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# Dynamic Games with (Almost) Perfect Information*

Wei He[†] Yeneng Sun[‡]

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**Abstract**

This paper aims to solve two fundamental problems on finite or infinite horizon dynamic games with complete information. Under some mild conditions, we prove the existence of subgame-perfect equilibria and the upper hemicontinuity of equilibrium payoffs in general dynamic games with simultaneous moves (i.e., almost perfect information), which go beyond previous works in the sense that stagewise public randomization and the continuity requirement on the state variables are not needed. For alternating move (i.e., perfect-information) dynamic games with uncertainty, we show the existence of *pure-strategy* subgame-perfect equilibria as well as the upper hemicontinuity of equilibrium payoffs, extending the earlier results on perfect-information deterministic dynamic games.

**JEL classification**: C62; C73

**Keywords**: Dynamic games, perfect information, almost perfect information, subgame-perfect equilibrium, atomless transition, atomless reference measure.

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# Contents

# 1 Introduction

Dynamic games with complete information, where players observe the history and move simultaneously or alternately in finite or infinite horizon, arise naturally in many situations. As noted in Chapter 4 of Fudenberg and Tirole (1991) and Harris, Reny and Robson (1995), these games form a general class of regular dynamic games with many applications in economics, political science, and biology. The associated notion of subgame-perfect equilibrium is a fundamental game-theoretic concept. For alternating move games with finite actions, an early result on backward induction (subgame-perfect equilibrium) was presented in Zermelo (1913). For simultaneous move games with finitely many actions and stages, the existence of subgame-perfect equilibria was shown in Selten (1965) while the infinite horizon but finite-action case was covered by Fudenberg and Levine (1983) under the usual continuity at infinity condition.[1]

Since the agents in many economic models need to make continuous choices, it is important to consider dynamic games with general action spaces. However, a simple example without any subgame-perfect equilibrium was presented in Harris, Reny and Robson (1995), where the game has two players in each of the two stages with only one player having a continuous choice set.[2] Thus, the existence of subgame-perfect equilibria under some suitable conditions remains an open problem even for two-stage dynamic games. The purpose of this paper is to prove the existence of subgame-perfect equilibria for general dynamic games with simultaneous or alternating moves in finite or infinite horizon under some suitable conditions. We shall adopt the general forms of intertemporal utilities, requiring neither stationarity nor additive separability.[3]

For deterministic games with perfect information (i.e., the players move alternately), the existence of pure-strategy subgame-perfect equilibria was shown in Börgers (1989, 1991), Fudenberg and Levine (1983), Harris (1985), Hellwig and Leininger (1987), and Hellwig *et al.* (1990) with the model parameters being continuous in actions, extending the early work of Zermelo (1913).[4] However, if the "deterministic" assumption is dropped by introducing a passive player - Nature,

[1]Without the continuity at infinity condition, subgame-perfect equilibria may not exist in infinite-horizon dynamic games with finitely many actions, see Solan and Vieille (2003) for a counterexample.

[2]As noted in Section 2 of Harris, Reny and Robson (1995), such a dynamic game with continuous choices provides a minimal nontrivial counterexample; see also Exercise 13.4 in Fudenberg and Tirole (1991) for another counterexample of Harris.

[3]To be specific, we assume that players's payoffs are functions of the whole histories endowed with the product topology, which does not need to be the discounted summation of the stage payoffs. For details, see Section 2.

[4]Alós-Ferrer and Ritzberger (2016) considered an alternative formulation of dynamic games with perfect information and without Nature, and showed the existence of subgame-perfect equilibrium.

then a pure-strategy subgame-perfect equilibrium need not exist as shown by a four-stage game in Harris, Reny and Robson (1995). In fact, the nonexistence of a mixed-strategy subgame-perfect equilibrium in a five-stage alternating move game was provided by Luttmer and Mariotti (2003).[5] Thus, it is still an open problem to show the existence of (pure or mixed strategy) subgame-perfect equilibria in (finite or infinite horizon) perfect-information dynamic games with uncertainty under some general conditions.

Continuous dynamic games with almost perfect information in the sense that the players move simultaneously have been considered in Harris, Reny and Robson (1995). In such games, there are a finite number of active players and a passive player (Nature), and all the relevant model parameters are assumed to be continuous in both action and state variables (*i.e.*, Nature's moves). It was shown in Harris, Reny and Robson (1995) that subgame-perfect equilibria exist in those extended games obtained from the original games by introducing stagewise public randomization as a correlation device.[6] As mentioned above, they also demonstrated the possible nonexistence of subgame-perfect equilibrium through a simple example.

The first aim of this paper is to resolve the above two open problems (in both finite and infinite horizon) for the class of continuous dynamic games. We assume the state transition in each period (except for those periods with one active player) to be an atomless probability measure for any given history.[7] In Theorems 1 and 2 (and Proposition 1), we present the existence results for subgame-perfect equilibria, and also some regularity properties of the equilibrium payoff correspondences, including compactness and upper hemicontinuity.[8] Note that our model allows the state history to fully influence all the model parameters, and hence covers the case with stagewise public randomization in the sense that the state transition has an additional atomless component that is independently and identically distributed across time, and does not enter the payoffs, state transitions and action correspondences. As a result, we obtain the existence result in Harris, Reny and Robson (1995) as a special case. In addition, we also provide a new existence result for continuous stochastic games in Proposition 2; see Remark 3 for

[5] All those counterexamples show that various issues arise when one considers general dynamic games; see, for example, the discussions in Stinchcombe (2005). In the setting with incomplete information, even the equilibrium notion needs to be carefully treated; see Myerson and Reny (2018).

[6] See also Mariotti (2000) and Reny and Robson (2002).

[7] A probability measure on a separable metric space is atomless if every single point has measure zero.

[8] Such an upper hemicontinuity property in terms of correspondences of equilibrium payoffs, or outcomes, or correlated strategies has been the key for proving the relevant existence results in Börgers (1989, 1991), Harris (1985), Harris, Reny and Robson (1995), Hellwig and Leininger (1987), Hellwig *et al.* (1990), and Mariotti (2000).

discussions.

For dynamic games with almost perfect information, our results allow the players to take mixed strategies. However, for the special class of continuous dynamic games with perfect information,[9] we obtain the existence of *pure-strategy* subgame-perfect equilibria in Theorem 2. When Nature is present, there has been no general result on the existence of equilibria (even in mixed strategies) for continuous dynamic games with perfect information. Our Theorem 2 provides a new existence result in pure strategy, which extends the results of Börgers (1989), Fudenberg and Levine (1983), Harris (1985), Hellwig and Leininger (1987), Hellwig *et al.* (1990), and Zermelo (1913) to the case when Nature is present.

The condition of atomless transitions is minimal in the particular sense that the existence results for continuous dynamic games may fail to hold if (1) the passive player, Nature, is not present in the model as shown in Harris, Reny and Robson (1995), or (2) with the presence of Nature, the state transition is not atomless even at one point of history as shown in Luttmer and Mariotti (2003).

The second aim of this paper is to consider an important extension in which the relevant model parameters are assumed to be continuous in actions, but measurable in states.[10] In particular, we show the existence of a subgame-perfect equilibrium in a general dynamic game with almost perfect information under some suitable conditions on the state transitions. Theorems 3 and 4 below go beyond our results on continuous dynamic games by dropping the continuity requirement on the state variables.[11] We work with the condition that the state transition in each period (except for those periods with one active player) have a component with a suitable density function with respect to some atomless reference measure.

In Appendix A, we provide a complete proof of Theorem 1, and point out those changes that are needed for proving Theorem 2 and Proposition 1. We follow the standard three-step procedure in obtaining subgame-perfect equilibria of dynamic games, namely, backward induction, forward induction, and approxi-

[9] Dynamic games with perfect information also have wide applications. For example, see Amir (1996) and Phelps and Pollak (1968) for an intergenerational bequest game, and Goldman (1980) and Peleg and Yaari (1973) for intrapersonal games in which consumers have changing preferences.

[10] Since the agents need to make optimal choices, the continuity assumption in terms of actions is natural and widely adopted. However, the state variable is not a choice variable, and thus it is unnecessary to impose the state continuity requirement in a general model. Note that the state measurability assumption is the minimal regularity condition one would expect for the model parameters. We may also point out that the proof for the case with state continuity in Appendix A is much simpler than the proof for the general case in Appendix B. For discussions on subgame perfect $\epsilon$-equilibria in dynamic games without the continuity conditions in actions, see Solan and Vieille (2003), Flesch *et al.* (2010), Laraki, Maitra and Sudderth (2013), Flesch and Predtetchinski (2016), and the references therein.

[11] In Appendix B, we also present a new existence results on subgame-perfect equilibria for a general stochastic game.

mation of infinite horizon by finite horizon. Because we drop the stagewise public randomization, new technical difficulties arise in the proofs. The main purpose of the step of backward induction is to show that if the payoff correspondence at a given stage satisfies certain regularity properties, then the equilibrium payoff correspondence at the previous stage is upper hemicontinuous. We notice that the condition of atomless transitions suffices for this purpose, and hence the exogenous stagewise public randomization is not needed for this step. For the step of forward induction, we need to obtain strategies that are jointly measurable in history. When there is a public randomization device, the joint measurability follows from the measurable version of Skorokhod's representation theorem and implicit function theorem respectively as in Harris, Reny and Robson (1995) and Reny and Robson (2002). Here we need to work with the deep "measurable" measurable choice theorem of Mertens (2003).

In Appendix B, we prove Theorem 3 first, and then describe the needed changes for proving Theorem 4 and Proposition B.1. The proofs for the results in measurable dynamic games are much more difficult than those in the case of continuous dynamic games. In the step of backward induction, we obtain a new existence result for discontinuous games with stochastic endogenous sharing rules, which extends the main result of Simon and Zame (1990) by allowing the payoff correspondence to be measurable (instead of upper hemicontinuous) in states.[12] In order to extend the results to the infinite horizon setting, we need to handle various subtle measurability issues due to the lack of continuity in the state variables in the more general model, which is the most difficult part of the proof for Theorem 3.[13]

The rest of the paper is organized as follows. The model is presented in Section 2. In Section 3, we provide a variation of the counterexample in Luttmer and Mariotti (2003) to demonstrate the key issues. The results for continuous dynamic games are given in Section 4. Section 5 extends continuous dynamic games to the setting in which the model parameters may only be measurable in the state variables. The proofs for the results of continuous dynamic games and measurable dynamic games are left in Appendices A and B, respectively.

[12]In Simon and Zame (1990), the payoff is assumed to be a correspondence that is bounded, upper hemicontinuous, with nonempty, convex, and compact values. Note that the upper hemicontinuity condition on a correspondence is equivalent to the fact that the lower inverse of any closed set is closed; see Aliprantis and Border (2006, Lemma 17.4). On the other hand, the measurability condition on a correspondence means that the lower inverse of any closed set is measurable (see Section 6.1). Thus, an upper hemicontinuous correspondence is automatically measurable. For more discussions of the approach in Simon and Zame (1990), see Harris, Stinchcombe and Zame (2005) and Stinchcombe (2005).

[13]Because our relevant model parameters are only measurable in the state variables, the usual method of approximating a limit continuous dynamic game by a sequence of finite games, as used in Börgers (1991), Harris, Reny and Robson (1995) and Hellwig *et al.* (1990), is not applicable in this setting.

# 2 Model

In this section, we shall present the model for an infinite-horizon dynamic game with almost perfect information.

The set of players is $I_0 = \{0, 1, \ldots, n\}$, where the players in $I = \{1, \ldots, n\}$ are active and player 0 is Nature. All the players move simultaneously. Time is discrete, and indexed by $t = 0, 1, 2, \ldots$.

The set of starting points is a product space $H_0 = X_0 \times S_0$, where $X_0$ is a compact metric space and $S_0$ is a Polish space (*i.e.*, a complete separable metric space).[14] At stage $t \geq 1$, player $i$'s action will be chosen from a subset of a Polish space $X_{ti}$ for each $i \in I$, and $X_t = \prod_{i \in I} X_{ti}$. Nature's action is chosen from a Polish space $S_t$. Let $X^t = \prod_{0 \leq k \leq t} X_k$ and $S^t = \prod_{0 \leq k \leq t} S_k$. The Borel $\sigma$-algebras on $X_t$ and $S_t$ are denoted by $\mathcal{B}(X_t)$ and $\mathcal{B}(S_t)$, respectively. Given $t \geq 0$, a history up to the stage $t$ is a vector[15]

$$h_t = (x_0, s_0, x_1, s_1, \ldots, x_t, s_t) \in X^t \times S^t.$$

The set of all such possible histories is denoted by $H_t$. For any $t \geq 0$, $H_t \subseteq X^t \times S^t$.

For any $t \geq 1$ and $i \in I$, let $A_{ti}$ be a measurable, nonempty and compact valued correspondence[16] from $H_{t-1}$ to $X_{ti}$ such that $A_{ti}(h_{t-1})$ is the set of available actions for player $i \in I$ given the history $h_{t-1}$. Let $A_t = \prod_{i \in I} A_{ti}$. Then $H_t = \mathrm{Gr}(A_t) \times S_t$, where $\mathrm{Gr}(A_t)$ is the graph of $A_t$.

For any $x = (x_0, x_1, \ldots) \in X^\infty$, let $x^t = (x_0, \ldots, x_t) \in X^t$ be the truncation of $x$ up to the period $t$. Truncations for $s \in S^\infty$ can be defined similarly. Let $H_\infty$ be the subset of $X^\infty \times S^\infty$ such that $(x, s) \in H_\infty$ if $(x^t, s^t) \in H_t$ for any $t \geq 0$. Then $H_\infty$ is the set of all possible histories in the game.[17] Hereafter, let $H_\infty$ be endowed with the product topology. For any $t \geq 1$, Nature's action is given by a Borel measurable mapping $f_{t0}$ from the history $H_{t-1}$ to $\mathcal{M}(S_t)$, where $\mathcal{M}(S_t)$ denotes the set of all Borel probability measures on $S_t$ and is endowed with the topology of weak convergence of measures on $S_t$.

For each $i \in I$, the payoff function $u_i$ is a bounded Borel measurable mapping from $H_\infty$ to $\mathbb{R}_{++}$. Without loss of generality, we can assume that the payoff

[14] In each stage $t \geq 1$, there will be a set of action profiles $X_t$ and a set of states $S_t$. Without loss of generality, we assume that the set of initial points is also a product space for notational consistency.

[15] By abusing the notation, we also view $h_t = (x_0, s_0, x_1, s_1, \ldots, x_t, s_t)$ as the vector $(x_0, x_1, \ldots, x_t, s_0, s_1, \ldots, s_t)$ in $X^t \times S^t$.

[16] Suppose that $Y$ and $Z$ are both Polish spaces, and $\Psi$ is a correspondence from $Y$ to $Z$. Hereafter, the measurability of $\Psi$, unless specifically indicated, is assumed to be the weak measurability with respect to the Borel $\sigma$-algebra $\mathcal{B}(Y)$ on $Y$. For the definitions and detailed discussions, see Section 6.1.

[17] A finite horizon dynamic game can be regarded as a special case of an infinite horizon dynamic game in the sense that the action correspondence $A_{ti}$ is point-valued for each player $i \in I$ and $t \geq T$ for some stage $T \geq 1$; see, for example, Börgers (1989) and Harris, Reny and Robson (1995).

function $u_i$ is bounded from above by some $\gamma > 0$ for each $i \in I$.

For player $i \in I$, a strategy $f_i$ is a sequence $\{f_{ti}\}_{t \geq 1}$ such that $f_{ti}$ is a Borel measurable mapping from $H_{t-1}$ to $\mathcal{M}(X_{ti})$ with $f_{ti}(A_{ti}(h_{t-1})|h_{t-1}) = 1$ for all $h_{t-1} \in H_{t-1}$. That is, player $i$ can only take the mixed strategy concentrated on the available set of actions $A_{ti}(h_{t-1})$ given the history $h_{t-1}$. A strategy profile $f = \{f_i\}_{i \in I}$ is a combination of strategies of all active players.

In any subgame, a strategy combination will generate a probability distribution over the set of possible histories. This probability distribution is called the path induced by the strategy combination in this subgame. Before describing how a strategy combination induces a path in Definition 1, we need to define some technical terms. Given a strategy profile $f = \{f_i\}_{i \in I}$, denote $\otimes_{i \in I_0} f_{(t'+1)i}$ as a transition probability from the set of histories $H_{t'}$ to $\mathcal{M}(X_{t'+1})$. For the notational simplicity later on, we assume that $\otimes_{i \in I_0} f_{(t'+1)i}(\cdot|h_{t'})$ represents the strategy profile in stage $t'+1$ for a given history $h_{t'} \in H_{t'}$, where $\otimes_{i \in I_0} f_{(t'+1)i}(\cdot|h_{t'})$ is the product of the probability measures $f_{(t'+1)i}(\cdot|h_{t'})$, $i \in I_0$. If $\lambda$ is a finite measure on $X$ and $\nu$ is a transition probability from $X$ to $Y$, then $\lambda \diamond \nu$ is a measure on $X \times Y$ such that $\lambda \diamond \nu(A \times B) = \int_A \nu(B|x)\lambda(\mathrm{d}x)$ for any measurable subsets $A \subseteq X$ and $B \subseteq Y$.

