [0, 1]\),
\[
\rho(\lambda X) = \rho(\lambda X + (1 - \lambda)0) \\ 
\geq \min\{\rho(X), \rho(0)\} \\ 
= \rho(X),
\]
since, by definition, \(\rho(0) = \bar{\rho}\) for a satisficing measure. For a CSM, this bound is tight. In other words, for a CSM, any decrease in potential losses below the aspiration level is offset by symmetrically increased gains above the aspiration level. We can therefore view CSMs as diversification-rewarding measures which retain the “target-oriented” nature of probability measures. Or, put another way, CSMs are quasi-concave “relaxations” of probability measures.

### 2.2 Examples

We now provide some examples of satisficing measures. We intentionally attempt to provide a number of examples here to illustrate how the concepts of QSMs and CSMs cut across a variety of approaches to decision-making under uncertainty, including risk measures from finance and ideas from utility theory. Some of these examples require proofs, which we have relegated to the appendix.

**Example 1. (Probability measure):**

As we have mentioned, \(\rho(X) = \mathbb{P}\{X \geq 0\}\) is a satisficing measure. Notice that it satisfies all the axioms of coherence except quasi-concavity.

**Example 2. (Sharpe ratio):**

Let \(\mathcal{X}\) be a linear space of a finite set of random variables including constants such that \(X \in \mathcal{X}\) is either a constant or a random variable with infinite support. This includes a set of independently distributed random variables that with infinite support such as those with normal distributions. The following Sharpe ratio,
\[
\rho_S(X) = \sup_{a \in [0, 1]} \left\{ a : -\mathbb{E}(X) + \sqrt{\frac{a}{1 - a}} \sigma(X) \leq 0 \right\},
\]
(or 0 if no such \(a\) exists) is a coherent satisficing measure on \(\mathcal{X}\). Moreover,
\[
\rho_S(X) \leq \mathbb{P}\{X \geq 0\}.
\]

Note that the function \(g(a) = \sqrt{\frac{a}{1 - a}}\) is an increasing function of \(a \in [0, 1]\), hence, whenever \(\sigma(X) > 0\) and \(\mathbb{E}(X) \geq 0\), \(g(\rho_S(X)) = \mathbb{E}(X)/\sigma(X)\), which is the standard definition of the Sharpe ratio (without
a risk free asset). Note that the Sharpe ratio is generally not a coherent satisficing measure on random variables with finite support. For instance, consider the random variable $X \sim U[0,1]$, which has the Sharpe ratio $\sqrt{3}$ and hence, $\rho_S(X) = 3/4$. However, since $X \geq 0$ the satisficing measure, $\rho_S$ violates the axioms of attainment content and monotonicity (since $\rho_S(X) < \rho_S(0) = 1$).

This satisficing measure is explored in more detail by Pinar and Tütüncü [20].

Example 3. (Optimized Sharpe ratio): Let

$$X = \{X : \exists(\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1} : X = \lambda_0 + \lambda_1 X_1 + \cdots + \lambda_n X_n\},$$

where $X_1, \ldots, X_n$ are random variables defined on $\Omega$ with positive definite covariance, and without loss of generality, zero means. The following optimized Sharpe ratio

$$\rho_{OS}(X) = \sup_{a \in [0,1)} \left\{ a : \min_{V \in X} \left\{ -E(X - V) + \sqrt{\frac{a}{1-a}} \sigma(X - V) : V \geq 0 \right\} \leq 0 \right\},$$

(or 0 if no such $a$ exists) on $X$ is a coherent satisficing measure on $X$. Moreover,

$$\rho_S(X) \leq \rho_{OS}(X) \leq \mathbb{P}\{X \geq 0\}.$$

Note that we can express $\mu_a(X) = \min_{V \in X} \left\{ -E(X - V) + \sqrt{\frac{a}{1-a}} \sigma(X - V) : V \geq 0 \right\}$ as an optimization problem over a $n$-dimensional convex cone. Moreover, its objective approaches to infinity whenever, $E(V)$ or $\sigma(V)$ approaches infinity. Since the covariance of $X_1, \ldots, X_n$ is positive definite, the optimal solution must be bounded and be achieved.

In the same example in which $X \sim U[0,1]$, we note that for all $a \in (0,1)$,

$$\mu_a(X) \leq \min_{V \in X} \left\{ -E(X - V) + \sqrt{\frac{a}{1-a}} \sigma(X - V) : V \geq 0, X = V \right\} = 0.$$

Hence, $\rho_{OS}(X) = 1$. Thus, $\rho_{OS}$ provides a better bound to probability measure than $\rho_S$.

Example 4. (Maximized risk aversion):

Consider the function

$$C_a(X) = \sup \{ m \in \mathbb{R} : E(u_a(X - m)) \geq 0 \},$$

where $\{u_a : a > 0\}$ is a family of concave, nondecreasing utility functions nonincreasing in $a > 0$. The scalar $a$ is a risk aversion parameter, with larger $a$ reflecting greater risk aversion. $C_a(X)$ is a certainty equivalent which may be interpreted as the maximum buying price of position $X$. Then the function

$$\rho(X) = \sup \{ a > 0 : C_a(X) \geq 0 \}$$

(or 0 if no such $a$ exists) is a QSM. As an example of this, when $u_a(x) = (1/a)(1 - \exp\{-ax\})$, we have

$$\rho(X) = \sup\{a > 0 : -\frac{1}{a} \ln(E(\exp(-aX))) \geq 0 \}$$

(or 0 if no such $a$ exists). It maximizes the risk aversion so that its certainty equivalence of $X \in X$ under an exponential utility achieves the break even point (this QSM is explored in Chua et al. [7]).
Example 5. (Family of concave utilities):

Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous concave nondecreasing function with \( f(0) = 0 \) and \( \lim_{x \to \infty} f(x) = 1 \). Then

\[
\rho(X) = \lim_{\epsilon \to 0} \sup_{a \in \mathbb{R}^+} \mathbb{E}(f(a(X + \epsilon)))
\]

is a coherent satisficing measure. Moreover, \( 0 \leq \rho(X) \leq \mathbb{P}\{X \geq 0\} \). The choice of function, \( f(x) = \min\{x, 1\} \) has recently been explored by Chen and Sim [6] and will be revisited in Section 2.1.

Most of the examples we have provided in this section are of the form of choosing the largest value of a parameter subject to some sort of risk constraint. In the case of Examples 2 and 3, for instance, we are choosing the largest value of a scalar \( a \) such that \(-\mathbb{E}(X) + g(a)\sigma(X) \leq 0\), which is a mean-variance risk constraint. In Example 4, we are choosing the largest value of a risk aversion parameter subject to its certainty equivalent under a concave utility function being nonnegative at that level. Though it is not immediately obvious, we can also write Example 5 in a similar form. In fact, it turns out that we can write any satisficing measure in the form of the largest value of a parameter subject to a risk constraint, as we will now show.

3 A dual representation of satisficing via risk measures

If we were to choose positions based on higher or maximal values of the satisficing measure \( \mathbb{P}\{X \geq 0\} \), intuition suggests that we are somehow embedding risk aversion into our selection process; we are hoping to find portfolios which have a greater likelihood of achieving the aspiration level.