**Definition 1.** *Suppose that a strategy profile $f = \{f_i\}_{i \in I}$ and a history $h_t \in H_t$ are given for some $t \geq 0$. Let $\tau_t = \delta_{h_t}$, where $\delta_{h_t}$ is the probability measure concentrated at the point $h_t$. If $\tau_{t'} \in \mathcal{M}(H_{t'})$ has already been defined for some $t' \geq t$, then let*

$$\tau_{t'+1} = \tau_{t'} \diamond (\otimes_{i \in I_0} f_{(t'+1)i}).$$

*Finally, let $\tau \in \mathcal{M}(H_\infty)$ be the unique probability measure on $H_\infty$ such that $\mathrm{Marg}_{H_{t'}} \tau = \tau_{t'}$ for all $t' \geq t$. Then $\tau$ is called the path induced by $f$ in the subgame $h_t$. For all $i \in I$, $\int_{H_\infty} u_i \,\mathrm{d}\tau$ is the payoff of player $i$ in this subgame.*

# 3 An example

As mentioned in the introduction, Luttmer and Mariotti (2003) presented a simple five-stage alternating move game which does not possess any subgame-perfect equilibrium. Below, we shall modify their counterexample to illustrate what could go wrong in a continuous dynamic game, and use this example to demonstrate some key issues.

Fix $0 \leq \epsilon \leq 1$. The game $G_\epsilon$ proceeds in five stages.

- In stage 1, player 1 chooses an action $a_1 \in [0, 1]$.
- In stage 2, player 2 chooses an action $a_2 \in [0, 1]$.

- In stage 3, Nature chooses some $x \in [-2-\epsilon+a_1+a_2, 2+\epsilon-a_1-a_2]$ based on the uniform distribution $\eta^{\epsilon}_{(a_1,a_2)}$.
- After Nature's choice, players 3 and 4 move sequentially. The subgame following a history $(a_1, a_2, x)$ and the associated payoffs for all four active players are shown in Figure 1, where

$$\gamma(x,\epsilon) = \begin{cases} x+\epsilon, & \text{if } x < -\epsilon; \\ x-\epsilon, & \text{if } x > \epsilon; \\ 0, & x \in [-\epsilon, \epsilon]. \end{cases}$$

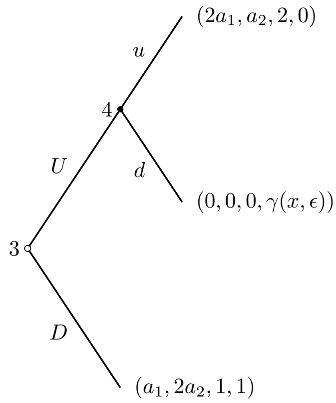


Figure 1: The subgame $(a_1, a_2, x)$.

In the following, let $\alpha$ and $\beta$ be the probabilities with which players 3 and 4 choose $U$ and $u$, respectively. Consider a subgame $(a_1, a_2, x)$. Let $\tilde{P}^{\epsilon}_3(a_1, a_2, x)$ (resp. $\tilde{P}^{\epsilon}_2(a_1, a_2)$) be the set of expected payoffs for players 1 and 2 in stage 3 (resp. stage 2).[18]

- If $x < -\epsilon$, then the equilibrium continuation path is $(U, u)$ (*i.e.*, $\alpha = 1$ and $\beta = 1$), and $\tilde{P}^{\epsilon}_3(a_1, a_2, x) = \{(2a_1, a_2)\}$.
- If $x > \epsilon$, then the equilibrium continuation path is $D$ (*i.e.*, $\alpha = 0$ and $\beta = 0$), and $\tilde{P}^{\epsilon}_3(a_1, a_2, x) = \{(a_1, 2a_2)\}$.
- If $x \in [-\epsilon, \epsilon]$, then the set of equilibrium continuation paths is characterized by three segments of mixing probabilities: $\alpha = 0$ and $\beta \in [0, \frac{1}{2}]$; $\alpha \in [0, 1]$ and $\beta = \frac{1}{2}$; and $\alpha = 1$ and $\beta \in [\frac{1}{2}, 1]$. Then

$$\tilde{P}^{\epsilon}_3(a_1, a_2, x) = \big\{(a_1, (2-\frac{3}{2}\alpha)a_2) | \alpha \in [0,1]\big\} \cup \big\{(2a_1\beta, a_2\beta) | \beta \in [\frac{1}{2}, 1]\big\},$$

which is not convex when $a_1 > 0$ and $a_2 > 0$.

[18] For simplicity, we focus on the equilibrium payoffs of players 1 and 2.

Denote $D = \{(a_1, (2 - \frac{3}{2}\alpha)a_2) | \alpha \in [0,1]\} \cup \{(2a_1\beta, a_2\beta) | \beta \in [\frac{1}{2}, 1]\}$, and Co$(D)$ as the convex hull of the set $D$. Below, for $a_1 > 0$ and $a_2 > 0$, $D$ is the union of the two segments in Figure 2, and Co$(D)$ is the dashed area with boundaries in Figure 3.

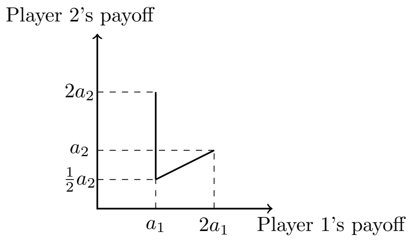


Figure 2: $D$

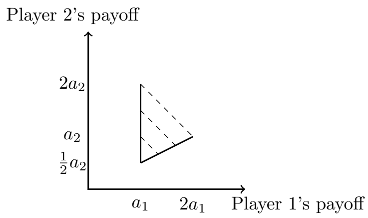


Figure 3: Co$(D)$

Fix any $\epsilon > 0$. Nature's move $x$ is uniformly distributed on the nondegenerate interval $[-2 - \epsilon + a_1 + a_2, 2 + \epsilon - a_1 - a_2]$, which is symmetric around zero for any $(a_1, a_2)$. The correspondence $\tilde{P}_3^\epsilon(a_1, a_2, x)$ is upper hemicontinuous, but is not convex valued when $x \in [-\epsilon, \epsilon]$ and $a_1, a_2 > 0$ (the set $D$ is not convex in this case). Given any $(a_1, a_2)$, the set of expected equilibrium continuation payoffs for players 1 and 2 is:[19]

$$
\begin{aligned}
\tilde{P}_2^\epsilon(a_1, a_2) &= \int_{-3}^{3} \tilde{P}_3^\epsilon(a_1, a_2, x)\eta^\epsilon_{(a_1,a_2)}(\mathrm{d}x) \\
&= \int_{-3}^{-\epsilon} \tilde{P}_3^\epsilon(a_1, a_2, x)\eta^\epsilon_{(a_1,a_2)}(\mathrm{d}x) + \int_{-\epsilon}^{\epsilon} \tilde{P}_3^\epsilon(a_1, a_2, x)\eta^\epsilon_{(a_1,a_2)}(\mathrm{d}x) \\
&\quad + \int_{\epsilon}^{3} \tilde{P}_3^\epsilon(a_1, a_2, x)\eta^\epsilon_{(a_1,a_2)}(\mathrm{d}x) \\
&= \frac{\epsilon}{2 + \epsilon - a_1 - a_2}\mathrm{Co}(D) + \frac{2 - a_1 - a_2}{2 + \epsilon - a_1 - a_2}\left\{(\frac{3}{2}a_1, \frac{3}{2}a_2)\right\} \\
&= \int_{-3}^{-\epsilon} \mathrm{Co}(\tilde{P}_3^\epsilon(a_1, a_2, x))\eta^\epsilon_{(a_1,a_2)}(\mathrm{d}x) + \int_{-\epsilon}^{\epsilon} \mathrm{Co}(\tilde{P}_3^\epsilon(a_1, a_2, x))\eta^\epsilon_{(a_1,a_2)}(\mathrm{d}x) \\
&\quad + \int_{\epsilon}^{3} \mathrm{Co}(\tilde{P}_3^\epsilon(a_1, a_2, x))\eta^\epsilon_{(a_1,a_2)}(\mathrm{d}x) \\
&= \int_{-3}^{3} \mathrm{Co}(\tilde{P}_3^\epsilon(a_1, a_2, x))\eta^\epsilon_{(a_1,a_2)}(\mathrm{d}x).
\end{aligned}
$$

For $x \in [-2 - \epsilon + a_1 + a_2, -\epsilon)$ or $(\epsilon, 2 + \epsilon - a_1 - a_2]$, $\tilde{P}_3^\epsilon(a_1, a_2, x)$ is a singleton, and hence is convex valued and coincides with Co$(\tilde{P}_3^\epsilon(a_1, a_2, x))$. For $x \in [-\epsilon, \epsilon]$, $\tilde{P}_3^\epsilon(a_1, a_2, x)$ is the set $D$. Its integration on $[-\epsilon, \epsilon]$ under the uniform distribution

[19] Given two sets $D_1, D_2 \subseteq \mathbb{R}^l$, $D_1 + D_2 = \{d_1 + d_2 : d_i \in D_i, i = 1, 2\}$; for $c \in \mathbb{R}$, $cD_1 = \{cd_1 : d_1 \in D_1\}$.

is simply $\mathrm{Co}(D)$, and hence coincides with $\mathrm{Co}(\tilde{P}_3^\epsilon(a_1,a_2,x))$. It is also clear that $\tilde{P}_2^\epsilon(a_1,a_2)$ is upper hemicontinuous in $(a_1,a_2)\in[0,1]\times[0,1]$.

There are two general observations: (1) the integral of a correspondence coincides with the integral of the convex hull of the correspondence based on an atomless measure, and (2) the integral of a convex valued, upper hemicontinuous correspondence based on a continuous transition probability is still upper hemicontinuous.[20] As a result, the integral of an upper hemicontinuous correspondence is still upper hemicontinuous and convex valued based on an *atomless* continuous transition probability.[21] In the particular case of this example, the above paragraph shows that even though $\tilde{P}_3^\epsilon(a_1,a_2,x)$ is not always convex valued on the nondegenerate set $[-\epsilon,\epsilon]$, $\tilde{P}_2^\epsilon(a_1,a_2)$ is still convex valued and upper hemicontinuous. Such a result also follows from the general observations since Nature's move $\eta^\epsilon_{(a_1,a_2)}$ is atomless and continuous in $(a_1,a_2)$. Here is a pure-strategy subgame-perfect equilibrium in the game $G_\epsilon$ for $\epsilon>0$: players 1 and 2 choose $a_1=1$ and $a_2=1$, players 3 and 4 choose $U$ and $u$ when $x<0$, $D$ and $d$ when $x\geq 0$. In this equilibrium, both players 1 and 2 get the payoff $\frac{3}{2}$.

For the case $\epsilon=0$, the game $G_0$ is the counterexample in Luttmer and Mariotti (2003), which does not have any subgame-perfect equilibrium. If $a_1+a_2<2$, then Nature's move $x$ is uniformly distributed on the nondegenerate interval $[-2+a_1+a_2, 2-a_1-a_2]$. As $x=0$ is drawn with probability 0, the non-convexity of the set of continuation payoffs for players 1 and 2 at $x=0$ does not matter. The expected continuation payoffs for players 1 and 2 are $\frac{3}{2}a_1$ and $\frac{3}{2}a_2$, respectively. That is, $\tilde{P}_2^0(a_1,a_2)=\{(\frac{3}{2}a_1,\frac{3}{2}a_2)\}$ when $a_1+a_2<2$. If $a_1+a_2=2$ (*i.e.*, $a_1=a_2=1$), then Nature's move must be $x=0$, and hence

$$\tilde{P}_2^0(1,1)=\tilde{P}_3^0(1,1,0)=\big\{(1,(2-\frac{3}{2}\alpha))|\alpha\in[0,1]\big\}\cup\big\{(2\beta,\beta)|\beta\in[\frac{1}{2},1]\big\}.$$

Whenever $a_1+a_2<2$, both players 1 and 2 have the incentive to choose their actions as close to 1 as possible, which gives them the expected payoff arbitrarily close to $\frac{3}{2}$. However, when both players 1 and 2 choose the action 1, some of them shall get a payoff no more than 1. This implies that there does not exist any subgame-perfect equilibrium.

As shown above, for any $\epsilon>0$, both players 1 and 2 in the game $G_\epsilon$ have $\frac{3}{2}$ as their equilibrium payoffs. Since $(\frac{3}{2},\frac{3}{2})$ cannot be the equilibrium payoffs of players 1 and 2 in the game $G_0$, the equilibrium payoff correspondence of the games $G_\epsilon,\epsilon\geq 0$ is not upper hemicontinuous at $\epsilon=0$.

Note that Nature's move $\eta^0_{(a_1,a_2)}$ is continuous in $(a_1,a_2)\in[0,1]\times[0,1]$, and

[20] See Lemma 7 in Section 6.1.

[21] Note that when the transition probability has an atom in its values, both properties may not be true. This is demonstrated in the case $\epsilon=0$ below.

atomless except for the one point $(a_1, a_2) = (1, 1)$. Footnote 21 indicates that the integral of an upper hemicontinuous correspondence with respect to such a transition probability may not be upper hemicontinuous and convex valued. Indeed, even though $\tilde{P}_3^0$ is an upper hemicontinuous correspondence, $\tilde{P}_2^0$ is neither upper hemicontinuous nor convex at the point $(1, 1)$. In particular, $(\frac{3}{2}a_1, \frac{3}{2}a_2) \in \tilde{P}_2^0(a_1, a_2)$ when $a_1 + a_2 < 2$, while its limit $(\frac{3}{2}, \frac{3}{2}) \notin \tilde{P}_2^0(1, 1)$ when both $a_1$ and $a_2$ converge to 1.

# 4 Continuous dynamic games

In this section, we consider continuous dynamic games in the sense that all the model parameters (the payoff functions, state transitions and action correspondences) are continuous in both action and state variables. We shall show that subgame-perfect equilibria exist for continuous dynamic games under the condition of atomless transitions. In Sections 4.1, we first consider dynamic games with almost perfect information, and show the existence of subgame-perfect equilibria. In Section 4.2, we consider dynamic games with perfect information in the sense that players move sequentially, and prove the existence of *pure-strategy* subgame-perfect equilibria. In Subsetion 4.3, we provide a roadmap for proving Theorems 1 and 2. The details of the proofs are left in Appendix A. In Section 4.4, we extend the model so that the previous existence results for continuous dynamic games with perfect and almost perfect information are covered as special cases. As a byproduct, we provide a new existence result for continuous stochastic games.

## 4.1 Continuous dynamic games with almost perfect information

In this subsection, we study an infinite-horizon continuous dynamic game with almost perfect information. Intuitively, we work with the class of games in which all the relevant parameters of the game, including action correspondences, Nature's move and payoff functions, vary smoothly with respect to the state and action variables. In particular, a dynamic game is said to be "continuous" if for each $t$ and $i$,

1. the action correspondence $A_{ti}$ is continuous on $H_{t-1}$;[22]
2. the transition probability $f_{t0}$ is a continuous mapping from $H_{t-1}$ to $\mathcal{M}(S_t)$, where $\mathcal{M}(S_t)$ is endowed with the topology of weak convergence (also called

[22] A correspondence is said to be continuous if it is both upper hemicontinuous and lower hemicontinuous. For definitions and detailed discussion, see Section 6.1.

the weak star topology); that is, for any bounded continuous function $\psi$ on $S_t$, the integral

$$\int_{S_t} \psi(s_t) f_{t0}(\mathrm{d}s_t | h_{t-1})$$

is continuous in $h_{t-1}$;

3. the payoff function $u_i$ is continuous on $H_\infty$.

Below, we propose the condition of "atomless transitions" on the state space, which means that Nature's move is an atomless probability measure in any stage.

**Assumption 1** (Atomless Transitions)**.** *For each $t \geq 1$, $f_{t0}(h_{t-1})$ is an atomless Borel probability measure for each $h_{t-1} \in H_{t-1}$.*

The notion of subgame-perfect equilibrium is given below. It requires each player's strategy to be optimal in every subgame given the strategies of all other players.

**Definition 2** (SPE)**.** *A subgame-perfect equilibrium is a strategy profile $f$ such that for all $i \in I$, $t \geq 0$, and all $h_t \in H_t$, player $i$ cannot improve his payoff in the subgame $h_t$ by a unilateral change in his strategy.*

Let $E_t(h_{t-1})$ be the set of subgame-perfect equilibrium payoffs in the subgame $h_{t-1}$. The following result shows that a subgame-perfect equilibrium exists, and the equilibrium correspondence $E_t$ satisfies certain desirable compactness and upper hemicontinuity properties.