Indeed, probability measures do in fact connect to the classical definition of risk known as value-at-risk. One definition of value-at-risk is

\[
\text{VaR}_\alpha(X) \equiv \inf \{t \in \mathbb{R} : \mathbb{P}\{t + X \geq 0\} \geq 1 - \alpha\}.
\]  

(4)

This quantity can be interpreted as the smallest amount of capital \( t \) necessary to add to \( X \) to ensure that the augmented portfolio \( X + t \) breaks even with probability at least \( 1 - \alpha \). Thus, all other things being equal, a portfolio with a lower value-at-risk at a pre-specified \( \alpha \) is preferred. In addition, value-at-risk is decreasing with \( \alpha \), i.e., lower \( \alpha \) corresponds to more conservatism.

When the distribution has discontinuities, there are various definitions of VaR which take into account potential limiting conditions at these discontinuities. VaR has received considerable attention among both practitioners and academics alike; see, for instance, Duffie and Pan [9] for one treatment of value-at-risk.

In fact, probability measures \( \rho \) as defined in (1) are simply dual forms of VaR. In particular, it is not hard to see that

\[
\rho(X) = \sup \{1 - \alpha : \text{VaR}_\alpha(X) \leq 0\}.
\]  

(5)
Indeed, observe that \( \text{VaR}_1(X) = -\infty \) and for \( \alpha < 1 \), the infimum in Problem (4) is achievable since \( \mathbb{P}\{X + t \geq 0\} \) is a right continuous, non-decreasing function with respect to \( t \). Hence,

\[
\sup\{1 - \alpha : \text{VaR}_\alpha(X) \leq 0\} = \sup\{1 - \alpha : \inf\{t : \mathbb{P}\{t + X \geq 0\} \geq 1 - \alpha\} \leq 0\} \\
= \sup\{1 - \alpha : \exists t \leq 0 : \mathbb{P}\{t + X \geq 0\} \geq 1 - \alpha\} \\
= \max\{1 - \alpha : \mathbb{P}\{X \geq 0\} \geq 1 - \alpha\} \\
= \mathbb{P}\{X \geq 0\} \\
= \rho(X)
\]

Notice that (5) is a dual of (4) in which \( \tau \), the aspiration level, is specified and \( \alpha \), the tolerance level, is chosen to be as small as possible. \( \text{VaR} \), on the flip side, has the tolerance level fixed with the target (or augmenting capital) chosen to be as small as possible. Thus, value-at-risk measures and probability measures are dual forms.

One may ask whether this relationship extends to general satisficing measures; in other words, whether any satisficing measure can be represented in dual form over a family of risk measures. The answer, as we will show, is yes. First, we define formally the concept of a risk measure.

**Definition 3.** A function \( \mu : X \rightarrow \mathbb{R} \) is a risk measure if it satisfies the following for all \( X, Y \in X \):

1. Monotonicity: If \( X \geq Y \), then \( \mu(X) \leq \mu(Y) \).
2. Translation invariance: If \( c \in \mathbb{R} \), then \( \mu(X + c) = \mu(X) - c \).

The interpretation of risk measures is, like value-at-risk, the smallest amount of capital necessary to augment a position by in order to make it “acceptable” according to some standard. As such, the properties above are clear; if one position never performs worse than another, then it cannot be any riskier. In addition, if we augment our position by a guaranteed amount \( c \), then our capital requirement is reduced correspondingly by \( c \) as well. See, for instance, Föllmer and Schied [12] for more on risk measures.\(^4\)

We are now ready for our first result.

**Theorem 1.** A function \( \rho : X \rightarrow [0, \bar{\rho}] \), where \( \bar{\rho} \in \{1, \infty\} \), is a satisficing measure if and only if there exists a family of risk measures \( \{\mu_k : k \in (0, \bar{\rho})\} \), nondecreasing in \( k \), and \( \mu_0 = -\infty \) such that

\[
\rho(X) = \sup\{k \in [0, \bar{\rho}] : \mu_k(X) \leq 0\},
\]

**Proof.** See Appendix. \( \square \)

Theorem 1 states that every satisficing measure can be written as the largest value of a parameter subject to a risk constraint at that parameter value over a parametric family of risk measures. Moreover, the proof of the result is constructive, so given a satisficing measure, we can always specify exactly the structure of the corresponding, parametric family of risk measures.

---

\(^4\)These authors use the same definition but refer to risk measures as “monetary measures of risk,” which is perhaps more descriptive; for convenience we’ll simply use the term “risk measures.”
The interpretation is that, rather than specifying risk aversion parameters (e.g., choosing an \( \alpha \) for VaR), the satisficing measure approach is always equivalent to fixing an aspiration level, then choosing the largest value of the aversion parameter such that the risk evaluated at that parameter is no greater than the aspiration level. In other words, satisficing measures quantify how risk averse a position allows us to be while still maintaining the goal of achieving our aspiration level.

### 3.1 Representation of QSMs and CSMs

Of course, without additional, structural properties on the satisficing measure, there are no other properties that the family of risk measures must satisfy. We now examine structural properties of the family of risk measures when we are dealing with QSMs and CSMs, respectively.

In order to relate these satisficing measures to risk measures, we now formally define a class of risk measures introduced by Artzner et al. [1] and then generalized by Föllmer and Schied [11]. In particular, we have the following.

**Definition 4.** A function \( \mu : \mathcal{X} \rightarrow \mathbb{R} \) is a convex risk measure if, in addition to Definition 3, it satisfies, for all \( X, Y \in \mathcal{X} \):

1. Convexity: If \( \lambda \in [0, 1] \), then \( \mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y) \).

If, in addition, we have

2. Positive homogeneity: If \( \lambda \geq 0 \), then \( \mu(\lambda X) = \lambda \mu(X) \),

we say that \( \mu \) is a coherent risk measure.

It is known that every coherent risk measure may be written in the form

\[
\mu(X) = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q(-X) \tag{7}
\]

for a family of generating measures \( \mathcal{Q} \) (see, for instance, Huber [13] for a proof of this in a different context). Föllmer and Schied [11] show more generally that any convex risk measure may be written in the form

\[
\mu(X) = \sup_{Q \ll \mathbb{P}} \{ \mathbb{E}_Q(-X) - \alpha(Q) \}, \tag{8}
\]

where \( \alpha \) is a closed, convex function. It is then not hard to show that a convex risk measure is coherent if and only if the corresponding \( \alpha \) in its representation is an indicator function on a subset of measures \( \mathcal{Q} \).

We remark that we will assume, without loss of generality, that a finite convex risk measure is normalized such that \( \mu(0) = 0 \); this implies, for instance, that \( \alpha(Q) \geq 0 \) for all \( Q \ll \mathbb{P} \). It is without loss of generality because, when \( \mu \) is convex, \( \mu(X) - \mu(0) \) is also convex.

As the names of QSMs and CSMs betray, they are intimately related to these classes of risk measures. In fact, one can see that the satisficing measure

\[
\rho(X) = \sup \{ k \in [0, \bar{\rho}] : \mu_k(X) \leq 0 \},
\]
where $\mu_0 = -\infty$ and the $\mu_k$ are a family of convex risk measures nondecreasing in $k$, is indeed quasi-
concave; moreover, $\rho$ is a CSM when the $\mu_k$ are also coherent.

Guided by Theorem 1, we may wonder whether every QSM and CSM has such a representation; in
fact this is the case.

**Theorem 2.** A satisficing measure $\rho$ is quasi-concave if and only if the family $\{\mu_k : k \in (0, \bar{\rho})\}$ in
Theorem 1 is a family of convex risk measures. Similarly, it is coherent if and only if the family is a
family of coherent risk measures.

**Proof.** See Appendix. \hfill \square

### 3.2 Optimization over QSMs

In contrast with maximizing the success probability over a convex set of random variables, which is
computationally intractable, we show that maximizing a QSM can be reduced to solving a sequence of
convex optimization problems. We consider the problem of

$$\rho^* = \max \{\rho(X) : X \in \mathcal{X}\} \tag{9}$$

where $\mathcal{X}$ is a convex set of random variables. From a computational perspective, finding a feasible
solution in a convex set is relatively easy compared to a non-convex one. In particular, given a QSM,$\rho$, the following set

$$S(k) = \{\rho(X) : \rho(X) \geq k, X \in \mathcal{X}\} \tag{10}$$

is a convex set of random variables. Indeed, if $X \in S(k)$ and $Y \in S(k)$, it is easy to see that $\lambda X +
(1 - \lambda)Y \in S(k)$ for all $\lambda \in [0, 1]$. Assuming that there admits a computationally efficient method for
finding a feasible random variable in the set $S(k)$, and assuming that $\rho^* \in [a, b]$, $a > 0$ and $b < \bar{\rho}$, we
propose the following binary search to obtain a solution, $Z \in \mathcal{X}$ satisfying $\rho^* - \zeta \leq \rho(Z) \leq \rho^*$.

**Algorithm 1.** (Binary Search)

**Input:** A routine that returns a feasible random variable in the set $S(k)$ or report infeasible; real,
nonnegative numbers $a$, $b$ and $\zeta$.

**Output:** A random variable $Z$.

1. Set $\gamma_1 := a$ and $\gamma_2 := b$.
2. If $\gamma_2 - \gamma_1 < \zeta$, stop. Output: $Z$
3. Let $\gamma := \frac{\gamma_1 + \gamma_2}{2}$.
4. If $S(\gamma)$ is infeasible, update $\gamma_2 := \gamma$. Otherwise, update $\gamma_1 := \gamma$ and find $Z \in S(\gamma)$.
5. Go to Step 2.
If the satisficing measure is given in its dual form (6), we can evaluate the feasibility of \( S(k) \) by solving the following convex optimization problem

\[
r(k) = \min \{ \mu_k(X) : X \in \mathcal{X} \}.
\]

(11)

Indeed, checking the feasibility of \( S(k) \) is the same if checking \( r(k) \leq 0 \), which is finding the random variable with the smallest risk under the convex risk measure \( \mu_k \).

In sum, we can find a \( \zeta \)-optimal position \( X^* \) to (9) in \( \log_2((b-a)/\zeta) \) attempts to solve (11); provided we can solve (11), then, we can also solve (9) to high precision without too much additional difficulty.

### 3.3 An ambiguity robust representation of CSMs

Theorem 2 immediately implies an interesting interpretation for any coherent satisficing measure.

**Corollary 1.** A satisficing measure \( \rho \) is coherent if and only if there exists a family of sets of probability measures \( \{ Q(k) : k \in (0, \bar{\rho}] \} \) (absolutely continuous with respect to \( \mathbb{P} \)) satisfying \( Q(k_1) \subseteq Q(k_2) \) for all \( k_1, k_2 \in (0, \bar{\rho}] \), \( k_2 \geq k_1 \), such that

\[
\rho(X) = \sup \{ k \in (0, \bar{\rho}] : \text{E}_Q(X) \geq 0 \ \forall \ Q \in Q(k) \}
\]

(12)

or 0 if no such \( k \) exists.

**Proof.** Implied by Theorem 2 and the representation theorem (7) for coherent risk measures.

This interpretation states that coherent satisficers are expected value maximizers in an *ambiguous* world in which probabilities are not known exactly. Their measure of satisfaction with a position is the maximum level of robustness for which the position breaks at least even in expected value across all probability measures in the ambiguous set. This perspective connects directly back to Simon’s stance that real-world agents do not know probability distributions exactly and therefore are making decisions in inherently ambiguous environments.

As we have noted, the only CSM axiom which probability measure fails to meet is quasi-concavity. Another interpretation, therefore, is that the approach of taking a “convex envelope” of probability measures and the approach of being expected value optimizers in a robust setting in which the probability distribution is ambiguous are in fact one and the same.

### 3.4 CVaR measure

In this section, we consider a CSM with some very interesting properties. This satisficing measure is a dual form of the following coherent risk measure, popularized by Rockafellar and Uryasev [21], among others.

**Definition 5.** For any \( \alpha \in (0, 1] \), the coherent risk measure

\[
\text{CVaR}_\alpha(X) = \inf_{\nu \in \mathbb{R}} \left\{ \nu + \frac{1}{\alpha} \text{E}(\nu - X)^+ \right\}
\]

(13)

is known as the conditional value-at-risk at level \( \alpha \).
It is well-known that, when $X$ has a continuous distribution, that
\[
\text{CVaR}_\alpha(X) = -\mathbb{E}(X \mid X \leq -\text{VaR}_\alpha(X)).
\]
Roughly speaking, we can interpret CVaR as the expected value over the lower $\alpha$-tail of the distribution. As it is a coherent risk measure, it induces a CSM, which we now define.

**Definition 6.** The coherent satisficing measure $\rho : \mathcal{X} \to [0, 1]$
\[
\rho_{\text{CVaR}}(X) = \sup \{1 - \alpha : \text{CVaR}_\alpha(X) \leq 0\}
\]
(or 0 if no such $\alpha$ exists) is called the conditional value-at-risk measure (or CVaR measure) on the target premium $X \in \mathcal{X}$.

Given the interpretation of CVaR, we can, roughly speaking, interpret $\rho_{\text{CVaR}}(X)$ as one minus the smallest left quantile such that the expected value of $X$ over this quantile is nonnegative. If the distribution is continuous, this interpretation is exact.

We consider the interesting case when $\mathbb{P}\{X < 0\} > 0$. Noting that $v + \frac{1}{\alpha} \mathbb{E}((-X - v)^+) > 0$ for all $v \geq 0$, we can express CVaR measures in the form of an expected utility of a concave function as follows:
\[
\rho_{\text{CVaR}}(X) = \sup \{1 - \alpha : \text{CVaR}_\alpha(X) \leq 0\}
= \sup \{1 - \alpha : \exists v < 0 : v + \frac{1}{\alpha} \mathbb{E}((-X - v)^+) \leq 0\}
= \sup \{1 - \alpha : \exists v < 0 : 1 - \alpha \leq 1 - \mathbb{E}((-X/(-v) + 1)^+) \leq 0\}
= \sup_{a \geq 0} \{1 - \mathbb{E}((aX + 1)^+)\}
= \sup_{a \geq 0} \{\mathbb{E}(\min\{aX, 1\})\}.
\]
As such, it is clearer that CVaR measures do not fail to ignore the magnitude of losses, as opposed to probability measures.