**Theorem 1.** *If a continuous dynamic game with almost perfect information has atomless transitions, then it possesses a subgame-perfect equilibrium. In addition, $E_t$ is nonempty and compact valued, and upper hemicontinuous on $H_{t-1}$ for any $t \geq 1$.*

**Remark 1.** *Theorem 1 goes beyond the main result of Harris, Reny and Robson (1995) for continuous dynamic games, where the existence of subgame-perfect equilibria was shown for those extended games obtained from the original games by introducing stagewise public randomization as a correlation device. Such a correlation device does not influence the payoffs, transitions or action correspondences. It is clear that the extended games with stagewise public randomization as in Harris, Reny and Robson (1995) automatically satisfy the condition of atomless transitions. The states in our model are completely endogenous in the sense that they can affect all the model parameters such as payoffs, transitions, and action correspondences.*

## 4.2 Continuous dynamic games with perfect information

In this subsection, we consider another important class of continuous dynamic games, namely continuous dynamic games with perfect information (with or without Nature). In such games, players move sequentially. We show the existence of pure-strategy subgame perfect equilibria. In particular, the condition of atomless transitions is imposed only when Nature moves.

In a continuous dynamic game with perfect information, there is only one player moving in each stage. In stage $t$, if $A_{ti}$ is not point valued for some player $i \in I$, then $A_{tj}$ is point valued for any $j \in I$ as long as $j \neq i$, and $f_{t0}(h_{t-1}) \equiv \delta_{s_t}$ for some $s_t$. That is, only player $i$ is active in stage $t$, while all the other players are inactive. If the state transition $f_{t0}$ does not put probability 1 on some point, then $A_{ti}$ must be point valued for any $i \in I$. That is, only Nature can move in stage $t$, and all the players $i \in I$ are inactive in this stage. A continuous dynamic game with perfect information is said to have atomless transitions if $f_{t0}(h_{t-1})$ is an atomless Borel probability measure when only Nature moves in the stage $t$.

**Theorem 2.** *If a continuous dynamic game with perfect information has atomless transitions, then it possesses a pure-strategy subgame-perfect equilibrium. In addition, $E_t$ is nonempty and compact valued, and upper hemicontinuous on $H_{t-1}$ for any $t \geq 1$.*

**Remark 2.** *As shown in Börgers (1989), Fudenberg and Levine (1983), Harris (1985), Hellwig and Leininger (1987), Hellwig et al. (1990), and Zermelo (1913), pure-strategy subgame-perfect equilibria exist in deterministic (i.e., without Nature) continuous dynamic games with perfect information. Theorem 2 extends those existence results to the case with Nature. We may point out that the condition of atomless transitions in Theorem 2 is minimal. In particular, the games in Section 3 can be viewed as an alternating move game with a starting point $\epsilon \in [0,1]$, where the transition probability $\eta^{\epsilon}_{(a_1,a_2)}$ in the third period is continuous in $(\epsilon, a_1, a_2) \in [0,1]^3$, and atomless except for the one point $(\epsilon, a_1, a_2) = (0,1,1)$. The violation of our condition of atomless transitions at just one point leads to the failure of the conclusions of Theorem 2.*[23]

[23]On the other hand, Remark 4 indicates that Theorem 2 can be generalized to the case when the state transitions either are atomless, or have the support inside a fixed finite set irrespective of the history at a particular stage.

## 4.3 A roadmap for proving Theorems 1 and 2

The existence results are established in three steps. The backward induction step aims to show that if the equilibrium payoff correspondence $Q_t$ in stage $t$ is well-behaved (bounded, nonempty and compact valued, and upper hemicontinuous), then these desirable properties can be preserved for the equilibrium payoff correspondence $Q_{t-1}$ in the previous stage $t-1$. As will be explained, the atomless transition condition plays an important role in this step. Next, given the equilibrium payoff correspondences across different periods, one needs to construct the equilibrium strategy profile stage by stage that is consistent with the equilibrium payoff correspondences. This is done in the forward induction step. The first two steps together prove the equilibrium existence results for finite-horizon dynamic games. The last step relates finite-horizon dynamic games to infinite-horizon dynamic games based on the condition of continuity at infinity. We shall sketch the main ideas of the proof based on simultaneous-move games, and point out the modifications for alternating-move games whenever necessary.

(1) We explain the first (backward induction) step via a $T$-stage dynamic game. Let $Q_{(T+1)}(h_T)$ be the singleton set with one element vector $(u_1(h_T), \ldots, u_n(h_T))$ for any $T$-stage history $h_T$, where $u_i$ is the payoff function of player $i$ at the last stage $T$ (a bounded continuous function from the space of complete histories $H_T$ to $\mathbb{R}$). Hence, $Q_{(T+1)i}$ is a bounded, nonempty and compact valued, and upper hemicontinuous correspondence. Given a history $h_{T-1}$ at stage $T-1$, the state $s_T$ at stage $T$ follows the distribution $f_{T0}(\cdot|h_{T-1})$. For an action profile $x_T$ and a state $s_T$ at stage $T$, $(h_{T-1}, x_T, s_T)$ is a history at stage $T$; let

$$P_T(h_{T-1}, x_T) = \int_{S_T} Q_{T+1}(h_{T-1}, x_T, s_T) f_{T0}(\mathrm{d}s_T|h_{T-1}).$$

Then $P_T(h_{T-1}, \cdot)$ is the set of expected possible payoff vectors in the subgame $h_{T-1}$.

Let $\Phi(Q_{T+1})(h_{T-1})$ be the set of all mixed-strategy Nash equilibrium payoffs for the game with the action set $A_{Ti}(h_{T-1})$ and the payoff function $P_T(h_{T-1}, \cdot)$. Then $\Phi(Q_{T+1})$ is a bounded, nonempty and compact valued, and upper hemicontinuous correspondence from $H_{T-1}$ to $\mathbb{R}^n$. Intuitively, $\Phi(Q_{T+1})(h_{T-2}, x_{T-1}, s_{T-1})$ represents the set of all possible payoff vectors in the subgame $h_{T-2}$ when Nature's move is $s_{T-1}$ and the players choose the action profile $x_{T-1}$ in stage $T-1$.[24] Note

[24] For $h_{T-1} = (h_{T-2}, x_{T-1}, s_{T-1})$, $\Phi(Q_{T+1})$ is a correspondence from $H_{T-1}$ to $\mathbb{R}^n$. Given $(h_{T-2}, x_{T-1})$, we slightly abuse the notation by viewing $\Phi(Q_{T+1})(h_{T-2}, x_{T-1}, \cdot)$ as a correspondence from $S_{T-1}$ to $\mathbb{R}^n$.

that in the subgame $h_{T-2}$,

$$P_{T-1}(h_{T-2}, x_{T-1}) = \int_{S_{T-1}} \Phi(Q_{T+1})(h_{T-2}, x_{T-1}, s_{T-1}) f_{(T-1)0}(\mathrm{d}s_{T-1}|h_{T-2})$$

is the set of payoff vectors. It is shown in Simon and Zame (1990) that if a payoff correspondence $P_{T-1}$ is bounded, nonempty, convex and compact valued, and upper hemicontinuous, then it possesses a Borel (possibly discontinuous) selection such that there exists an equilibrium in mixed strategy by taking this selection as the payoff function.[25] The difficulties here are that (1) the correspondence $P_{T-1}$ is no longer single-valued, and may not be convex valued; (2) even though $\Phi(Q_{T+1})$ is an upper hemicontinuous correspondence, it is not clear whether $P_{T-1}$ is upper hemicontinuous or not.[26] By introducing the condition of "atomless transition," we show that (1) even though $\Phi(Q_{T+1})$ may not be convex valued, if $f_{(T-1)0}(h_{T-2})$ is atomless, then

$$\begin{aligned}&\int_{S_{T-1}} \Phi(Q_{T+1})(h_{T-2}, x_{T-1}, s_{T-1}) f_{(T-1)0}(\mathrm{d}s_{T-1}|h_{T-2})\\ =&\int_{S_{T-1}} \mathrm{co}\Phi(Q_{T+1})(h_{T-2}, x_{T-1}, s_{T-1}) f_{(T-1)0}(\mathrm{d}s_{T-1}|h_{T-2});\end{aligned}$$

hence, $P_{T-1}(h_{T-2}, x_{T-1})$ is convex; (2) $P_{T-1}(h_{T-2}, x_{T-1})$ is upper hemicontinuous because the correspondence $\mathrm{co}\Phi(Q_{T+1})$ under the integral is convex valued and upper hemicontinuous. We can then repeat this backward induction argument from $\Phi(Q_{T+1})$ until the first stage.

Note that the key in the backward induction step is to preserve the convexity and upper hemicontinuity of the correspondences. As one arrives at the first stage, there is no need to conduct the backward induction again. Thus, our result can be strengthened by relaxing the condition of atomless transitions in the first stage. In simultaneous-move games, the argument in the previous paragraph requires that Nature be active and have an atomless transition in every stage (except the first stage). In alternating-move games, we only require that Nature's move be

[25] It was demonstrated in Stinchcombe (2005, Example 2.2) that given an exogenous payoff correspondence with two measurable selections $v$ and $u$, an equilibrium strategy of a player for the game with $v$ as the payoff functions may be a strictly dominated strategy for the game with $u$ as the payoff functions. Such an issue does not arise in our setting. Our primitives for the payoffs are the payoff functions (not payoff correspondences) of the players. When a full history is given, the players have a unique payoff vector in our setting. The payoff correspondence in our backward induction step is endogenous. As shall be explained in Steps 2 and 3, each payoff vector given by the payoff correspondence corresponds to a subgame perfect equilibrium in the original dynamic game. On the other hand, it is clear that any subgame perfect equilibrium strategy of a player in a dynamic game cannot be a strictly dominated strategy of that game.

[26] As illustrated in Section 3, the upper hemicontinuity property may not be preserved if the transition probability has an atom in its value.

atomless whenever Nature is active (except the first stage). In particular, when Nature is inactive in some stage $t$, $P_{t-1}$ is indeed $\Phi(Q_{t+1})$, a bounded, nonempty and compact valued, and upper hemicontinuous correspondence. The one who is the only active player in that stage faces a single-player decision problem. The key observation is that the only active player must possess an optimal choice in pure strategy even though $P_{t-1}$ may not be convex valued. When Nature is active, the reason why the backward induction argument holds is the same as that in the previous paragraph.

(2) We now describe the second (forward induction) step. Suppose that $q_t$ is a measurable selection of $\Phi(Q_{t+1})$ in some stage $t$. Given the construction in backward induction step, $q_t(h_{t-1})$ represents a possible payoff vector in the subgame $h_{t-1}$ if all the players follow some equilibrium strategy in the subsequent stages. The aim of this step is to identify those subsequent equilibrium strategies and the corresponding payoff functions.

Based on the construction of $\Phi(Q_{t+1})$, one can expect that in every subgame $h_{t-1}$, there exists a strategy profile $f_t(h_{t-1})$ and a payoff profile $g_t(h_{t-1},\cdot) \in P_t(h_{t-1},\cdot)$ such that for all $h_{t-1} \in H_{t-1}$,

1. $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} g_t(h_{t-1},x) f_t(\mathrm{d}x|h_{t-1})$;
2. $f_t(h_{t-1})$ is a Nash equilibrium in the subgame $h_{t-1}$ with the payoff $g_t(h_{t-1},\cdot)$ and action space $A_t(h_{t-1})$.

The key technical difficulties here are that (1) the payoff function $g_t$ needs to be jointly measurable in $(h_{t-1},x)$; and (2) one needs to further construct a jointly measurable selection $q_{t+1}$ of $Q_{t+1}$ (in $(h_{t-1},x_t,s_t)$) such that $g_t(h_{t-1},x_t) = \int_{S_t} q_{t+1}(h_{t-1},x_t,s_t) f_{t0}(\mathrm{d}s_t|h_{t-1})$ for all $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$. We solve the first issue by carefully modifying the argument in Reny and Robson (2002). For the second one, we show that a deep "measurable" measurable choice theorem of Mertens (2003) can be used to address this issue.[27]

By completing this step, we establish the relationship between the equilibrium payoff correspondence in stage $t+1$, and the equilibrium payoff correspondence in stage $t$ if all players play some equilibrium strategy in the subsequent stage. Together with the first step, the forward induction helps us obtain the equilibrium existence result in dynamic games with finite stages as follows. We can start with backward induction from the last period and stop at the initial period, then run forward induction from the initial period to the last period.

[27]In Harris, Reny and Robson (1995) and Reny and Robson (2002), the joint measurability follows from the measurable version of Skorokhod's representation theorem and implicit function theorem, respectively. These arguments are not applicable here.

(3) Step 3 proves the equilibrium existence result in infinite-horizon dynamic games via Lemmas 11-15.

Since there is no last stage in the infinite-horizon setting, it is not clear where one should start with backward induction argument. We pick an arbitrary stage $\tau \geq 1$ and let $Q^{\tau}_{\tau+1}$ be the expected payoff correspondence in stage $\tau$ if the players are free to choose any (not necessarily equilibrium) strategies in the future stages. Then run backward induction based on $Q^{\tau}_{\tau+1}$ from stage $\tau$, and denote $Q^{\tau}_{t}$ as the equilibrium payoff correspondence in stage $t$ for $t \leq \tau$. For $t \geq \tau+1$, let $Q^{\tau}_{t} = Q^{\tau}_{\tau+1}$. Lemmas 11 and 12 show that the set of possible equilibrium payoff vectors satisfy desirable properties. In particular, $Q^{\tau}_{t}$ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous.

It is easy to see that $Q^{\tau}_{t}(h_{t-1}) \subseteq Q^{\tau-1}_{t}(h_{t-1})$ for any $h_{t-1}$. That is, $\{Q^{\tau}_{t}\}_{\tau \geq 1}$ is a decreasing sequence in terms of $\tau$. Denote $Q^{\infty}_{t} = \cap_{\tau \geq 1} Q^{\tau}_{t}$. Lemma 13 shows that $Q^{\infty}_{t} = \Phi(Q^{\infty}_{t+1})$. By induction, $Q^{\infty}_{t} = \Phi^{\tau-t}(Q^{\infty}_{\tau})$ for any $\tau > t$. That is, given the payoff correspondence $Q^{\infty}_{\tau}$ in stage $\tau$ for $\tau > t$, $Q^{\infty}_{t}$ is the equilibrium payoff correspondence in stage $t$ due to the construction of backward induction. Because of the assumption of continuity at infinity, the strategies in the far future are not important. For fixed $t$, it means that $Q^{\infty}_{t}$ will be very close to the set by running backward induction from stage $\tau$ to stage $t$ based on the actual equilibrium payoff correspondence in stage $\tau$ if $\tau$ is sufficiently large. Since $Q^{\infty}_{t}$ is the intersection of all such $Q^{\infty}_{\tau}$, it is natural to expect that $Q^{\infty}_{t}$ is indeed the equilibrium payoff correspondence in stage $t$. Recall that $E_t(h_{t-1})$ is the set of payoff vectors of subgame-perfect equilibria in the subgame $h_{t-1}$. Given a measurable selection $c_t$ of $\Phi(Q^{\infty}_{t+1})$, Lemma 14 shows that $c_t(h_{t-1})$ is a subgame-perfect equilibrium payoff vector in the subgame $h_{t-1}$ by constructing the subsequent equilibrium strategies based on the forward induction; that is, $\Phi(Q^{\infty}_{t+1})(h_{t-1}) \subseteq E_t(h_{t-1})$. In Lemma 15, we show that $E_t(h_{t-1}) \subseteq Q^{\infty}_{t}(h_{t-1})$. Then we have $Q^{\infty}_{t}(h_{t-1}) = \Phi(Q^{\infty}_{t+1})(h_{t-1}) = E_t(h_{t-1})$. This completes the sketch for the infinite-horizon case.

## 4.4 An extension

In this subsection, we extend the model of continuous dynamic games as specified in the previous two subsections. The aim is to combine the models of dynamic games with perfect and almost perfect information, and cover an important class of dynamic games, namely stochastic games. We show the existence of a subgame-perfect equilibrium such that whenever there is only one active player at some stage, the player can play pure strategy as part of the equilibrium strategies. As a byproduct, we obtain a new existence result for continuous stochastic games.

For concreteness, we shall allow for the case in which (1) the state transition depends on the action profile in the current stage as well as on the previous history, and (2) the players may have perfect information in some stages. The first modification covers the model of stochastic games as a special case. The second change allows us to combine the models of dynamic games with perfect and almost perfect information.