We can illustrate this fact with some small examples. Indeed, consider the random variable $X \sim U[-1, 2]$. Clearly, $\mathbb{P}\{X \geq 0\} = 2/3$ and a quick calculation shows $\rho_{\text{CVaR}}(X) = 1/3$. Now consider the random variable $X' = (1/3)(2Y + Z)$, where $Y \sim U[0, 2]$ and $Z \sim U[-2, 0]$ and $Y$ and $Z$ are independent; the only difference between $X$ and $X'$ is that the left tail below zero has been spread out from $[-1, 0]$ in $X$ to $[-2, 0]$ in $X'$. Now, $\mathbb{P}\{X' \geq 0\} = 2/3 = \mathbb{P}\{X \geq 0\}$. On the other hand, we have
\[
\rho_{\text{CVaR}}(X') = \frac{\sqrt{2}(\sqrt{2} - 1)}{3} < \frac{1}{3} = \rho_{\text{CVaR}}(X),
\]
so CVaR measure does in fact recognize the differences in the left tail losses, whereas probability measure does not.

A similar example can be used to illustrate that CVaR measure is sensitive to right-tail changes as well. Consider the same $X$; now let $X'' = (1/3)(Y' + 2Z')$, where $Y' \sim U[-1, 0]$, $Z' \sim U[0, 4]$, and
$Y'$ and $Z'$ are independent. $X''$ is the same as $X$ with the right-tail (gains) expanded. As before, $\mathbb{P}\{X \geq 0\} = \mathbb{P}\{X'' \geq 0\} = 2/3$, but now we find

$$\rho_{\text{CVaR}}(X'') = \frac{4 - \sqrt{2}}{6} > 1/3 = \rho_{\text{CVaR}}(X).$$

In fact, it turns out that CVaR measure is a special QSM when it comes to bounding probability measure; we now define the following.

**Definition 7.** A satisficing measure $\rho$ on $\mathcal{X}$ is law-invariant if $\rho(X) = \rho(Y)$ whenever $X$ and $Y$ have the same distribution under $\mathbb{P}$.

This definition simply extends the notion of law-invariant risk measures defined in, e.g., Föllmer and Schied [12]. Clearly, $\rho_{\text{CVaR}}(X)$ and $\mathbb{P}\{X \geq 0\}$ are law-invariant satisficing measures.

It is well-known (e.g., [12]) that CVaR is the smallest law-invariant risk measure which dominates VaR for all $X \in \mathcal{X}$ over the space $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, P)$. By this we mean not only do we have $\text{CVaR}_\alpha(X) \geq \text{VaR}_\alpha(X)$ for all $X \in \mathcal{X}$, but if $\mu$ is any other convex risk measure, the implication

$$\mu(X) \geq \text{VaR}_\alpha(X) \quad \forall X \in \mathcal{X} \Rightarrow \quad \mu(X) \geq \text{CVaR}_\alpha(X) \quad \forall X \in \mathcal{X}$$

holds. We now show that this idea extends to satisficing measures, i.e., that CVaR measure is the largest lower bound to probability measure over all law-invariant QSMs.

**Theorem 3.** Consider the case when $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$; for any $X \in \mathcal{X}$, we have $\rho_{\text{CVaR}}(X) \leq \mathbb{P}\{X \geq 0\}$. Moreover, for any law-variant quasi-concave satisficing measure $\rho : \mathcal{X} \rightarrow [0, 1]$ the following implication holds

$$\rho(X) \leq \mathbb{P}\{X \geq 0\} \quad \forall X \in \mathcal{X} \Rightarrow \quad \rho(X) \leq \rho_{\text{CVaR}}(X) \quad \forall X \in \mathcal{X}. \quad (15)$$

**Proof.** See Appendix. \hfill \Box

As suggested by our small example above, the gap in this bound can be quite loose; for instance, for a random variable symmetrically distributed around zero, we have $\mathbb{P}\{X \geq 0\} = 1/2$, whereas $\rho_{\text{CVaR}}(X) = 0$. Still, Theorem 3 states that, among law-invariant QSMs, we cannot hope to do any better.

Despite the gap in this bound, if $\mathcal{X}$ is a family of normal random variables, then the set of $X \in \mathcal{X}$ that maximize the probability measure is the same as the set of solutions that maximizes the CVaR measure. Indeed, if $X$ is normally distributed and $\mathbb{E}(X) > 0$,

$$\text{CVaR}_\alpha(X) = -\mathbb{E}(X) + \frac{\phi(\Phi^{-1}(\alpha))}{\xi(\alpha)} \sigma(X),$$

where $\phi(\cdot)$ is the density of a standard normal, and $\Phi(\cdot)$ is the corresponding cumulative distribution function. Now, noting that $\xi(\alpha)$ is a decreasing function of $\alpha$, we see that any $X \in \mathcal{X}$ that maximizes CVaR measure also maximizes the Sharpe ratio, $\mathbb{E}(X)/\sigma(X)$, as well as the probability measure $\mathbb{P}\{X \geq 0\} = \Phi(\mathbb{E}(X)/\sigma(X))$.\hfill 15
4 Portfolio optimization using CSM

We consider an investor who selects a portfolio return that maximizes her probability of achieving at least the risk free rate, $r_f$, which is the return she would have had if she puts all her wealth in the risk free asset. Her portfolio comprises of a nonnegative fraction, $\alpha$, of her wealth in a selected risky asset with random return $X \in \mathcal{X}$ and the remaining portion in the risk free asset. Hence, her portfolio has random return $\alpha X + (1 - \alpha)r_f$. The fraction $\alpha$ is chosen such that the expected return of her effective portfolio achieves the level $\tau$, with $\tau > r_f$. Clearly, if $\tau \leq r_f$, she would invest only in the risk free asset. Therefore, it suffices to focus on the set of random variables $\mathcal{X}$ such that $E(X) > r_f$ for all $X \in \mathcal{X}$. The investor solves the following problem.

$$\max \ P \{\alpha X + (1 - \alpha)r_f \geq r_f\}$$
$$\alpha E(X) + (1 - \alpha)r_f = \tau$$
$$X \in \mathcal{X}, \alpha \geq 0.$$  \hfill (16)

If the returns are normally distributed, then Problem (16) is equivalent to the following:

$$\max \ \Phi \left( \frac{E(X) - r_f}{\sigma(X)} \right)$$
$$\alpha E(X) + (1 - \alpha)r_f = \tau$$
$$X \in \mathcal{X}, \alpha \geq 0,$$  \hfill (17)

where $\Phi(\cdot)$ is the distribution function of a standard normal. Noting that $\Phi(\cdot)$ is an increasing function, this is equivalent to

$$\max \ \frac{E(X) - r_f}{\sigma(X)}$$
$$\alpha E(X) + (1 - \alpha)r_f = \tau$$
$$X \in \mathcal{X}, \alpha \geq 0.$$  \hfill (18)

Notice that the objective does not depend on $\alpha$; this suggests that we can perhaps solve a problem without an expected value constraint of the form $\max \{(E(X) - r_f)/\sigma(X) : X \in \mathcal{X}\}$ then, given a desired expected value $\tau$, simply choose $\alpha$ and mix in the risk-free rate to achieve this desired $\tau$.

Generalizing this idea, we consider an investor who maximizes a CSM, $\rho(\cdot)$ as follows

$$\max \ \rho(\alpha X + (1 - \alpha)r_f - r_f)$$
$$\alpha E(X) + (1 - \alpha)r_f = \tau$$
$$X \in \mathcal{X}, \alpha \geq 0.$$  \hfill (19)

We now formalize the discussion above.

**Theorem 4.** Suppose $E(X) > r_f$ for all $X \in \mathcal{X}$. Let $\rho(\cdot)$ be a CSM and suppose the objective of Problem (19) is strictly positive for a given $\tau > r_f$. Then the corresponding optimal asset position $X^*$ is also optimal to the following problem.

$$\max \ \rho(X - r_f).$$
$$X \in \mathcal{X}.$$  \hfill (20)
Proof. See Appendix.

Theorem 4 implies that, when optimizing over a CSM with the risk-free aspiration level, we may find a single “tangent” portfolio and then mix it with the risk-free asset according to our desired expected return.

Note that if the risky assets are normally distributed, the problem

$$\max \ P \{X \geq r_f\}$$

$$X \in \mathcal{X}.$$  

is equivalent to

$$\max \ \frac{E(X) - r_f}{\sigma(X)}$$

$$X \in \mathcal{X}.$$  

which is the classical result of choosing an asset that maximizes the Sharpe ratio.

### 4.1 Computational example

In this section we solve a computational example based on asset allocation from January, 1986 until March, 2006 using 243 in-sample monthly returns for \( n = 36 \) asset classes. The asset classes we used are listed in Table 1. We solve the problem

$$\text{maximize} \ \ \rho(\tilde{r}'w)$$

$$\text{subject to} \ \ E(\tilde{r})'w \geq \gamma$$

$$e'w = 1$$  

over the portfolio weights decision variable \( w \in \mathbb{R}^{36} \) for various values of \( \gamma \geq 0 \) and for \( \rho = \rho_{\text{CVaR}} \) and \( \rho \) as the Sharpe ratio. Notice that the aspiration level here is effectively set to be zero. The results are shown in Figures 1-4. Note that all returns are expressed as monthly.

Figure 1 shows how the probability of losses increases for both approaches as the expected value parameter \( \gamma \) increases, both. The results reflect Theorem 3 at work, with \( \rho_{\text{CVaR}} \) resulting in substantially lower probabilities of a loss (in sample) compared to Sharpe ratio.

Similarly, Figure 2 shows the expected value of losses conditioned on a loss occurring. Generally, CVaR measure beats out the Sharpe ratio, though, for large enough desired expected returns, the performance does switch over. Finally, Figures 3 and 4 show the return characteristics in two different lights; obviously, in sample, both approaches should have the same overall return, provided the expected value constraints are tight.

The overall insight from this simple example is that CVaR measure, as suggested by Theorem 3, does substantially better than other measures at keeping the probability of loss low, while, at the same time, being simple to compute.

We emphasize that all of these results are in sample. Out of sample results were also explored but the preliminary findings in this case were very sensitive to the expected return level and the number of training samples used. The out of sample performance of CSMs is obviously an important issue, but it is a complicated one beyond the focus of this paper. It is an interesting subject for future research.
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References


**Appendix**

**Proofs**

**Example 2**

*Proof.* Scale invariance: straightforward.

Attainment content: Suppose $X \in \mathcal{X}$ satisfies $X \geq 0$; $X$ must be a constant. Hence, for all $a \in [0, 1)$, we have $-E(X) + \sqrt{\frac{a}{1-a}} \sigma(X) \leq 0$. Therefore, $\rho_S(X) = 1$.

Non-attainment apathy: Suppose $X \in \mathcal{X}$ satisfies $X < 0$; this implies $X$ is a constant. Hence, $-E(X) + \sqrt{\frac{a}{1-a}} \sigma(X) > 0$ for all $a \in [0, 1)$, which is infeasible. Therefore, $\rho_S(X) = 0$.

Gain continuity: Note that

$$
\rho_S(X + c) = \sup_{a \in [0,1)} \left\{ a : -E(X) + \sqrt{\frac{a}{1-a}} \sigma(X) \leq c \right\}
$$

and that $-E(X) + \sqrt{\frac{a}{1-a}} \sigma(X)$ is a nondecreasing function of $a$. Therefore, $\rho_S(X + c)$ is right continuous with respect to $c$.

Monotonicity: Suppose $X, Y \in \mathcal{X}$ satisfy $X \geq Y$. Since, $X - Y \in \mathcal{X}$ and $X - Y \geq 0$, $X - Y$ must therefore be a constant, or equivalently, $\sigma(X - Y) = 0$. Hence, $\sigma(X) \leq \sigma(X - Y) + \sigma(Y) = \sigma(Y)$. Therefore, for all $a \in [0, 1)$,

$$
-E(X) + \sqrt{\frac{a}{1-a}} \sigma(X) \leq -E(Y) + \sqrt{\frac{a}{1-a}} \sigma(Y),
$$

implying $\rho_S(X) \geq \rho_S(Y)$. 

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Quasi-concavity: Let $\beta = \min\{\rho_S(X), \rho_S(Y)\}$. Note that the condition for quasi-concavity is easily satisfied when $\beta = 0$. Moreover, if $\beta = 1$, either $X$ or $Y$ or both should be nonnegative. In which case, quasi-concavity follows from scale invariant and monotonicity, which we have shown earlier. Otherwise, i.e., $\beta \in (0, 1)$, we observe that

$$-E(X) + g(\beta)\sigma(X) \leq 0,$$

and

$$-E(Y) + g(\beta)\sigma(Y) \leq 0,$$

where $g(a) = \sqrt{\frac{a}{1-a}}$. Hence, for all $\lambda \in [0, 1]$ we have

$$-E(\lambda X + (1-\lambda)Y) + g(\beta)\sigma(\lambda X + (1-\lambda)Y) \leq 0.$$

Hence,

$$\rho_S(\lambda X + (1-\lambda)Y) \geq \beta = \min\{\rho_S(X), \rho_S(Y)\}.$$

Finally, to show the probability bound, we focus on the non trivial case when $\rho_S(X) \in (0, 1)$, in which the inequality

$$-E(X) + g(\rho_S(X))\sigma(X) \leq 0$$

hold. Observe that

$$\mathbb{P}\{X < 0\} \leq \mathbb{P}\{E(X) - X > g(\rho_S(X))\sigma(X)\} \leq \frac{1}{1 + g(\rho_S(X))^2} \quad \text{[one-sided Chebyshev inequality]}$$

$$= 1 - \rho_S(X).$$

Hence, $\mathbb{P}\{X \geq 0\} \geq \rho_S(X)$. 

\textbf{Example 3}

\textit{Proof.} Let us define

$$\mu_a(X) = \min_{V \in \mathcal{X}} \left\{-E(X - V) + \sqrt{\frac{a}{1-a}}\sigma(X - V) : V \geq 0\right\}$$

for all $a \in [0, 1)$.