1. For each $t \geq 1$, the choice of Nature depends not only on the history $h_{t-1}$, but also on the action profile $x_t$ in this stage. For any $t \geq 1$, suppose that $A_{t0}$ is a continuous, nonempty and closed valued correspondence from $\mathrm{Gr}(A_t)$ to $S_t$. Then $H_t = \mathrm{Gr}(A_{t0})$, and $H_\infty$ is the subset of $X^\infty \times S^\infty$ such that $(x, s) \in H_\infty$ if $(x^t, s^t) \in H_t$ for any $t \geq 0$.
2. Nature's action is given by a continuous mapping $f_{t0}$ from $\mathrm{Gr}(A_t)$ to $\mathcal{M}(S_t)$ such that $f_{t0}(A_{t0}(h_{t-1}, x_t)|h_{t-1}, x_t) = 1$ for all $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$.
3. For each $t \geq 1$, we use the notation $N_t$ to track whether there is a unique active player in stage $t$. In particular, let

$$N_t = \begin{cases} 1, & \text{if } f_{t0}(h_{t-1}, x_t) \equiv \delta_{s_t} \text{ for some } s_t \text{ and} \\ & |\{i \in I \colon A_{ti} \text{ is not point valued}\}| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

   where $|K|$ represents the number of points in the set $K$. Thus, if $N_t = 1$, then the player who is active in the period $t$ is the only active player and has perfect information. If $N_t = 0$, then Nature moves in this stage.

Similarly as in Section 4.2, we can drop the condition of atomless transition in those periods with only one active player in $I$.

**Assumption 2** (Atomless Transitions′). *1. For any $t \geq 1$ with $N_t = 1$, $S_t$ is a singleton set $\{\acute{s}_t\}$.*

*2. For each $t \geq 1$ with $N_t = 0$, $f_{t0}(h_{t-1}, x_t)$ is an atomless Borel probability measure for each $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$.*

The result on the equilibrium existence is presented below.

**Proposition 1.** *If a continuous dynamic game (as described above) satisfies the condition of atomless transitions′, then it possesses a subgame-perfect equilibrium $f$. In particular, for $j \in I$ and $t \geq 1$ such that $N_t = 1$ and player $j$ is the only active player in this period, $f_{tj}$ can be chosen to be deterministic. In addition, $E_t$ is nonempty and compact valued, and upper hemicontinuous on $H_{t-1}$ for $t \geq 1$.*

As the extension above covers the model of continuous stochastic games, a new equilibrium existence result can be stated below for continuous stochastic games.

**Proposition 2.** *If a continuous stochastic game has atomless transitions, then it possesses a subgame-perfect equilibrium.*

**Remark 3.** *Consider a standard stochastic game with uncountable states as in Mertens and Parthasarathy (2003), where the existence of a subgame-perfect equilibrium was shown by assuming the state transitions (not necessarily atomless) to be norm continuous (in the norm topology on the space of Borel measures) with respect to the actions in the previous stage.*[28] *It is noted in (Maitra and Sudderth, 2007, p. 712) that "This is a very strong condition". Maitra and Sudderth (2007) also indicated on the same page the desirability to weaken such a norm continuity condition: "it would be preferable to assume some sort of weak continuity ...". By restricting our result on general dynamic games to the setting of stochastic games, we obtain the existence of subgame-perfect equilibria in stochastic games whose state transitions are atomless and continuous in the weak star topology on the space of Borel measures. Our result cannot be covered by the result in Mertens and Parthasarathy (2003), and vice versa.*

*We provide a simple example to demonstrate that our continuity condition on the state transitions is indeed weaker than the norm continuity condition required by Mertens and Parthasarathy (2003). Consider the following state transitions.*

- *The action space is* $A_1 = A_2 = [0, 1]$.
- *The state space in stage* $t$ *is a product space:* $S_t = S_{t1} \times S_{t2} = [0, 1] \times [0, 1]$.
- *Given* $s_{t-1} = (s_{(t-1)1}, s_{(t-1)2})$ *and* $a_{t-1} = (a_{(t-1)1}, a_{(t-1)2})$, *the stage transition* $f_{t0}(\cdot|s_{t-1}, a_{t-1})$ *induces a product probability measure* $\psi_{t1}(\cdot|s_{t-1}, a_{t-1}) \otimes \psi_{t2}(\cdot|s_{t-1}, a_{t-1})$ *on* $S_{t1} \times S_{t2}$, *where* $\psi_{t2}(\cdot|s_{t-1}, a_{t-1})$ *is the uniform distribution on* $S_{t2} = [0, 1]$ *regardless of* $(s_{t-1}, a_{t-1})$, *and*

$$\psi_{t1}\left(s_{t1} = \frac{s_{(t-1)1} + s_{(t-1)2} + a_{(t-1)1} + a_{(t-1)2}}{4} | s_{t-1}, a_{t-1}\right) = 1.$$

  *That is, given* $(s_{t-1}, a_{t-1})$, $\psi_{t1}(\cdot|s_{t-1}, a_{t-1})$ *puts probability* $1$ *on their average.*

*Since* $\psi_{t2}$ *gives the uniform distribution on* $S_{t2}$, *the state transition is atomless. In addition, the state transition* $f_{t0}$ *is continuous as* $\psi_{t1}$ *is continuous and* $\psi_{t2}$ *is constant.*

*The state transition* $f_{t0}$ *is clearly not norm continuous. For example, we fix* $s_{t-1} = (s_{(t-1)1}, s_{(t-1)2})$ *and a sequence* $\{a^n_{t-1}\}_{n\geq 0}$ *such that* $s_{(t-1)1} + s_{(t-1)2} = \frac{1}{2}$, $(a^n_{(t-1)1}, a^n_{(t-1)2}) \to (a^0_{(t-1)1}, a^0_{(t-1)2})$ *as* $n \to \infty$, *and* $a^n_{(t-1)1} + a^n_{(t-1)2} = \frac{1}{2} - \frac{1}{2n}$ *for*

[28] For detailed discussions on general stochastic games, see Jaśkiewicz and Nowak (2016).

*any* $n \geq 1$. *Then*

$$f_{t0}\left(\{\frac{1}{4}\} \times [0,1] \middle| s_{t-1}, a^n_{t-1}\right) = 0 \nrightarrow 1 = f_{t0}\left(\{\frac{1}{4}\} \times [0,1] \middle| s_{t-1}, a^0_{t-1}\right).$$

# 5 Measurable dynamic games

In this section, we consider the more general setting in which the model parameters are jointly measurable in the action and state variables, but continuity is only required for the action variables. The proofs of the results in this section are left in Appendix B.

In Section 5.1, we adopt the model specified in Section 4.1, but relax the continuity requirement to measurability in the state variables. To obtain the existence of subgame-perfect equilibria, we strengthen the condition "atomless transitions" to the condition "atomless reference measure (ARM)" on the state transitions. The latter condition means that in each stage, there is an atomless reference measure and the state transitions are absolutely continuous with respect to this reference measure. In Section 5.2, we consider dynamic games with perfect information. The ARM type condition is imposed only when Nature moves. We show the existence of *pure-strategy* subgame-perfect equilibria. In Section 5.3, we provide a roadmap for proving Theorems 3 and 4. To omit the repetitive descriptions, we follow the argument in Section 4.3 and only highlight the necessary changes.

In Appendix B, we present a further extension by partially relaxing the ARM condition in two ways. First, we allow the possibility that there is only one active player (but no Nature) at some stages, where the ARM type condition is dropped. Second, we introduce an additional weakly continuous component on the state transitions at any other stages. In addition, we allow the state transition in each period to depend on the current actions as well as on the previous history. As the generalization of the model in Section 4.4, (1) we combine the models for measurable dynamic games with perfect and almost perfect information, (2) we show the existence of subgame perfect equilibria such that whenever there is only one active player at some stage, the player can play pure strategy as part of the equilibrium, and (3) a new existence result is obtained for stochastic games.

## 5.1 Measurable dynamic games with almost perfect information

We will follow the setting and notations in Section 4.1 as closely as possible, and only describe the changes we need to make on the model. In Section 4.1, we assume that the relevant model parameters (action correspondences, Nature's move, and payoff functions) are continuous in both actions and states. Here, we shall work with the class of games with sectionally continuous model parameters in the following sense. Suppose that $Y_1$, $Y_2$ and $Y_3$ are all Polish spaces, and $Z \subseteq Y_1 \times Y_2$. Denote $Z(y_1) = \{y_2 \in Y_2 \colon (y_1, y_2) \in Z\}$ for any $y_1 \in Y_1$. A function (resp. correspondence) $f \colon Z \to Y_3$ is said to be sectionally continuous on $Y_2$ if $f(y_1, \cdot)$ is continuous on $Z(y_1)$ for all $y_1$ with $Z(y_1) \neq \emptyset$. Similarly, one can define the sectional upper hemicontinuity for a correspondence.

Compared with continuous dynamic games with almost perfect information, the changes we need to make to describe measurable dynamic games are as follows.

1. For any $t \geq 1$ and $i \in I$, $A_{ti}$ is sectionally continuous on $X^{t-1}$.[29]
2. For any $t \geq 1$, $f_{t0}$ is sectionally continuous on $X^{t-1}$.
3. For each $i \in I$, the payoff function $u_i$ is sectionally continuous on $X^{\infty}$.

For each $t \geq 0$, suppose that $\lambda_t$ is a Borel probability measure on $S_t$ and $\lambda_t$ is atomless for $t \geq 1$. Let $\lambda^t = \otimes_{0 \leq k \leq t} \lambda_k$ for $t \geq 0$. We shall assume the following condition on the state transitions.

**Assumption 3** (Atomless Reference Measure (ARM)). *A dynamic game is said to satisfy the "atomless reference measure (ARM)" condition if for each $t \geq 1$,*

1. *the probability $f_{t0}(\cdot|h_{t-1})$ is absolutely continuous with respect to $\lambda_t$ on $S_t$ with the Radon-Nikodym derivative $\varphi_{t0}(h_{t-1}, s_t)$ for all $h_{t-1} \in H_{t-1}$;*[30]
2. *the mapping $\varphi_{t0}$ is Borel measurable and sectionally continuous on $X^{t-1}$, and integrably bounded in the sense that there is a $\lambda_t$-integrable function $\phi_t \colon S_t \to \mathbb{R}_+$ such that $\varphi_{t0}(h_{t-1}, s_t) \leq \phi_t(s_t)$ for any $h_{t-1} \in H_{t-1}$ and $s_t \in S_t$.*

When one considers a dynamic game with infinite horizon, the following "continuity at infinity" condition is standard.[31] This condition means that the actions and states in the far future would not matter that much for any player's payoff. In particular, all discounted repeated games or stochastic games satisfy this condition.

[29]Note that a history mixes the multiple components of states and actions in different periods. As noted in Footnote 15, one can also view a history $h_{t-1}$ as an element in $X^{t-1} \times S^{t-1}$ by abusing the notation.

[30]It is common to have a reference measure when one considers a game with uncountable states. For example, if $S_t$ is a convex subset of $\mathbb{R}^l$, then the uniform distribution on the convex set is a natural reference measure. In particular, the condition that the state transitions are absolutely continuous with respect to a reference measure is widely adopted in the literature on stochastic games; see, for example, Nowak (1985), Nowak and Raghavan (1992), Duffie *et al.* (1994) and He and Sun (2017).

[31]See, for example, Fudenberg and Levine (1983).

For any $T \geq 1$, let

$$w^T = \sup_{\substack{i \in I \\ (x,s) \in H_\infty \\ (\overline{x},\overline{s}) \in H_\infty \\ x^{T-1} = \overline{x}^{T-1} \\ s^{T-1} = \overline{s}^{T-1}}} |u_i(x,s) - u_i(\overline{x},\overline{s})|. \tag{1}$$

**Assumption 4** (Continuity at Infinity)**.** *A dynamic game is said to be "continuous at infinity" if $w^T \to 0$ as $T \to \infty$.*

We shall modify the notion of subgame-perfect equilibrium slightly. In particular, when the state space is uncountable and has a reference measure, it is natural to consider the optimality for almost all sub-histories in the probabilistic sense:[32] a property is said to hold for $\lambda^t$-almost all $h_t = (x^t, s^t) \in H_t$ if it is satisfied for $\lambda^t$-almost all $s^t \in S^t$ and all $x^t \in H_t(s^t)$.

**Definition 3** (SPE)**.** *A subgame-perfect equilibrium is a strategy profile $f$ such that for all $i \in I$, $t \geq 0$, and $\lambda^t$-almost all $h_t \in H_t$, player $i$ cannot improve his payoff in the subgame $h_t$ by a unilateral change in his strategy.*

The theorem below shows the existence of a subgame-perfect equilibrium under the conditions of ARM and continuity at infinity. Recall that $E_t(h_{t-1})$ is the set of all subgame-perfect equilibrium payoffs in the subgame $h_{t-1}$. The theorem also shows the compactness and upper hemicontinuity properties of the correspondence $E_t$. In particular, we shall work with the upper hemicontinuity property also in the probabilistic sense. Suppose that $Y_1$, $Y_2$ and $Y_3$ are all Polish spaces, and $Z \subseteq Y_1 \times Y_2$ and $\eta$ is a Borel probability measure on $Y_1$. Denote $Z(y_1) = \{y_2 \in Y_2 \colon (y_1, y_2) \in Z\}$ for any $y_1 \in Y_1$. A function (resp. correspondence) $f \colon Z \to Y_3$ is said to be essentially sectionally continuous on $Y_2$ if $f(y_1, \cdot)$ is continuous on $Z(y_1)$ for $\eta$-almost all $y_1$. Similarly, one can define the essential sectional upper hemicontinuity for a correspondence.

**Theorem 3.** *If a dynamic game with almost perfect information satisfies the ARM condition and is continuous at infinity, then it possesses a subgame-perfect equilibrium. In addition, $E_t$ is nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^{t-1}$.*

## 5.2 Measurable dynamic games with perfect information

In this subsection, we consider dynamic games with perfect information (with or without Nature). We will follow the setting and notations in Section 4.2, and make

[32] See, for example, Abreu, Pearce and Stacchetti (1990) and Footnote 4 therein.

the same changes as those in Section 5.1. In particular, the continuity requirement in the state variables are dropped.

In dynamic games with perfect information where players move sequentially, we show the existence of pure-strategy subgame-perfect equilibria. The ARM condition is imposed when Nature moves, and is dropped in those periods with one active player from the set $I$.[33]

**Theorem 4.** *If a dynamic game with perfect information satisfies the ARM condition and is continuous at infinity, then it possesses a pure-strategy subgame-perfect equilibrium. In addition, $E_t$ is nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^{t-1}$ for any $t \geq 1$.*

## 5.3 A roadmap for proving Theorems 3 and 4

The logic for the proofs of Theorems 3 and 4 is similar to that for the proofs of Theorems 1 and 2. The existence results are also established in three steps. However, new subtle difficulties arise. In this subsection, we summarize the main changes for proving Theorems 3 and 4. For simplicity, we omit the repetitive descriptions and adopt the same notations as in Section 4.3.

In the first (backward induction) step, following the same argument as in Section 4.3, one can construct the correspondence $P_t$. Recall that the key role of the atomless transition condition is to guarantee that $P_t$ is convex valued and upper hemicontinuous. With the condition of atomless reference measure, one is still able to show that $P_t$ is convex valued. However, though the correspondence $P_t$ remains upper hemicontinuous in actions, it is only measurable with respect to the states. As a result, the existence result in Simon and Zame (1990) is not readily applicable. We extend their existence result by allowing the payoff correspondence to be upper hemicontinuous in actions, but measurable in states. The key of this extension is to approximate the measurable correspondence by continuous correspondences based on Lusin's theorem (see Lemma 3).

In the forward induction step, an important observation is that the set of histories $H_{t-1}$ at stage $t$ can be divided into countably many Borel subsets $\{H_{t-1}^m\}_{m \geq 0}$ with desirable properties. In particular,

1. $H_{t-1} = \cup_{m \geq 0} H_{t-1}^m$ and $\frac{\lambda^{t-1}(\cup_{m \geq 1} \mathrm{proj}_{S^{t-1}}(H_{t-1}^m))}{\lambda^{t-1}(\mathrm{proj}_{S^{t-1}}(H_{t-1}))} = 1$, where $\mathrm{proj}_{S^{t-1}}(H_{t-1}^m)$ and $\mathrm{proj}_{S^{t-1}}(H_{t-1})$ are projections of $H_{t-1}^m$ and $H_{t-1}$ on $S^{t-1}$;

[33] As noted in Remark 4, Theorem 4 can be generalized to the case when the state transitions either satisfy the ARM condition, or have the support inside a fixed finite set irrespective of the history at a particular stage.