Scale invariance: We show that $\mu_a$ is positive homogenous. Indeed, for all $k > 0$

$$\mu_a(kX) = \min_{V \in \mathcal{X}} \left\{-E(kX - V) + \sqrt{\frac{a}{1-a}}\sigma(kX - V) : V \geq 0\right\}$$

$$= \min_{kV \in \mathcal{X}} \left\{-E(kX - kV) + \sqrt{\frac{a}{1-a}}\sigma(kX - kV) : kV \geq 0\right\}$$

$$= \mu_a(kX).$$

Hence, $\mu_a(X) \leq 0$ if and only if $\mu_a(kX) \leq 0$ for some $k > 0$. 

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Attainment content: Suppose $X \geq 0$. Observe that for all $a \in [0, 1),$

$$\mu_a(X) = \min_{V \in \mathcal{X}} \left\{ -E(X - V) + \sqrt{\frac{a}{1-a}} \sigma(X - V) : V \geq 0 \right\}$$

$$\leq \min_{V \in \mathcal{X}} \left\{ -E(X - V) + \sqrt{\frac{a}{1-a}} \sigma(X - V) : V \geq 0, X = V \right\}$$

$$= 0.$$ 

Hence, $\rho_{OS}(X) = 1.$

Non-attainment apathy: Suppose $X < 0$, then it is easy to see that $\mu_a(X) > 0$ for all $a \in [0, 1)$, which leads to infeasibility. Therefore, $\rho_S(X) = 0$.

Gain continuity: Note that for any constants, $c$,

$$\mu_a(X + c) = \min_{V \in \mathcal{X}} \left\{ -E(X - V + c) + \sqrt{\frac{a}{1-a}} \sigma(X - V + c) : V \geq 0 \right\}$$

$$= \min_{V \in \mathcal{X}} \left\{ -E(X - V) + \sqrt{\frac{a}{1-a}} \sigma(X - V) : V \geq 0 \right\} - c.$$ 

Hence,

$$\rho_S(X + c) = \sup_{a \in [0,1)} \{ a : \mu_a(X) \leq c \}.$$ 

Therefore, $\rho_S(X + c)$ is right continuous with respect to $c$ follows from $\mu_a(X)$ being nondecreasing in $a$.

Monotonicity: Suppose $X \geq Y$. Observed that

$$\mu_a(X) = \min_{V \in \mathcal{X}} \left\{ -E(X - V) + \sqrt{\frac{a}{1-a}} \sigma(X - V) : V \geq 0 \right\}$$

$$= \min_{V \in \mathcal{X}} \left\{ -E(Y - (Y - X + V)) + \sqrt{\frac{a}{1-a}} \sigma(X - (Y - X + V)) : V \geq 0 \right\}$$

$$= \min_{W \in \mathcal{X}} \left\{ -E(Y - W) + \sqrt{\frac{a}{1-a}} \sigma(X - W) : W + X - Y \geq 0 \right\}$$

$$\leq \min_{W \in \mathcal{X}} \left\{ -E(Y - W) + \sqrt{\frac{a}{1-a}} \sigma(X - W) : W \geq 0 \right\}$$

$$= \mu_a(Y),$$

implying $\rho_{OS}(X) \geq \rho_{OS}(Y)$.

Quasi-concavity: Let $\beta = \min\{\rho_S(X), \rho_S(Y)\}$. Note that the condition for quasi-concavity is easily satisfied when $\beta = 0$. In this case, quasi-concavity follows from scale invariance and monotonicity, which we have shown earlier. Therefore, for all $\alpha \in (0, \beta)$, there exist $V_\alpha \geq 0$ and $W_\alpha \geq 0$ such

$$-E(X - V_\alpha) + g(\alpha)\sigma(X - V_\alpha) \leq 0$$

and

$$-E(Y - W_\alpha) + g(\alpha)\sigma(X - W_\alpha) \leq 0$$
where \( g(a) = \sqrt{\frac{a}{1-a}} \). Hence, for all \( \lambda \in [0,1] \) we have

\[
-\mathbb{E}(\lambda X + (1-\lambda)Y - Z) + g(\alpha)\sigma(\lambda X + (1-\lambda)Y - Z) \leq 0,
\]

in which \( Z = \lambda V_\alpha + (1-\lambda)W_\alpha \geq 0 \). Hence,

\[
\rho_{OS}(\lambda X + (1-\lambda)Y) \geq \beta = \min\{\rho_{OS}(X), \rho_{OS}(Y)\}.
\]

We note that \( \mu_a(X) \leq -\mathbb{E}(X) + g(a)\sigma(X) \). Hence, \( \rho_{OS}(X) \geq \rho_{OS}(X) \). Finally, to show the probability bound, we focus on the non trivial case when \( \rho_{OS}(X) \in (0,1) \), in which there exists, \( V \geq 0 \) such that

\[
-\mathbb{E}(X - V) + g(\rho_{OS}(X))\sigma(X - V) \leq 0
\]

hold. Observe that

\[
\begin{align*}
\mathbb{P}\{X < 0\} & \leq \mathbb{P}\{X - V < 0\} \\
& \leq \mathbb{P}\{\mathbb{E}(X - V) - (X - V) > g(\rho_{OS}(X))\sigma(X - V)\} \\
& \leq \frac{1}{1 + g(\rho_{OS}(X))^2} \quad \text{[one-sided Chebyshev inequality]} \\
& = 1 - \rho_{OS}(X).
\end{align*}
\]

Hence, \( \mathbb{P}\{X \geq 0\} \geq \rho_{OS}(X) \).

\[\square\]

**Example 4**

**Proof.** This is a special case of Theorem 2 (which we have yet to prove), noting that

\[\{-C_a(X) : a > 0\}\]

is a family of convex risk measures nondecreasing on \( a > 0 \).

\[\square\]

**Example 5**

**Proof.** It is straightforward to see that \( \rho(X) \in [0,1] \). Consider the following step function, \( \mathbb{P}\{X \geq 0\} = \mathbb{E}(u(X)) \), where

\[
u(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{otherwise.}
\end{cases}
\]

Observe that for all \( a \geq 0 \) we have \( f(aX) \leq u(X) \). Hence

\[
\mathbb{P}\{X \geq 0\} = \mathbb{E}(u(X)) \geq \mathbb{E}(f(aX)).
\]

and \( \mathbb{P}\{X \geq 0\} \geq \rho(X) \). Now, we verify that it is a CSM.

Gain continuity: Obviously true.
Attainment content: Suppose $X \geq 0$, we have, $E(f(a(X + \epsilon))) \geq f(a(\epsilon))$. Hence, for all $\epsilon > 0$, we have
\[
\sup_{a \geq 0} E(f(a(X + \epsilon))) \geq \sup_{a \geq 0} f(a(\epsilon)) = 1.
\]
Hence, $\rho(X) = 1$.

Non-attainment apathy: Suppose, $X < 0$, with $f$ being concave, we have $E(f(a(X + \epsilon))) \leq f(aE(X + \epsilon))$ for all $a \geq 0$ and $0 < \epsilon < -E(X)$. The bound attained with $a = 0$. Hence, $\rho(X) = 0$.