2. for $m \geq 1$, $H_{t-1}^m$ is compact, $\Phi_t$ is upper hemicontinuous on $H_{t-1}^m$, and $P_t$ is upper hemicontinuous on

$$\{(h_{t-1}, x_t) \colon h_{t-1} \in H_{t-1}^m, x_t \in A_t(h_{t-1})\}.$$

Note that within each compact subset $H_{t-1}^m$ for $m \geq 1$, the correspondences $\Phi_t$ and $P_t$ are well behaved. One can apply forward induction argument from the proof for continuous dynamic games to each $H_{t-1}^m$, which enables us to obtain a strategy defined on this subset $H_{t-1}^m$. The forward induction step for measurable dynamic games is then completed by combining the equilibrium strategies obtained on $H_{t-1}^m$, $m \geq 0$ (subject to slight modifications).

The last step (extending the finite-horizon setting to infinite-horizon setting) follows a similar logic as that explained in the third step of Section 4.3. The main challenge is to handle various subtle measurability issues due to the lack of continuity in the state variables. As described in Section 4.3, the idea of this step is to show the upper hemicontinuity of equilibrium payoff correspondences in infinite horizon. In the case of continuous dynamic games, this property is shown, based on a few technical lemmas on upper hemicontinuous correspondences. For the class of measurable dynamic games as considered here, we need to extend those technical lemmas to the more difficult case of measurable correspondences.

# 6 Appendix A

In Section 6.1, we present several lemmas as the mathematical preparations for proving Theorems 1, 2 and Proposition 1. Since correspondences will be used extensively in the proofs, we collect, for the convenience of the reader, several known results on various properties of correspondences.[34] One can skip Section 6.1 first and go to the proofs in Sections 6.2-6.4 directly, and refer to those technical lemmas in Section 6.1 whenever necessary.

The proof of Theorem 1 is provided in Section 6.2. In Sections 6.3 and 6.4, we give the proofs of Theorem 2 and Proposition 1, respectively. We will only describe the necessary changes in comparison with the proofs presented in Sections 6.2.1-6.2.3.

[34] These technical lemmas are stated in a general form so that they can still be used in the proofs in Appendix B for the case of measurable dynamic games.

## 6.1 Technical preparations

Let $(S, \mathcal{S})$ be a measurable space and $X$ a topological space with its Borel $\sigma$-algebra $\mathcal{B}(X)$. A correspondence $\Psi$ from $S$ to $X$ is a function from $S$ to the space of all subsets of $X$. A mapping $\psi$ is said to be a selection of $\Psi$ if $\psi(s) \in \Psi(s)$ for any $s \in S$. The upper inverse $\Psi^u$ of a subset $A \subseteq X$ is

$$\Psi^u(A) = \{s \in S \colon \Psi(s) \subseteq A\}.$$

The lower inverse $\Psi^l$ of a subset $A \subseteq X$ is

$$\Psi^l(A) = \{s \in S \colon \Psi(s) \cap A \neq \emptyset\}.$$

The correspondence $\Psi$ is

1. weakly measurable, if $\Psi^l(O) \in \mathcal{S}$ for each open subset $O \subseteq X$;
2. measurable, if $\Psi^l(K) \in \mathcal{S}$ for each closed subset $K \subseteq X$.

The graph of $\Psi$ is denoted by $\mathrm{Gr}(\Psi) = \{(s, x) \in S \times X \colon s \in S, x \in \Psi(s)\}$. The correspondence $\Psi$ is said to have a measurable graph if $\mathrm{Gr}(\Psi) \in \mathcal{S} \otimes \mathcal{B}(X)$.

If $S$ is a topological space, then $\Psi$ is

1. upper hemicontinuous, if $\Psi^u(O)$ is open for each open subset $O \subseteq X$;
2. lower hemicontinuous, if $\Psi^l(O)$ is open for each open subset $O \subseteq X$;
3. continuous, if it is both upper hemicontinuous and lower hemicontinuous.

The following two lemmas present some basic measurability and continuity properties for correspondences.

**Lemma 1.** *Let $(S, \mathcal{S})$ be a measurable space, $X$ a Polish space endowed with the Borel $\sigma$-algebra $\mathcal{B}(X)$, and $\mathcal{K}$ the space of nonempty compact subsets of $X$ endowed with its Hausdorff metric topology. Suppose that $\Psi \colon S \to X$ is a nonempty and closed valued correspondence.*

1. *If $\Psi$ is weakly measurable, then it has a measurable graph.*
2. *If $\Psi$ is compact valued, then the following statements are equivalent.*
   - *(a) The correspondence $\Psi$ is weakly measurable.*
   - *(b) The correspondence $\Psi$ is measurable.*
   - *(c) The function $f \colon S \to \mathcal{K}$, defined by $f(s) = \Psi(s)$, is Borel measurable.*
3. *Suppose that $S$ is a topological space. If $\Psi$ is compact valued, then the function $f \colon S \to \mathcal{K}$ defined by $f(s) = \Psi(s)$ is continuous if and only if the correspondence $\Psi$ is continuous.*

4. *Suppose that* $(S, \mathcal{S}, \lambda)$ *is a complete probability space. Then* $\Psi$ *is* $\mathcal{S}$*-measurable if and only if it has a measurable graph.*

5. *For a correspondence* $\Psi\colon S \to X$ *between two Polish spaces, the following statements are equivalent.*

   (a) *The correspondence* $\Psi$ *is upper hemicontinuous at a point* $s \in S$ *and* $\Psi(s)$ *is compact.*

   (b) *If a sequence* $(s_n, x_n)$ *in the graph of* $\Psi$ *satisfies* $s_n \to s$*, then the sequence* $\{x_n\}$ *has a limit in* $\Psi(s)$*.*

6. *For a correspondence* $\Psi\colon S \to X$ *between two Polish spaces, the following statements are equivalent.*

   (a) *The correspondence* $\Psi$ *is lower hemicontinuous at a point* $s \in S$*.*

   (b) *If* $s_n \to s$*, then for each* $x \in \Psi(s)$*, there exist a subsequence* $\{s_{n_k}\}$ *of* $\{s_n\}$ *and elements* $x_k \in \Psi(s_{n_k})$ *for each* $k$ *such that* $x_k \to x$*.*

7. *Given correspondences* $F\colon X \to Y$ *and* $G\colon Y \to Z$*, the composition* $F$ *and* $G$ *is defined by*

$$G(F(x)) = \cup_{y \in F(x)} G(y).$$

   *The composition of upper hemicontinuous correspondences is upper hemicontinuous. The composition of lower hemicontinuous correspondences is lower hemicontinuous.*

*Proof.* Properties (1), (2), (3), (5), (6) and (7) are Theorems 18.6, 18.10, 17.15, 17.20, 17.21 and 17.23 of Aliprantis and Border (2006), respectively. Property (4) follows from Proposition 4 in page 61 of Hildenbrand (1974). □

**Lemma 2.** 1. *A correspondence* $\Psi$ *from a measurable space* $(S, \mathcal{S})$ *into a topological space* $X$ *is weakly measurable if and only if its closure correspondence* $\overline{\Psi}$ *is weakly measurable, where* $\overline{\Psi}(s)$ *is the closure of the set* $\Psi(s)$ *in* $X$ *for each* $s \in S$*.*

2. *For a sequence* $\{\Psi_m\}$ *of correspondences from a measurable space* $(S, \mathcal{S})$ *into a Polish space, the union correspondence* $\Psi(s) = \cup_{m \geq 1} \Psi_m(s)$ *is weakly measurable if each* $\Psi_m$ *is weakly measurable. If each* $\Psi_m$ *is weakly measurable and compact valued, then the intersection correspondence* $\Phi(s) = \cap_{m \geq 1} \Psi_m(s)$ *is weakly measurable.*

3. *A weakly measurable, nonempty and closed valued correspondence from a measurable space into a Polish space admits a measurable selection.*

4. *A correspondence with closed graph between compact metric spaces is measurable.*

5. *A nonempty and compact valued correspondence* $\Psi$ *from a measurable space* $(S, \mathcal{S})$ *into a Polish space is weakly measurable if and only if there exists a sequence* $\{\psi_1, \psi_2 \ldots\}$ *of measurable selections of* $\Psi$ *such that* $\Psi(s) = \overline{\Phi}(s)$*, where* $\Phi(s) = \{\psi_1(s), \psi_2(s), \ldots\}$ *for each* $s \in S$.

6. *The image of a compact set under a compact valued upper hemicontinuous correspondence is compact.*[35] *If the domain is compact, then the graph of a compact valued upper hemicontinuous correspondence is compact.*

7. *The intersection of a correspondence with closed graph and an upper hemicontinuous compact valued correspondence is upper hemicontinuous.*

8. *If the correspondence* $\Psi \colon S \to \mathbb{R}^l$ *is compact valued and upper hemicontinuous, then the convex hull of* $\Psi$ *is also compact valued and upper hemicontinuous.*

*Proof.* Properties (1)-(7) are Lemmas 18.3 and 18.4, Theorems 18.13 and 18.20, Corollary 18.15, Lemma 17.8 and Theorem 17.25 in Aliprantis and Border (2006), respectively. Property 8 is Proposition 6 in page 26 of Hildenbrand (1974). □

Parts 1 and 2 of the following lemma are the standard Lusin's Theorem and Michael's continuous selection theorem, while the other two parts are about properties involving mixture of measurability and continuity of correspondences.

**Lemma 3.** 1. *Lusin's Theorem: Suppose that* $S$ *is a Borel subset of a Polish space,* $\lambda$ *is a Borel probability measure on* $S$ *and* $\mathcal{S}$ *is the completion of* $\mathcal{B}(S)$ *under* $\lambda$*. Let* $X$ *be a Polish space. If* $f$ *is an* $\mathcal{S}$*-measurable mapping from* $S$ *to* $X$*, then for any* $\epsilon > 0$*, there exists a compact subset* $S_1 \subseteq S$ *with* $\lambda(S \setminus S_1) < \epsilon$ *such that the restriction of* $f$ *to* $S_1$ *is continuous.*

2. *Michael's continuous selection theorem: Let* $S$ *be a metrizable space, and* $X$ *a complete metrizable closed subset of some locally convex space. Suppose that* $F \colon S \to X$ *is a lower hemicontinuous, nonempty, convex and closed valued correspondence. Then there exists a continuous mapping* $f \colon S \to X$ *such that* $f(s) \in F(s)$ *for all* $s \in S$.

3. *Let* $(S, \mathcal{S}, \lambda)$ *be a finite measure space,* $X$ *a Polish space, and* $Y$ *a locally convex linear topological space. Let* $F \colon S \to X$ *be a closed-valued correspondence such that* $Gr(F) \in \mathcal{S} \otimes \mathcal{B}(X)$*, and* $f \colon Gr(F) \to Y$ *a measurable function which is sectionally continuous on* $X$*. Then there exists a measurable function* $f' \colon S \times X \to Y$ *such that (1)* $f'$ *is sectionally continuous on*

[35] Given a correspondence $F \colon X \to Y$ and a subset $A$ of $X$, the image of $A$ under $F$ is defined to be the set $\cup_{x \in A} F(x)$.

$X$, *(2) for* $\lambda$*-almost all* $s \in S$, $f'(s,x) = f(s,x)$ *for all* $x \in F(s)$ *and* $f'(s,X) \subseteq cof(s,F(s))$.[36]

4. *Let* $(S,\mathcal{S})$ *be a measurable space, and* $X$ *and* $Y$ *Polish spaces. Let* $\Psi\colon S \times X \to \mathcal{M}(Y)$ *be an* $\mathcal{S} \otimes \mathcal{B}(X)$*-measurable, nonempty, convex and compact valued correspondence which is sectionally continuous on* $X$*, where the compactness and continuity are with respect to the weak* topology on* $\mathcal{M}(Y)$*. Then there exists an* $\mathcal{S} \otimes \mathcal{B}(X)$*-measurable selection* $\psi$ *of* $\Psi$ *that is sectionally continuous on* $X$*.*

*Proof.* Lusin's theorem is Theorem 7.1.13 in Bogachev (2007). Michael's continuous selection theorem can be found in Michael (1966) and the last paragraph of page 228 of Bogachev (2007). Properties (3) is Theorem 2.7 in Brown and Schreiber (1989). Property (4) follows from Theorem 1 and the main lemma of Kucia (1998). □

The following lemma presents the convexity, compactness and continuity properties on integration of correspondences.

**Lemma 4.** *Let* $(S,\mathcal{S},\lambda)$ *be an atomless probability space,* $X$ *a Polish space, and* $F$ *a correspondence from* $S$ *to* $\mathbb{R}^l$*. Denote*

$$\int_S F(s)\lambda(\mathrm{d}s) = \left\{ \int_S f(s)\lambda(\mathrm{d}s) \colon f \text{ is an integrable selection of } F \text{ on } S \right\}.$$

1. *If* $F$ *is measurable, nonempty and closed valued, and* $\lambda$*-integrably bounded by some integrable function* $\psi\colon S \to \mathbb{R}_+$ *in the sense that for* $\lambda$*-almost all* $s \in S$, $\|y\| \le \psi(s)$ *for any* $y \in F(s)$*, then* $\int_S F(s)\lambda(\mathrm{d}s)$ *is nonempty, convex and compact, and*
$$\int_S F(s)\lambda(\mathrm{d}s) = \int_S coF(s)\lambda(\mathrm{d}s).$$

2. *If* $G$ *is a measurable, nonempty and closed valued correspondence from* $S \times X \to \mathbb{R}^l$ *such that (1)* $G(s,\cdot)$ *is upper (resp. lower) hemicontinuous on* $X$ *for all* $s \in S$*, and (2)* $G$ *is* $\lambda$*-integrably bounded by some integrable function* $\psi\colon S \to \mathbb{R}_+$ *in the sense that for* $\lambda$*-almost all* $s \in S$, $\|y\| \le \psi(s)$ *for any* $x \in X$ *and* $y \in G(s,x)$*, then* $\int_S G(s,x)\lambda(\mathrm{d}s)$ *is upper (resp. lower) hemicontinuous on* $X$*.*

*Proof.* See Theorems 2, 3 and 4, Propositions 7 and 8, and Problem 6 in Section D.II.4 of Hildenbrand (1974). □

The following result proves a measurable version of Lyapunov's theorem, which is taken from Mertens (2003). Let $(S,\mathcal{S})$ and $(X,\mathcal{X})$ be measurable spaces. A

[36]For any set $A$ in a linear topological space, coA denotes the convex hull of $A$.

transition probability from $S$ to $X$ is a mapping $f$ from $S$ to the space $\mathcal{M}(X)$ of probability measures on $(X, \mathcal{X})$ such that $f(B|\cdot) : s \to f(B|s)$ is $\mathcal{S}$-measurable for each $B \in \mathcal{X}$.

**Lemma 5.** *Let $f(\cdot|s)$ be a transition probability from a measurable space $(S, \mathcal{S})$ to another measurable space $(X, \mathcal{X})$ ($\mathcal{X}$ is separable).*[37] *Let $Q$ be a measurable, nonempty and compact valued correspondence from $S \times X$ to $\mathbb{R}^l$, which is $f$-integrable in the sense that for any measurable selection $q$ of $Q$, $q(s, \cdot)$ is $f(\cdot|s)$-absolutely integrable for any $s \in S$. Let $\int Q \, \mathrm{d}f$ be the correspondence from $S$ to subsets of $\mathbb{R}^l$ defined by*

$$M(s) = \left(\int Q \, \mathrm{d}f\right)(s) = \left\{\int_X q(s, x) f(\mathrm{d}x|s) \colon q \text{ is a measurable selection of } Q\right\}.$$

*Denote the graph of $M$ by $J$. Let $\mathcal{J}$ be the restriction of the product $\sigma$-algebra $\mathcal{S} \otimes \mathcal{B}(\mathbb{R}^l)$ to $J$.*

*Then*

1. *$M$ is a measurable, nonempty and compact valued correspondence;*
2. *there exists a measurable, $\mathbb{R}^l$-valued function $g$ on $(X \times J, \mathcal{X} \otimes \mathcal{J})$ such that $g(x, e, s) \in Q(x, s)$ and $e = \int_X g(x, e, s) f(\mathrm{d}x|s)$.*

The proof of Lemma 6 in Reny and Robson (2002) leads to the following result.

**Lemma 6.** *Suppose that $H$ and $X$ are Polish spaces. Let $P \colon H \times X \to \mathbb{R}^n$ be a measurable, and nonempty and compact valued correspondence, and the mappings $f \colon H \to \mathcal{M}(X)$ and $\mu \colon H \to \triangle(X)$ be measurable, where $\triangle(X)$ is the set of all finite Borel measures on $X$. In addition, suppose that $\mu(\cdot|h) = p(h, \cdot) \circ f(\cdot|h)$ such that $p(h, \cdot)$ is a measurable selection of $P(h, \cdot)$ for each $h$.*[38] *Then there exists a jointly Borel measurable selection $g$ of $P$ such that $\mu(\cdot|h) = g(h, \cdot) \circ f(\cdot|h)$; that is, $g(h, x) = p(h, x)$ for $f(\cdot|h)$-almost all $x$.*

Suppose that $(S_1, \mathcal{S}_1)$ is a measurable space, $S_2$ is a Polish space endowed with the Borel $\sigma$-algebra, and $S = S_1 \times S_2$ which is endowed with the product $\sigma$-algebra $\mathcal{S}$. Let $D$ be an $\mathcal{S}$-measurable subset of $S$ such that $D(s_1)$ is compact for any $s_1 \in S_1$. The $\sigma$-algebra $\mathcal{D}$ is the restriction of $\mathcal{S}$ on $D$. Let $X$ be a Polish space, and $A$ a $\mathcal{D}$-measurable, nonempty and closed valued correspondence from $D$ to $X$ which is sectionally continuous on $S_2$. The following lemma considers the property of upper hemicontinuity for the correspondence $M$ as defined in Lemma 5.

[37] A $\sigma$-algebra is said to be separable if it is generated by a countable collection of sets.
[38] The finite measure $\mu(\cdot|h) = p(h, \cdot) \circ f(\cdot|h)$ if $\mu(E|h) = \int_E p(h, x) f(\mathrm{d}x|h)$ for any Borel set $E$.

**Lemma 7.** *Let $f(\cdot|s)$ be a transition probability from $(D, \mathcal{D})$ to $\mathcal{M}(X)$ such that $f(A(s)|s) = 1$ for any $s \in D$, which is sectionally continuous on $S_2$. Let $G$ be a bounded, measurable, nonempty, convex and compact valued correspondence from $Gr(A)$ to $\mathbb{R}^l$, which is sectionally upper hemicontinuous on $S_2 \times X$. Let $\int G \, \mathrm{d}f$ be the correspondence from $D$ to subsets of $\mathbb{R}^l$ defined by*

$$M(s) = \left( \int G \, \mathrm{d}f \right)(s) = \left\{ \int_X g(s,x) f(\mathrm{d}x|s) \colon g \text{ is a measurable selection of } G \right\}.$$

*Then $M$ is $\mathcal{S}$-measurable, nonempty and compact valued, and sectionally upper hemicontinuous on $S_2$.*

*Proof.* Define a correspondence $\tilde{G} \colon S \times X \to \mathbb{R}^l$ as

$$\tilde{G} = \begin{cases} G(s,x), & \text{if } (s,x) \in \mathrm{Gr}(A); \\ \{0\}, & \text{otherwise.} \end{cases}$$

Then $M(s) = \left( \int \tilde{G} \, \mathrm{d}f \right)(s) = \left( \int G \, \mathrm{d}f \right)(s)$. The measurability, nonemptiness and compactness follows from Lemma 5. Given $s_1 \in S_1$ such that (1) $D(s_1) \neq \emptyset$, (2) $f(s_1, \cdot)$ and $G(s_1, \cdot, \cdot)$ is upper hemicontinuous. The upper hemicontinuity of $M(s_1, \cdot)$ follows from Lemma 2 in Simon and Zame (1990) and Lemma 4 in Reny and Robson (2002). □

From now onwards, whenever we work with mappings taking values in the space $\mathcal{M}(X)$ of Borel probability measures on some separable metric space $X$, the relevant continuity or convergence is assumed to be in terms of the topology of weak convergence of measures on $\mathcal{M}(X)$ unless otherwise noted.

In the following lemma, we state some properties for transition correspondences.

**Lemma 8.** *Suppose that $Y$ and $Z$ are Polish spaces. Let $G$ be a measurable, nonempty, convex and compact valued correspondence from $Y$ to $\mathcal{M}(Z)$. Define a correspondence $G'$ from $\mathcal{M}(Y)$ to $\mathcal{M}(Z)$ as*

$$G'(\nu) = \left\{ \int_Y g(y) \nu(\mathrm{d}y) \colon g \text{ is a Borel measurable selection of } G \right\}.$$[39]

1. *The correspondence $G'$ is measurable, nonempty, convex and compact valued.*
2. *The correspondence $G$ is upper hemicontinuous if and only if $G'$ is upper hemicontinuous. In addition, if $G$ is continuous, then $G'$ is continuous.*

[39] The integral $\int_Y g(y)\nu(\mathrm{d}y)$ defines a Borel probability measure $\tau$ on $Z$ such that for any Borel set $C$ in $Z$, $\tau(C) = \int_Y g(y)(C)\nu(\mathrm{d}y)$. The measure $\tau$ is also equal to the Gelfand integral of $g$ with respect to the measure $\nu$ on $Y$, when $\mathcal{M}(Z)$ is viewed as a set in the dual space of the space of bounded continuous functions on $Z$; see Definition 11.49 of Aliprantis and Border (2006).

*Proof.* (1) is Lemma 19.29 of Aliprantis and Border (2006). By Theorem 19.30 therein, $G$ is upper hemicontinuous if and only if $G'$ is upper hemicontinuous. We need to show that $G'$ is lower hemicontinuous if $G$ is lower hemicontinuous.

Let $Z$ be endowed with a totally bounded metric, and $U(Z)$ the space of bounded, real-valued and uniformly continuous functions on $Z$ endowed with the supremum norm, which is obviously separable. Pick a countable set $\{f_m\}_{m\geq 1} \subseteq U(Z)$ such that $\{f_m\}$ is dense in the unit ball of $U(Z)$. It follows from Theorem 15.2 of Aliprantis and Border (2006) that the weak* topology of $\mathcal{M}(Z)$ is metrizable by the metric $d_z$, where

$$d_z(\mu_1, \mu_2) = \sum_{m=1}^{\infty} \frac{1}{2^m} \left| \int_Z f_m(z)\mu_1(\mathrm{d}z) - \int_Z f_m(z)\mu_2(\mathrm{d}z) \right|$$

for each pair of $\mu_1, \mu_2 \in \mathcal{M}(Z)$.

Suppose that $\{\nu_j\}_{j\geq 0}$ is a sequence in $\mathcal{M}(Y)$ such that $\nu_j \to \nu_0$ as $j \to \infty$. Pick an arbitrary point $\mu_0 \in G'(\nu_0)$. By the definition of $G'$, there exists a Borel measurable selection $g$ of $G$ such that $\mu_0 = \int_Y g(y)\nu_0(\mathrm{d}y)$.

For each $k \geq 1$, by Lemma 3 (Lusin's theorem), there exists a compact subset $D_k \subseteq Y$ such that $g$ is continuous on $D_k$ and $\nu_0(Y \setminus D_k) < \frac{1}{3k}$. Define a correspondence $G_k \colon Y \to \mathcal{M}(Z)$ as follows:

$$G_k(y) = \begin{cases} \{g(y)\}, & y \in D_k; \\ G(y), & y \in Y \setminus D_k. \end{cases}$$

Then $G_k$ is nonempty, convex and compact valued, and lower hemicontinuous. By Theorem 3.22 in Aliprantis and Border (2006), $Y$ is paracompact. Then by Lemma 3 (Michael's selection theorem), it has a continuous selection $g_k$.

For each $k$, since $\nu_j \to \nu_0$ and $g_k$ is continuous, $\int_Y g_k(y)\nu_j(\mathrm{d}y) \to \int_Y g_k(y)\nu_0(\mathrm{d}y)$ in the sense that for any $m \geq 1$,

$$\int_Y \int_Z f_m(z) g_k(\mathrm{d}z|y)\nu_j(\mathrm{d}y) \to \int_Y \int_Z f_m(z) g_k(\mathrm{d}z|y)\nu_0(\mathrm{d}y).$$

Thus, there exists a point $\nu_{j_k}$ such that $\{j_k\}$ is an increasing sequence and

$$d_z\left(\int_Y g_k(y)\nu_{j_k}(\mathrm{d}y), \int_Y g_k(y)\nu_0(\mathrm{d}y)\right) < \frac{1}{3k}.$$

In addition, since $g_k$ coincides with $g$ on $D_k$ and $\nu_0(Y \setminus D_k) < \frac{1}{3k}$,

$$d_z\left(\int_Y g_k(y)\nu_0(\mathrm{d}y), \int_Y g(y)\nu_0(\mathrm{d}y)\right) < \frac{2}{3k}.$$

Thus,

$$d_z\left(\int_Y g_k(y)\nu_{j_k}(\mathrm{d}y), \int_Y g(y)\nu_0(\mathrm{d}y)\right) < \frac{1}{k}.$$

Let $\mu_{j_k} = \int_Y g_k(y)\nu_{j_k}(\mathrm{d}y)$ for each $k$. Then $\mu_{j_k} \in G'(\nu_{j_k})$ and $\mu_{j_k} \to \mu_0$ as $k \to \infty$. By Lemma 1, $G'$ is lower hemicontinuous. □

The next lemma presents some properties for the composition of two transition correspondences in terms of the product of transition probabilities.

**Lemma 9.** *Let $X$, $Y$ and $Z$ be Polish spaces, and $G$ a measurable, nonempty and compact valued correspondence from $X$ to $\mathcal{M}(Y)$. Suppose that $F$ is a measurable, nonempty, convex and compact valued correspondence from $X \times Y$ to $\mathcal{M}(Z)$. Define a correspondence $\Pi$ from $X$ to $\mathcal{M}(Y \times Z)$ as follows:*

$$\Pi(x) = \{g(x) \diamond f(x) \colon g \text{ is a Borel measurable selection of } G,$$
$$f \text{ is a Borel measurable selection of } F\}.$$

1. *If $F$ is sectionally continuous on $Y$, then $\Pi$ is a measurable, nonempty and compact valued correspondence.*
2. *If there exists a function $g$ from $X$ to $\mathcal{M}(Y)$ such that $G(x) = \{g(x)\}$ for any $x \in X$, then $\Pi$ is a measurable, nonempty and compact valued correspondence.*
3. *If both $G$ and $F$ are continuous correspondences, then $\Pi$ is a nonempty and compact valued, and continuous correspondence.*[40]
4. *If $G(x) \equiv \{\lambda\}$ for some fixed Borel probability measure $\lambda \in \mathcal{M}(Y)$ and $F$ is sectionally continuous on $X$, then $\Pi$ is a continuous, nonempty and compact valued correspondence.*

*Proof.* (1) Define three correspondences $\tilde{F} \colon X \times Y \to \mathcal{M}(Y \times Z)$, $\hat{F} \colon \mathcal{M}(X \times Y) \to \mathcal{M}(Y \times Z)$ and $\check{F} \colon X \times \mathcal{M}(Y) \to \mathcal{M}(Y \times Z)$ as follows:

$$\tilde{F}(x,y) = \{\delta_y \otimes \mu \colon \mu \in F(x,y)\},$$

$$\hat{F}(\tau) = \left\{\int_{X \times Y} f(x,y)\tau(\mathrm{d}(x,y)) \colon f \text{ is a Borel measurable selection of } \tilde{F}\right\},$$

$$\check{F}(x,\mu) = \hat{F}(\delta_x \otimes \mu).$$

Since $F$ is measurable, nonempty, convex and compact valued, $\tilde{F}$ is measurable, nonempty, convex and compact valued. By Lemma 8, the correspondence $\hat{F}$ is

[40] In Lemma 29 of Harris, Reny and Robson (1995), they showed that $\Pi$ is upper hemicontinuous if both $G$ and $F$ are upper hemicontinuous.

measurable, nonempty, convex and compact valued, and $\check{F}(x, \cdot)$ is continuous on $\mathcal{M}(Y)$ for any $x \in X$.

Since $G$ is measurable and compact valued, there exists a sequence of Borel measurable selections $\{g_k\}_{k\geq 1}$ of $G$ such that $G(x) = \overline{\{g_1(x), g_2(x), \dots\}}$ for any $x \in X$ by Lemma 2 (5). For each $k \geq 1$, define a correspondence $\Pi^k$ from $X$ to $\mathcal{M}(Y \times Z)$ by letting $\Pi^k(x) = \check{F}(x, g_k(x)) = \hat{F}(\delta_x \otimes g_k(x))$. Then $\Pi^k$ is measurable, nonempty, convex and compact valued.

Fix any $x \in X$. It is clear that $\Pi(x) = \check{F}(x, G(x))$ is a nonempty valued. Since $G(x)$ is compact, and $\check{F}(x, \cdot)$ is compact valued and continuous, $\Pi(x)$ is compact by Lemma 2. Thus, the closure $\overline{\bigcup_{k\geq 1} \Pi^k(x)}$ of $\bigcup_{k\geq 1} \Pi^k(x)$ is a subset of $\Pi(x)$.

Fix any $x \in X$ and $\tau \in \Pi(x)$. There exists a point $\nu \in G(x)$ such that $\tau \in \check{F}(x, \nu)$. Since $\{g_k(x)\}_{k\geq 1}$ is dense in $G(x)$, it has a subsequence $\{g_{k_m}(x)\}$ such that $g_{k_m}(x) \to \nu$. As $\check{F}(x, \cdot)$ is continuous, $\check{F}(x, g_{k_m}(x)) \to \check{F}(x, \nu)$. That is,

$$\tau \in \overline{\bigcup_{k\geq 1} \check{F}(x, g_k(x))} = \overline{\bigcup_{k\geq 1} \Pi^k(x)}.$$

Therefore, $\overline{\bigcup_{k\geq 1} \Pi^k(x)} = \Pi(x)$ for any $x \in X$. Lemma 2 (1) and (2) imply that $\Pi$ is measurable.

(2) As in (1), the correspondence $\hat{F}$ is measurable, nonempty, convex and compact valued. If $G = \{g\}$ for some measurable function $g$, then $\Pi(x) = \hat{F}(\delta_x \otimes g(x))$, which is measurable, nonempty and compact valued.

(3) We continue to work with the two correspondences $\tilde{F} \colon X \times Y \to \mathcal{M}(Y \times Z)$ and $\hat{F} \colon \mathcal{M}(X \times Y) \to \mathcal{M}(Y \times Z)$ as in Part (1). By the condition on $F$, it is obvious that the correspondence $\tilde{F}$ is continuous, nonempty, convex and compact valued. Lemma 8 implies the properties for the correspondence $\hat{F}$. Define a correspondence $\hat{G} \colon X \to \mathcal{M}(X \times Y)$ as $\hat{G}(x) = \delta_x \otimes G(x)$.[41] Since $\hat{G}$ and $\hat{F}$ are both nonempty valued, $\Pi(x) = \hat{F}(\hat{G}(x))$ is nonempty. As $\hat{G}$ is compact valued and $\hat{F}$ is continuous, $\Pi$ is compact valued by Lemma 2. As $\hat{G}$ and $\hat{F}$ are both continuous, $\Pi$ is continuous by Lemma 1 (7).

(4) Let $Y' = Y$ and define a correspondence $\breve{F} \colon X \times Y \to \mathcal{M}(Y' \times Z)$ as follows:

$$\breve{F}(x, y) = \delta_y \otimes F(x, y) = \{\delta_y \otimes \mu \colon \mu \in F(x, y)\}.$$

Then $\breve{F}$ is also measurable, nonempty, convex and compact valued, and sectionally

[41] Given a finite measure $\nu$ on $X$ and a set $D$ of finite measures on $Y$, $\nu \otimes D$ denotes the set of finite measures $\{\nu \otimes \mu : \mu \in D\}$.

upper hemicontinuous on $X$.

Let $d$ be a totally bounded metric on $Y' \times Z$, and $U(Y' \times Z)$ the space of bounded, real-valued and uniformly continuous functions on $Y' \times Z$ endowed with the supremum norm, which is obviously separable. It follows from Theorem 15.2 of Aliprantis and Border (2006) that the space of Borel probability measures on $Y' \times Z$ with the topology of weak convergence of measures can be viewed as a subspace of the dual space of $U(Y' \times Z)$ with the weak* topology. By Corollary 18.37 of Aliprantis and Border (2006), $\Pi(x) = \int_Y \breve{F}(x, y)\lambda(\mathrm{d}y)$ is nonempty, convex and compact for any $x \in X$.[42]

Now we shall show the upper hemicontinuity. If $x_n \to x_0$ and $\mu_n \in \Pi(x_n)$, we need to prove that there exists some $\mu_0 \in \Pi(x_0)$ such that a subsequence of $\{\mu_n\}$ weakly converges to $\mu_0$. Suppose that for $n \geq 1$, $f_n$ is a Borel measurable selection of $F(x_n, \cdot)$ such that $\mu_n = \lambda \diamond f_n$.