Scale invariance: We have
\[
\rho(kX) = \lim_{\epsilon \to 0} \sup_{a \geq 0} E(f(a(k(X + \epsilon)))) = \lim_{\epsilon / k \to 0} \max_{a k > 0} E(f(a(k(X + \epsilon / k)))) = \rho(X).
\]

Monotonicity: Suppose $X \geq Y$, then since $f \in F$ is a nondecreasing function, we have $E(f(a(X + \epsilon))) \geq E(f(a(Y + \epsilon)))$ for all $a \geq 0$, $\epsilon > 0$. Therefore, $\rho(X) \geq \rho(Y)$.

Quasi-concavity: Observe that from the above axioms, quasi-concavity, that is,
\[
\rho(\lambda X + (1 - \lambda)Y) \geq \min\{\rho(X), \rho(Y)\} \quad \forall \lambda \in [0, 1]
\]
is trivially true whenever $X \geq 0$ or $Y \geq 0$. Consequently, we assume $\mathbb{P}\{X < 0\} > 0$ and $\mathbb{P}\{Y < 0\} > 0$. It suffices to show that for all $0 < \epsilon \leq \bar{\epsilon}$, for some $\bar{\epsilon}$ such that
\[
\sup_{a \geq 0} E(f(a(\lambda X + (1 - \lambda)Y + \epsilon))) \leq \min\left\{\sup_{a \geq 0} E(f(a(X + \epsilon))), \sup_{a \geq 0} E(f(a(Y + \epsilon)))\right\}
\]  
(22)

Note that $f(-\infty) = -\infty$. Suppose $\bar{\epsilon}$ satisfies $\mathbb{P}\{X + \bar{\epsilon} < 0\} > 0$ and $\mathbb{P}\{Y + \bar{\epsilon} < 0\} > 0$. We then have
\[
\lim_{a \to -\infty} E(f(a(X + \epsilon))) = \lim_{a \to -\infty} E(f(a(Y + \epsilon))) = -\infty \quad \forall 0 \leq \epsilon \leq \bar{\epsilon}
\]
Moreover, as the functions $E(f(a(X + \epsilon)))$, $E(f(a(Y + \epsilon)))$ are concave in $a$, there exist $\alpha_\epsilon, \beta_\epsilon \geq 0$ such that
\[
E(f(\alpha_\epsilon(X + \epsilon))) = \sup_{a \geq 0} E(f(a(X + \epsilon)))
\]
and
\[
E(f(\beta_\epsilon(Y + \epsilon))) = \sup_{a \geq 0} E(f(a(Y + \epsilon))).
\]
Note that the inequality (22) holds whenever, $\alpha_\epsilon$ or $\beta_\epsilon$ equals zero; hence, we consider $\alpha_\epsilon, \beta_\epsilon > 0$. For any $\lambda \in [0, 1]$, $\alpha_\epsilon, \beta_\epsilon > 0$, let
\[
a_\lambda = \frac{\alpha_\epsilon \beta_\epsilon}{\lambda \beta_\epsilon + (1 - \lambda)\alpha_\epsilon}, \quad \zeta = \frac{\lambda \alpha_\lambda}{\alpha_\epsilon} \geq 0
\]
\[
1 - \zeta = 1 - \frac{\lambda \alpha_\lambda}{\alpha_\epsilon} = \frac{(1 - \lambda)\alpha_\lambda}{\beta_\epsilon} \geq 0.
\]
Hence,
\[
\sup_{a \geq 0} E(f(a(\lambda X + (1 - \lambda)Y + \epsilon))) = \sup_{a \geq 0} E(f(a(\lambda(X + \epsilon) + (1 - \lambda)(Y + \epsilon)))) \\
\geq E(f(a\lambda(X + \epsilon) + a\lambda(1 - \lambda)(Y + \epsilon))) \\
= E(f(\zeta\alpha(X + \epsilon) + (1 - \zeta)\beta(Y + \epsilon))) \\
\geq \min\{E(f(\alpha(X + \epsilon))), E(f(\beta(Y + \epsilon)))\}.
\]

Theorem 1

Proof. Assume that \(\rho\) is in the form (6). Clearly, since \(k = 0\) is always feasible in Problem (6), we have \(\rho(X) \in [0, \bar{\rho}]\) for all \(X \in \mathcal{X}\). Observe that monotonicity follows from \(\mu_k\) being nondecreasing on \(k\). Note that for all \(X < 0\), there exist an \(\epsilon < 0\) such that \(X \leq \epsilon\). Therefore, for \(k \in (0, \bar{\rho}]\), we have
\[
\mu_k(X) \geq \mu_k(\epsilon) = \mu_k(0) - \epsilon > 0,
\]
which is infeasible in problem (6). Hence, \(\rho(X) = 0\) whenever \(X < 0\), satisfying the non-attainment apathy. To satisfy the attainment content axiom, we note that for all \(X \geq 0\),
\[
\mu_{\bar{\rho}}(X) \leq \mu_{\bar{\rho}}(0) = 0.
\]
Hence, \(\rho(X) = \bar{\rho}\) for all \(X \geq 0\). Finally, to show gain continuity, observe that from translation invariance, we have
\[
\rho(X + a) = \sup \{k : \mu_k(X) \leq a, k \in [0, \bar{\rho}]\}.
\]
Noting that \(\mu_k(X)\) is a nondecreasing function with respect to \(k\), it is easy to see that the function \(\rho(X + a)\) is continuous from the right with respect to \(a\).

For the other direction, we define a family of risk measures \(\{\mu_k : k \in [0, \bar{\rho}]\}\) such that
\[
\mu_k(X) = \inf \{a : \rho(X + a) \geq k\},
\]
where \(\rho\) is a satisfying measure. It is clear that \(\mu_k\) is nondecreasing on \(k \in [0, \bar{\rho}]\). Moreover, is easily verified that \(\mu_0 = -\infty\). To verify that \(\mu_k\) is a risk measure, we note the following:

1. Translation invariance: For all \(c\),
\[
\mu_k(X + c) = \inf \{a : \rho(X + c + a) \geq k\} \\
= \inf \{a - c : \rho(X + a) \geq k\} \\
= \mu_k(X) - c.
\]
Finally, to complete the proof we need to show that
\[ \rho(X) = \sup \{ k : \mu_k(X) \leq 0, k \in [0, \bar{\rho}] \}. \]

Indeed, since \( \rho(X + a) \) is right continuous with respect to \( a \), the limit of Problem (23) is achievable. Therefore,
\[
\sup \{ k : \mu_k(X) \leq 0, k \in [0, \bar{\rho}] \} = \sup \{ \rho(X + a) : a \leq 0 \}
\]
which completes the proof.

**Theorem 2**

*Proof.* Assume that \( \rho \) is in the form (6) with \( \{ \mu_k : k \in (0, \bar{\rho}) \} \) a family of convex risk measures. To show that \( \rho \) is a QSM, we only need to show that quasi-concavity holds.