Fix any $y \in Y$, let $J(y) = \overline{\mathrm{co}}\{f_n(y) \otimes \delta_{x_n}\}_{n \geq 1}$, which is the closure of the convex hull of $\{f_n(y) \otimes \delta_{x_n}\}_{n \geq 1}$. It is obvious that $J(y)$ is nonempty and convex. It is also clear that $J(y)$ is the closure of the countable set

$$\left\{ \sum_{i=1}^{n} \alpha_i f_i(y) \otimes \delta_{x_i} : n \geq 1, \alpha_i \in \mathbb{Q}_+, i = 1, \ldots, n, \sum_{i=1}^{n} \alpha_i = 1 \right\},$$

where $\mathbb{Q}_+$ is the set of non-negative rational numbers. Let $F'(x) = \{\mu \otimes \delta_x : \mu \in F(x, y)\}$ for any $x \in X$. Then, $F'$ is continuous on $X$. Since $\{x_n : n \geq 0\}$ is a compact set, Lemma 2 (6) implies that $\bigcup_{n \geq 0} F'(x_n)$ is compact. Hence, $\{f_n(y) \otimes \delta_{x_n}\}_{n \geq 1}$ is relatively compact. By Theorem 5.22 of Aliprantis and Border (2006), $\{f_n(y) \otimes \delta_{x_n}\}_{n \geq 1}$ is tight. That is, for any positive real number $\epsilon$, there is a compact set $K_\epsilon$ in $Z \times X$ such that for any $n \geq 1$, $f_n(y) \otimes \delta_{x_n}(K_\epsilon) > 1 - \epsilon$. Thus, $\sum_{i=1}^{n} \alpha_i f_i(y) \otimes \delta_{x_i}(K_\epsilon) > 1 - \epsilon$ for any $n \geq 1$, and for any $\alpha_i \in \mathbb{Q}_+$, $i = 1, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$. Hence, $J(y)$ is compact by Theorem 5.22 of Aliprantis and Border (2006) again.

For any $n \geq 1$, and for any $\alpha_i \in \mathbb{Q}_+$, $i = 1, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$, it is clear that $\sum_{i=1}^{n} \alpha_i f_i(y) \otimes \delta_{x_i}$ is measurable in $y \in Y$. Lemma 2 (5) implies that $J$ is also a measurable correspondence from $Y$ to $\mathcal{M}(Z \times X)$. By the argument in the second paragraph of the proof of Part (4), the set

$$\Lambda = \{\lambda \diamond \zeta : \zeta \text{ is a Borel measurable selection of } J\}$$

is compact.

Since $\lambda \diamond (f_n \otimes \delta_{x_n}) \in \Lambda$ for each $n$, there exists some Borel measurable selection

[42] Note that the integral $\int_Y \breve{F}(x, y)\lambda(\mathrm{d}y)$ can be viewed as the Gelfand integral of the correspondence in the dual space of $U(Y' \times Z)$; see Definition 18.36 of Aliprantis and Border (2006).

$\zeta$ of $J$ such that a subsequence of $\{\lambda \diamond (f_n \otimes \delta_{x_n})\}$, say itself, weakly converges to $\lambda \diamond \zeta \in \Lambda$. Let $\zeta_X(y)$ be the marginal probability of $\zeta(y)$ on $X$ for each $y$. Since $x_n$ converges to $x_0$, $\zeta_X(y) = \delta_{x_0}$ for $\lambda$-almost all $y \in Y$. As a result, there exists a Borel measurable function $f_0$ such that $\zeta = f_0 \otimes \delta_{x_0}$, where $f_0(y) \in \overline{\mathrm{co}}\mathrm{Ls}_n\{f_n(y)\}$ for $\lambda$-almost all $y \in Y$. Since $F$ is convex and compact valued, and upper hemicontinuous on $X$, $f_0$ is a measurable selection of $F(x_0, \cdot)$. Let $\mu_0 = \lambda \diamond f_0$. Then $\mu_n$ weakly converges to $\mu_0$, which implies that $\Pi$ is upper hemicontinuous.

Next we shall show the lower hemicontinuity of $\Pi$. Suppose that $x_n \to x_0$ and $\mu_0 \in \Pi(x_0)$. Then there exists a Borel measurable selection $f_0$ of $F(x_0, \cdot)$ such that $\mu_0 = \lambda \diamond f_0$. Since $F$ is sectionally lower hemicontinuous on $X$ and compact valued, for each $n \geq 1$, there exists a measurable selection $f_n$ for $F(x_n, \cdot)$ such that $f_n(y)$ weakly converges to $f_0(y)$ for each $y \in Y$.[43] Denote $\mu_n = \lambda \diamond f_n$. For any bounded continuous function $\psi$ on $Y \times Z$, $\int_Z \psi(y,z) f_n(\mathrm{d}z|y)$ converges to $\int_Z \psi(y,z) f_0(\mathrm{d}z|y)$ for any $y \in Y$. Thus, we have

$$\begin{aligned}\int_{Y \times Z} \psi(y,z)\mu_n(\mathrm{d}(y,z)) &= \int_Y \int_Z \psi(y,z) f_n(\mathrm{d}z|y)\lambda(\mathrm{d}y) \\ &\to \int_Y \int_Z \psi(y,z) f_0(\mathrm{d}z|y)\lambda(\mathrm{d}y)\end{aligned}$$

by the Dominated Convergence Theorem. Therefore, $\Pi$ is lower hemicontinuous. The proof is thus complete. □

The following lemma is taken from Simon and Zame (1990) (see also Lemma 4 in Reny and Robson (2002)).

1. $S$ and $Y$ are Polish spaces, $D$ is a closed subset of $S \times Y$, where $D(s)$ is compact for all $s \in S$;
2. $X = \prod_{1 \leq i \leq n} X_i$, where each $X_i$ is a Polish space;
3. for each $i$, $A_i$ is a nonempty and compact valued, continuous correspondence from $D$ to $X_i$, $A = \prod_{1 \leq i \leq n} A_i$ and $E = \mathrm{Gr}(A)$;
4. $P$ is a bounded, nonempty, convex and compact valued, upper hemicontinuous correspondence from $E$ to $\mathbb{R}^n$.

**Lemma 10.** *Consider the correspondence $\Phi \colon D \to \mathbb{R}^n \times \mathcal{M}(X) \times \triangle(X)$ defined as follows: $(v, \alpha, \mu) \in \Phi(s,y)$ if $p$ is a selection of $P$ such that $p(s,y,\cdot)$ is Borel measurable for any $(s,y)$, and*

*1. $v = \int_X p(s,y,x)\alpha(\mathrm{d}x)$;*

[43] See Proposition 4.2 in Sun (1997). Note that the atomless Loeb probability measurable space assumption is not needed for the result of lower hemicontinuity as in Theorem 10 therein.

*2.* $\alpha \in \otimes_{i\in I}\mathcal{M}(A_i(s,y))$ *is a Nash equilibrium in the subgame* $(s,y)$ *with payoff profile* $p(s,y,\cdot)$*, and action space* $A_i(s,y)$ *for each player* $i$*;*

*3.* $\mu = p(s,y,\cdot)\circ\alpha$*.*

*Then* $\Phi$ *is nonempty and compact valued, and upper hemicontinuous on* $D$*.*

## 6.2 Proof of Theorem 1

### 6.2.1 Backward induction

As explained in Section 4.3, the backward induction step aims to show that some desirable properties of the equilibrium payoff correspondences can be preserved when one works backwards along the game tree.

Given $t \geq 1$, let $Q_{t+1}$ be a bounded, nonempty and compact valued, and upper hemicontinuous correspondence from $H_t$ to $\mathbb{R}^n$. For any $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$, let

$$P_t(h_{t-1},x_t) = \int_{S_t} Q_{t+1}(h_{t-1},x_t,s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}).$$

Since $f_{t0}(\cdot|h_{t-1})$ is atomless and $Q_{t+1}$ is nonempty and compact valued, by Lemma 4,

$$P_t(h_{t-1},x_t) = \int_{S_t} \mathrm{co}Q_{t+1}(h_{t-1},x_t,s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}),$$

where $\mathrm{co}Q_{t+1}(h_{t-1},x_t,s_t)$ is the convex hull of $Q_{t+1}(h_{t-1},x_t,s_t)$. By Lemma 4, $P_t$ is bounded, nonempty, convex and compact valued. To show that $P_t$ is upper hemicontinuous on $\mathrm{Gr}(A_t)$, by Lemma 1 (5), it is sufficient to show that $P_t$ is upper hemicontinuous on $\{(h_{t-1}^k, x_t^k)\}_{k\geq 0}$, where $\{(h_{t-1}^k, x_t^k)\}_{k\geq 0}$ is a sequence such that $(h_{t-1}^k, x_t^k) \to (h_{t-1}^0, x_t^0)$ as $k \to \infty$. Note that $\{(h_{t-1}^k, x_t^k)\}_{k\geq 0}$ is indeed a compact set. Then the upper hemicontinuity of $P_t$ on $\{(h_{t-1}^k, x_t^k)\}_{k\geq 0}$ follows from Lemma 7.

By Lemma 10, there exists a bounded, measurable, nonempty and compact valued correspondence $\Phi_t$ from $H_{t-1}$ to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \triangle(X_t)$ such that $\Phi_t$ is upper hemicontinuous, and for all $h_{t-1} \in H_{t-1}$, $(v,\alpha,\mu) \in \Phi_t(h_{t-1})$ if $p_t$ is a selection of $P_t$ such that $p_t(h_{t-1},\cdot)$ is Borel measurable, and

1. $v = \int_{A_t(h_{t-1})} p_t(h_{t-1},x)\alpha(\mathrm{d}x)$;
2. $\alpha \in \otimes_{i\in I}\mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame $h_{t-1}$ with payoff $p_t(h_{t-1},\cdot)$ and action space $\prod_{i\in I} A_{ti}(h_{t-1})$;
3. $\mu = p_t(h_{t-1},\cdot)\circ\alpha$.

Denote the restriction of $\Phi_t$ on the first component $\mathbb{R}^n$ as $\Phi(Q_{t+1})$, which is a correspondence from $H_{t-1}$ to $\mathbb{R}^n$. By Lemma 10, $\Phi(Q_{t+1})$ is bounded, nonempty and compact valued, and upper hemicontinuous.

### 6.2.2 Forward induction

If one views $Q_t$ as some payoff correspondence for the players in stage $t$, then the correspondence $\Phi_t$ obtained in the backward induction step collects all the equilibrium strategies, the corresponding payoff vectors and the induced probabilities in stage $t-1$. The issue here is that one needs to construct jointly measurable payoff functions (as selections of $Q_t$) and strategy profiles in stage $t$ which induce the equilibrium payoffs in $\Phi_t$. This is done in the forward induction step. Specifically, we shall prove the following proposition.

**Proposition 3.** *For any $t \geq 1$ and any Borel measurable selection $q_t$ of $\Phi(Q_{t+1})$, there exists a Borel measurable selection $q_{t+1}$ of $Q_{t+1}$ and a Borel measurable mapping $f_t\colon H_{t-1} \to \otimes_{i\in I}\mathcal{M}(X_{ti})$ such that for all $h_{t-1} \in H_{t-1}$,*

1. *$f_t(h_{t-1}) \in \otimes_{i\in I}\mathcal{M}(A_{ti}(h_{t-1}))$;*
2. *$q_t(h_{t-1}) = \int_{A_t(h_{t-1})}\int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}) f_t(\mathrm{d}x_t|h_{t-1})$;*
3. *$f_t(\cdot|h_{t-1})$ is a Nash equilibrium in the subgame $h_{t-1}$ with action spaces $\{A_{ti}(h_{t-1})\}_{i\in I}$ and the payoff functions*

$$\int_{S_t} q_{t+1}(h_{t-1}, \cdot, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}).$$

*Proof.* We divide the proof into three steps. In step 1, we show that there exist Borel measurable mappings $f_t\colon H_{t-1} \to \otimes_{i\in I}\mathcal{M}(X_{ti})$ and $\mu_t\colon H_{t-1} \to \triangle(X_t)$ such that $(q_t, f_t, \mu_t)$ is a selection of $\Phi_t$. In step 2, we obtain a Borel measurable selection $g_t$ of $P_t$ such that for all $h_{t-1} \in H_{t-1}$,

1. $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} g_t(h_{t-1}, x) f_t(\mathrm{d}x|h_{t-1})$;
2. $f_t(h_{t-1})$ is a Nash equilibrium in the subgame $h_{t-1}$ with payoff $g_t(h_{t-1}, \cdot)$ and action space $A_t(h_{t-1})$.[44]

In step 3, we show that there exists a Borel measurable selection $q_{t+1}$ of $Q_{t+1}$ such that for all $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$,

$$g_t(h_{t-1}, x_t) = \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}).$$

Combining Steps 1-3, the proof is complete.

Step 1. Let $\Psi_t\colon \mathrm{Gr}(\Phi(Q_{t+1})) \to \mathcal{M}(X_t) \times \triangle(X_t)$ be

$$\Psi_t(h_{t-1}, v) = \{(\alpha, \mu)\colon (v, \alpha, \mu) \in \Phi_t(h_{t-1})\}.$$

[44]One cannot simply use $p_t$ in the previous subsection instead of $g_t$ here. Note that $p_t$ may not be jointly Borel measurable in $(h_{t-1}, x)$ even though $p_t(h_{t-1}, \cdot)$ is Borel measurable for each fixed $h_{t-1}$.

For any $\{(h_{t-1}^k, v^k)\}_{1\le k\le\infty} \subseteq \mathrm{Gr}(\Phi(Q_{t+1}))$ such that $(h_{t-1}^k, v^k)$ converges to $(h_{t-1}^\infty, v^\infty)$, pick $(\alpha^k, \mu^k)$ such that $(v^k, \alpha^k, \mu^k) \in \Phi_t(h_{t-1}^k)$ for $1 \le k < \infty$. Since $\Phi_t$ is upper hemicontinuous and compact valued, there exists a subsequence of $(v^k, \alpha^k, \mu^k)$, say itself, such that $(v^k, \alpha^k, \mu^k)$ converges to some $(v^\infty, \alpha^\infty, \mu^\infty) \in \Phi_t(h_{t-1}^\infty)$ due to Lemma 1 (5). Thus, $(\alpha^\infty, \mu^\infty) \in \Psi_t(h_{t-1}^\infty, v^\infty)$, which implies that $\Psi_t$ is also upper hemicontinuous and compact valued. By Lemma 2 (3), $\Psi_t$ has a Borel measurable selection $\psi_t$. Given a Borel measurable selection $q_t$ of $\Phi(Q_{t+1})$, let $\phi_t(h_{t-1}) = (q_t(h_{t-1}), \psi_t(h_{t-1}, q_t(h_{t-1})))$. Then $\phi_t$ is a Borel measurable selection of $\Phi_t$. Let $f_t$ and $\mu_t$ be the second and third dimension of $\phi_t$, respectively. By the construction of $\Phi_t$, for all $h_{t-1} \in H_{t-1}$,

1. $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} p_t(h_{t-1}, x) f_t(\mathrm{d}x|h_{t-1})$ such that $p_t(h_{t-1}, \cdot)$ is a Borel measurable selection of $P_t(h_{t-1}, \cdot)$;
2. $f_t(h_{t-1}) \in \otimes_{i\in I}\mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame $h_{t-1}$ with payoff $p_t(h_{t-1}, \cdot)$ and action space $\prod_{i\in I} A_{ti}(h_{t-1})$;
3. $\mu_t(\cdot|h_{t-1}) = p_t(h_{t-1}, \cdot) \circ f_t(\cdot|h_{t-1})$.

Step 2. Since $P_t$ is upper hemicontinuous on $\{(h_{t-1}, x_t) \colon h_{t-1} \in H_{t-1}, x_t \in A_t(h_{t-1})\}$, due to Lemma 6, there exists a Borel measurable mapping $g$ such that (1) $g(h_{t-1}, x_t) \in P_t(h_{t-1}, x_t)$ for any $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$, and (2) $g(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all $x_t$.

In a subgame $h_{t-1} \in H_{t-1}$, let

$$B_{ti}(h_{t-1}) = \{y_i \in A_{ti}(h_{t-1}) \colon$$

$$\int_{A_{t(-i)}(h_{t-1})} g_i(h_{t-1}, y_i, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) > \int_{A_t(h_{t-1})} p_{ti}(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1})\}.$$

Since $g(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all $x_t$,

$$\int_{A_t(h_{t-1})} g(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1}) = \int_{A_t(h_{t-1})} p_t(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1}).$$

Thus, $B_{ti}$ is a measurable correspondence from $H_{t-1}$ to $A_{ti}(h_{t-1})$. Then $B_{ti}$ has a Borel measurable graph. As $f_t(h_{t-1})$ is a Nash equilibrium in the subgame $h_{t-1} \in H_{t-1}$ with payoff $p_t(h_{t-1}, \cdot)$, $f_{ti}(B_{ti}(h_{t-1})|h_{t-1}) = 0$.