To do this, let \( k^* = \min \{ \rho(X), \rho(Y) \} \). Note that \( \lim_{k \uparrow k^*} \mu_k(X) \leq 0 \) and \( \lim_{k \uparrow k^*} \mu_k(Y) \leq 0 \). Then, using convexity of \( \mu_k \), we have
\[
\rho(\lambda X + (1 - \lambda) Y) = \sup \{ k : \mu_k(\lambda X + (1 - \lambda) Y) \leq 0, k \in [0, \bar{\rho}] \}
\]
\[
\geq \sup \{ k : \lambda \mu_k(X) + (1 - \lambda) \mu_k(Y) \leq 0, k \in [0, \bar{\rho}] \}
\]
\[
\geq k^*
\]
\[
= \min \{ \rho(X), \rho(Y) \}.
\]

When the \( \mu_k \) are, in addition, coherent, scale invariance clearly holds by virtue of positive homogeneity of \( \mu_k \).

For the other direction, we define, as before, a family of risk measures \( \{ \mu_k : k \in [0, \bar{\rho}] \} \) such that
\[
\mu_k(X) = \inf \{ a : \rho(X + a) \geq k \},
\]
where \( \rho \) is a QSM. We have already verified that \( \mu_k \) is a risk measure; we simply need to verify convexity.

Given \( X, Y \in \mathcal{X} \), notice that, by monotonicity of \( \rho \) and the definition of \( \mu_k \), we have for all \( \epsilon > 0 \),
\[
\rho(X + \mu_k(X) + \epsilon) \geq k
\]
and
\[
\rho(Y + \mu_k(Y) + \epsilon) \geq k.
\]

For every \( \lambda \in [0, 1] \), define \( a_\lambda = \lambda \mu_k(X) + (1 - \lambda) \mu_k(Y) \). Then for all \( \epsilon > 0 \),
\[
\rho(\lambda X + (1 - \lambda) Y + a_\lambda + \epsilon) = \rho(\lambda(X + \mu_k(X) + \epsilon) + (1 - \lambda)(Y + \mu_k(Y) + \epsilon))
\]
\[
\geq \min \{ \rho(X + \mu_k(X) + \epsilon), \rho(Y + \mu_k(Y) + \epsilon) \}
\]
\[
\geq k.
\]
Then
\[
\mu_k(\lambda X + (1 - \lambda)Y) = \inf \{ a : \rho(\lambda X + (1 - \lambda)Y + a) \geq k \} \\
\leq a_{\lambda} \\
= \lambda \mu_k(X) + (1 - \lambda) \mu_k(Y).
\]

When \( \rho \) is in addition a CSM, scale invariance also holds, implying
\[
\mu_k(\lambda X) = \inf \{ a : \rho(\lambda X + a) \geq k \} \\
= \inf \{ \lambda a : \rho(\lambda X + \lambda a) \geq k \} \\
= \inf \{ \lambda a : \rho(X + a) \geq k \} \\
= \lambda \mu_k(X),
\]
so \( \mu_k \) is coherent, and we are done.

Theorem 3

Proof. If \( \rho \) is a QSM, Theorem 2 admits a description of the form
\[
\rho(X) = \sup \{ 1 - \alpha : \mu_\alpha(X) \leq 0 \}.
\]
Recall that \( \mathbb{P} \{ X \geq 0 \} = \sup \{ 1 - \alpha : \text{VaR}_\alpha(X) \leq 0 \} \). The relation \( \rho(X) \leq \mathbb{P} \{ X \geq 0 \} \) for all \( X \in \mathcal{X} \) is therefore equivalent to the relation
\[
\left\{ \begin{array}{l}
\sup \{ 1 - \alpha \} \\
\text{subject to} \mu_\alpha(X) \leq 0
\end{array} \right\} \leq \left\{ \begin{array}{l}
\sup \{ 1 - \alpha \} \\
\text{subject to} \text{VaR}_\alpha(X) \leq 0
\end{array} \right\}
\]
(24)
for all \( X \in \mathcal{X} \). We claim that (24) is equivalent to \( \mu_\alpha(X) \geq \text{VaR}_\alpha(X) \) for all \( X \in \mathcal{X}, \alpha \in (0, 1] \). One direction of this claim is obvious; for the other, assume instead that there exists an \( X \in \mathcal{X} \) and an \( \alpha \in (0, 1] \) such that \( \mu_\alpha(X) < \text{VaR}_\alpha(X) \). Now consider the random variable \( X' = X + \text{VaR}_\alpha(X) \); clearly \( X' \in \mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \). We have
\[
\text{VaR}_\alpha(X') = \text{VaR}_\alpha(X + \text{VaR}_\alpha(X)) = \text{VaR}_\alpha(X) - \text{VaR}_\alpha(X) = 0.
\]
On the other hand,
\[
\mu_\alpha(X') = \mu_\alpha(X + \text{VaR}_\alpha(X)) = \mu_\alpha(X) - \text{VaR}_\alpha(X) < 0.
\]
Therefore, there exists an \( \epsilon > 0 \) such that \( X'' = X' - \epsilon \in \mathcal{X} \) satisfies \( \mu_\alpha(X'') \leq 0 \); on the other hand, \( \text{VaR}_\alpha(X'') = \epsilon > 0 \), which implies that \( \mathbb{P} \{ X'' \geq 0 \} < 1 - \alpha \leq \rho(X) \), which verifies the claim.

The result now follows from the well-known fact (e.g., Föllmer and Schied, [12]) that CVaR is the smallest law-invariant convex risk measure which dominates VaR from above on \( L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \); hence any law-invariant QSM \( \rho \) satisfying the condition \( \rho(X) \leq \mathbb{P} \{ X \geq 0 \} \) is generated by a family of law-invariant convex risk measures satisfying \( \mu_\alpha(X) \geq \text{VaR}_\alpha(X) \). Since such a family must satisfy \( \mu_\alpha(X) \geq \text{CVaR}_\alpha(X) \), it follows from our claim that \( \rho(X) \leq \rho_{\text{CVaR}}(X) \).
**Theorem 4**

*Proof.* Let $Z_1$ and $Z_2$ be the optimal objectives of Problems (19) and (20). Suppose $X^*$ is optimal in Problem (19). Since $E(X^*) > r_f$ and $\alpha = \frac{\tau - r_f}{E(X^*) - r_f} > 0$, we have, by scale invariance,

$$\rho(\alpha X^* + (1 - \alpha)r_f - r_f) = \rho(\alpha(X^* - r_f)) = \rho(X^* - r_f).$$

Hence, $Z_2 \geq Z_1$. Now suppose $X^*$ is now the feasible solution to Problem (20). Since $\tau > r_f$ and $E(X) > r_f$, we have $\alpha = \frac{\tau - r_f}{E(X^*) - r_f} > 0$. Hence, $(\alpha, X^*)$ is clearly feasible in Problem (19) and correspondingly, it yields the same objective value of $Z_2$. Hence, $Z_2 = Z_1$. \qed
### Tables and figures

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</table>

Table 1: The various asset classes used in the computational experiment.
Figure 1: Loss probability vs. expected value target.

Figure 2: Expected value of a loss conditioned on a loss vs. expected value target.
Figure 3: Average return vs. expected value target.

Figure 4: Cumulative return over time for expected value target = 1.8%.