Denote $\beta_i(h_{t-1}, x_t) = \min P_{ti}(h_{t-1}, x_t)$, where $P_{ti}(h_{t-1}, x_t)$ is the projection of $P_t(h_{t-1}, x_t)$ on the $i$-th dimension. Then the correspondence $P_{ti}$ is measurable and compact valued, and $\beta_i$ is Borel measurable. Let $\Lambda_i(h_{t-1}, x_t) = \{\beta_i(h_{t-1}, x_t)\} \times [0, \gamma]^{n-1}$, where $\gamma > 0$ is the upper bound of $P_t$. Denote $\Lambda_i'(h_{t-1}, x_t) = \Lambda_i(h_{t-1}, x_t) \cap P_t(h_{t-1}, x_t)$. Then $\Lambda_i'$ is a measurable and compact valued correspondence, and hence has a Borel measurable selection $\beta_i'$. Note that $\beta_i'$ is a Borel measurable selection of $P_t$. Let

$$g_t(h_{t-1}, x_t) =$$

$$\begin{cases} \beta_i'(h_{t-1}, x_t) & \text{if } h_{t-1} \in H_{t-1}, x_{ti} \in B_{ti}(h_{t-1}) \text{ and } x_{tj} \notin B_{tj}(h_{t-1}), \forall j \neq i; \\ g(h_{t-1}, x_t) & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned} &\{(h_{t-1}, x_t) \in \mathrm{Gr}(A_t) \colon h_{t-1} \in H_{t-1}, x_{ti} \in B_{ti}(h_{t-1}) \text{ and } x_{tj} \notin B_{tj}(h_{t-1}), \forall j \neq i; \} \\ &= \mathrm{Gr}(A_t) \cap \cup_{i \in I} \left( (\mathrm{Gr}(B_{ti}) \times \prod_{j \neq i} X_{tj}) \setminus (\cup_{j \neq i} (\mathrm{Gr}(B_{tj}) \times \prod_{k \neq j} X_{tk})) \right), \end{aligned}$$

which is a Borel set. As a result, $g_t$ is a Borel measurable selection of $P_t$. Moreover, $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for all $h_{t-1} \in H_{t-1}$ and $f_t(\cdot|h_{t-1})$-almost all $x_t$.

Fix a subgame $h_{t-1} \in H_{t-1}$. We will verify that $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $g_t(h_{t-1}, \cdot)$ in the subgame $h_{t-1}$. Suppose that player $i$ deviates to some action $\tilde{x}_{ti}$.

If $\tilde{x}_{ti} \in B_{ti}(h_{t-1})$, then player $i$'s expected payoff is

$$\begin{aligned} &\int_{A_{t(-i)}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\ &= \int_{\prod_{j \neq i} B_{tj}^c(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\ &= \int_{\prod_{j \neq i} B_{tj}^c(h_{t-1})} \beta_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\ &\le \int_{\prod_{j \neq i} B_{tj}^c(h_{t-1})} p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\ &= \int_{A_{t(-i)}(h_{t-1})} p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\ &\le \int_{A_t(h_{t-1})} p_{ti}(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1}) \\ &= \int_{A_t(h_{t-1})} g_{ti}(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1}). \end{aligned}$$

The first and the third equalities hold since $f_{tj}(B_{tj}(h_{t-1})|h_{t-1}) = 0$ for each $j$, and hence $f_{t(-i)}(\prod_{j \neq i} B_{tj}^c(h_{t-1})|h_{t-1}) = f_{t(-i)}(A_{t(-i)}(h_{t-1})|h_{t-1})$. The second equality and the first inequality are due to the fact that $g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = \beta_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = \min P_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) \le p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)})$ for $x_{t(-i)} \in \prod_{j \neq i} B_{tj}^c(h_{t-1})$. The second inequality holds since $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $p_t(h_{t-1}, \cdot)$ in the subgame $h_{t-1}$. The fourth equality follows from the fact that $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all $x_t$.

If $\tilde{x}_{ti} \notin B_{ti}(h_{t-1})$, then player $i$'s expected payoff is

$$\begin{aligned}
&\int_{A_{t(-i)}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
&= \int_{\prod_{j \neq i} B^c_{tj}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
&= \int_{\prod_{j \neq i} B^c_{tj}(h_{t-1})} g_{i}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
&= \int_{A_{t(-i)}(h_{t-1})} g_{i}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
&\leq \int_{A_t(h_{t-1})} p_{ti}(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1}) \\
&= \int_{A_t(h_{t-1})} g_{ti}(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1}).
\end{aligned}$$

The first and the third equalities hold since

$$f_{t(-i)}\left(\prod_{j \neq i} B^c_{tj}(h_{t-1})|h_{t-1}\right) = f_{t(-i)}(A_{t(-i)}(h_{t-1})|h_{t-1}).$$

The second equality is due to the fact that $g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = g_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)})$ for $x_{t(-i)} \in \prod_{j \neq i} B^c_{tj}(h_{t-1})$. The first inequality follows from the definition of $B_{ti}$, and the fourth equality holds since $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all $x_t$.

Thus, player $i$ cannot improve his payoff in the subgame $h_t$ by a unilateral change in his strategy for any $i \in I$, which implies that $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $g_t(h_{t-1}, \cdot)$ in the subgame $h_{t-1}$.

Step 3. For any $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$,

$$P_t(h_{t-1}, x_t) = \int_{S_t} Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}).$$

By Lemma 5, there exists a Borel measurable mapping $q$ from $\mathrm{Gr}(P_t) \times S_t$ to $\mathbb{R}^n$ such that

1. $q(h_{t-1}, x_t, e, s_t) \in Q_{t+1}(h_{t-1}, x_t, s_t)$ for any $(h_{t-1}, x_t, e, s_t) \in \mathrm{Gr}(P_t) \times S_t$;
2. $e = \int_{S_t} q(h_{t-1}, x_t, e, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1})$ for any $(h_{t-1}, x_t, e) \in \mathrm{Gr}(P_t)$, where $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$.

Let

$$q_{t+1}(h_{t-1}, x_t, s_t) = q(h_{t-1}, x_t, g_t(h_{t-1}, x_t), s_t)$$

for any $(h_{t-1}, x_t, s_t) \in H_t$. Then $q_{t+1}$ is a Borel measurable selection of $Q_{t+1}$.

For $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$,

$$\begin{aligned} g_t(h_{t-1}, x_t) &= \int_{S_t} q(h_{t-1}, x_t, g_t(h_{t-1}, x_t), s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}) \\ &= \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}). \end{aligned}$$

Therefore, we have a Borel measurable selection $q_{t+1}$ of $Q_{t+1}$, and a Borel measurable mapping $f_t\colon H_{t-1} \to \otimes_{i\in I}\mathcal{M}(X_{ti})$ such that for all $h_{t-1} \in H_{t-1}$, properties (1)-(3) are satisfied. The proof is complete. □

If a dynamic game has only $T$ stages for some positive integer $T \geq 1$, then let $Q_{T+1}(h_T) = \{u(h_T)\}$ for any $h_T \in H_T$, and $Q_t = \Phi(Q_{t+1})$ for $1 \leq t \leq T-1$. We can start with the backward induction from the last period and stop at the initial period, then run the forward induction from the initial period to the last period. We obtain the following corollary.

**Corollary 1.** *If a finite-horizon continuous dynamic game with almost perfect information satisfies the condition of atomless transitions, then it has a subgame-perfect equilibrium.*[45]

### 6.2.3 Infinite horizon case

Pick a sequence $\xi = (\xi_1, \xi_2, \ldots)$ such that (1) $\xi_m$ is a transition probability from $H_{m-1}$ to $\mathcal{M}(X_m)$ for any $m \geq 1$, and (2) $\xi_m(A_m(h_{m-1})|h_{m-1}) = 1$ for any $m \geq 1$ and $h_{m-1} \in H_{m-1}$. Denote the set of all such $\xi$ by $\Upsilon$. Intuitively, $\xi$ can be viewed as a correlated strategy profile with each $\xi_t$ being the correlated strategy in stage $t$, and $\Upsilon$ is the set of all such correlated strategies.

Fix any $t \geq 1$, define a correspondence $\Delta_t^t$ as follows: in the subgame $h_{t-1}$,

$$\Delta_t^t(h_{t-1}) = \mathcal{M}(A_t(h_{t-1})) \otimes f_{t0}(h_{t-1}).$$

Then $\Delta_t^t(h_{t-1})$ is the set of probability measures on the space of action profiles of stage $t$, which is induced by all the possible correlated strategies among the active players and Nature's move in the subgame $h_{t-1}$. Inductively, we shall define the set of possible paths for correlated strategies in any subgame between stages $t$ and $m_1$ for $t < m_1 \leq \infty$.

[45]The condition of atomless transition is not needed at the last stage.

For any integer $m_1 > t$, suppose that the correspondence $\Delta_t^{m_1-1}$ has been defined. The correspondence $\Delta_t^{m_1}\colon H_{t-1} \to \mathcal{M}\left(\prod_{t\le m\le m_1}(X_m \times S_m)\right)$ is defined as follows:

$$\begin{aligned}\Delta_t^{m_1}(h_{t-1}) = &\{g(h_{t-1}) \diamond (\xi_{m_1}(h_{t-1},\cdot) \otimes f_{m_1 0}(h_{t-1},\cdot))\colon \\ & g \text{ is a Borel measurable selection of } \Delta_t^{m_1-1}, \\ & \xi_{m_1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_1})\},\end{aligned}$$

where the correspondence $\mathcal{M}(A_{m_1})$ takes value $\mathcal{M}\left(A_{m_1}(h_{m_1-1})\right)$ at subgame $h_{m_1-1}$. For any $m_1 \ge t$, let $\varrho^{m_1}_{(h_{t-1},\xi)} \in \Delta_t^{m_1}$ be the probability measure on $\prod_{t\le m\le m_1}(X_m \times S_m)$ induced by Nature's moves $\{f_{m0}\}_{t\le m\le m_1}$ and the correlated strategies $\{\xi_m\}_{t\le m\le m_1}$. Then $\varrho^{m_1}_{(h_{t-1},\xi)}$ is a possible path induced by $\xi$ in the subgame $h_{t-1}$ before stage $m_1$, and $\Delta_t^{m_1}(h_{t-1})$ is the set of all such possible paths $\varrho^{m_1}_{(h_{t-1},\xi)}$ in the subgame $h_{t-1}$. Note that $\varrho^{m_1}_{(h_{t-1},\xi)}$ can be regarded as a probability measure on $H_{m_1}(h_{t-1}) = \{(x_t, s_t, \ldots, x_{m_1}, s_{m_1}) : (h_{t-1}, x_t, s_t, \ldots, x_{m_1}, s_{m_1}) \in H_{m_1}\}$. Similarly, let $\varrho_{(h_{t-1},\xi)}$ be the probability measure on $\prod_{m\ge t}(X_m \times S_m)$ induced by Nature's moves $\{f_{m0}\}_{m\ge t}$ and the correlated strategies $\{\xi_m\}_{m\ge t}$ after the subgame $h_{t-1}$. The correspondence

$$\Delta_t\colon H_{t-1} \to \mathcal{M}(\prod_{m\ge t}(X_m \times S_m))$$

collects all the possible paths $\varrho_{(h_{t-1},\xi)}$.

We shall show that the correspondence $\Delta_t(h_{t-1})$, which contains all the possible paths induced by correlated strategies in the subgame $h_{t-1}$, is nonempty and compact valued, and continuous. The claim is proved by showing that $\Delta_t^{m_1}$ has such properties, and $\Delta_t$ can be approximated by $\Delta_t^{m_1}$ when $m_1$ is sufficiently large.

**Lemma 11.** *1. For any $t \ge 1$, the correspondence $\Delta_t^{m_1}$ is nonempty and compact valued, and continuous for any $m_1 \ge t$.*

*2. For any $t \ge 1$, the correspondence $\Delta_t$ is nonempty and compact valued, and continuous.*

*Proof.* (1) Consider the case $m_1 = t \ge 1$, where

$$\Delta_t^{m_1}(h_{t-1}) = \mathcal{M}(A_t(h_{t-1})) \otimes f_{t0}(h_{t-1}).$$

Since $A_t$ is nonempty and compact valued, and both $A_t$ and $f_{t0}$ are continuous, $\Delta_t^{m_1}$ is nonempty and compact valued, and continuous.

Suppose that $\Delta_t^{m_2}$ is nonempty and compact valued, and continuous for some

$m_2 \geq t \geq 1$. Note that

$$\begin{aligned}\Delta_t^{m_2+1}(h_{t-1}) = \{ & g(h_{t-1}) \diamond (\xi_{m_2+1}(h_{t-1}, \cdot) \otimes f_{(m_2+1)0}(h_{t-1}, \cdot)) : \\ & g \text{ is a Borel measurable selection of } \Delta_t^{m_2}, \\ & \xi_{m_2+1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_2+1})\}.\end{aligned}$$

By Lemma 9 (3), $\Delta_t^{m_2+1}$ is nonempty and compact valued, and continuous.

(2) It is obvious that $\Delta_t$ is nonempty valued, we shall first prove that it is upper hemicontinuous and compact valued.

Given sequences $\{h_{t-1}^k\}$ and $\tau^k \subseteq \Delta_t(h_{t-1}^k)$, there exists a sequence of $\{\xi^k\}_{k\geq 1}$ such that $\xi^k = (\xi_1^k, \xi_2^k, \ldots) \in \Upsilon$ and $\tau^k = \varrho_{(h_{t-1}^k, \xi^k)}$ for each $k \geq 1$. Suppose that $h_{t-1}^k \to h_{t-1}^\infty$. By (1), $\Delta_t^t$ is compact valued and upper hemicontinuous. Then there exists a measurable mapping $g_t$ such that (1) $g^t = (\xi_1^1, \ldots, \xi_{t-1}^1, g_t, \xi_{t+1}^1, \ldots) \in \Upsilon$, and (2) a subsequence of $\{\varrho^t_{(h_{t-1}^k, \xi^k)}\}$, say $\{\varrho^t_{(h_{t-1}^{k_{1l}}, \xi^{k_{1l}})}\}_{l\geq 1}$, weakly converges to $\varrho^t_{(h_{t-1}^\infty, g^t)}$. Note that $\{\xi_{t+1}^k\}$ is a Borel measurable selection of $\mathcal{M}(A_{t+1})$. By Lemma 9 (3), there is a Borel measurable selection $g_{t+1}$ of $\mathcal{M}(A_{t+1})$ such that there is a subsequence of $\{\varrho^{t+1}_{(h_{t-1}^{k_{1l}}, \xi^{k_{1l}})}\}_{l\geq 1}$, say $\{\varrho^{t+1}_{(h_{t-1}^{k_{2l}}, \xi^{k_{2l}})}\}_{l\geq 1}$, weakly converges to $\varrho^{t+1}_{(h_{t-1}^\infty, g^{t+1})}$, where $g^{t+1} = (\xi_1^1, \ldots, \xi_{t-1}^1, g_t, g_{t+1}, \xi_{t+2}^1, \ldots) \in \Upsilon$. Repeat this procedure, one can construct a Borel measurable mapping $g$ such that $\varrho_{(h_{t-1}^{k_{11}}, \xi^{k_{11}})}, \varrho_{(h_{t-1}^{k_{22}}, \xi^{k_{22}})}, \varrho_{(h_{t-1}^{k_{33}}, \xi^{k_{33}})}, \ldots$ weakly converges to $\varrho_{(h_{t-1}^\infty, g)}$. That is, $\varrho_{(h_{t-1}^\infty, g)}$ is a convergent point of $\{\varrho_{(h_{t-1}^k, \xi^k)}\}$. By Lemma 1 (5), $\Delta_t$ is compact valued and upper hemicontinuous.

Next, we consider the lower hemicontinuity of $\Delta_t$. Suppose that $\tau^0 \in \Delta_t(h_{t-1}^0)$. Then there exists some $\xi \in \Upsilon$ such that $\tau^0 = \varrho_{(h_{t-1}^0, \xi)}$. Denote $\tilde{\tau}^m = \varrho^m_{(h_{t-1}^0, \xi)} \in \Delta_t^m(h_{t-1}^0)$ for $m \geq t$. As $\Delta_t^m$ is continuous, for each $m$, there exists some $\xi^m \in \Upsilon$ such that $d(\varrho^m_{(h_{t-1}^{k_m}, \xi^m)}, \tilde{\tau}^m) \leq \frac{1}{m}$ for $k_m$ sufficiently large, where $d$ is the Prokhorov metric. Let $\tau^m = \varrho_{(h_{t-1}^{k_m}, \xi^m)}$. Then $\tau^m$ weakly converges to $\tau^0$, which implies that $\Delta_t$ is lower hemicontinuous. □

Below, we shall define a correspondence $Q_t^\tau$ from $H_{t-1}$ to $\mathbb{R}_{++}^n$ inductively for any stages $t, \tau \geq 1$. When $\tau < t$, $Q_t^\tau(h_{t-1})$ is the set of all the payoffs based on correlated strategies in the subgame $h_{t-1}$, which does not depend on $\tau$. As a result, for any stage $\tau$, $Q_{\tau+1}^\tau$ can be defined. Then for $\tau \geq t$, $Q_t^\tau$ is the correspondence obtained by repeating the backward induction from the correspondence $Q_{\tau+1}^\tau$. Specifically,

$$Q_t^\tau(h_{t-1}) =$$

$$\begin{cases} \{\int_{\prod_{m\geq t}(X_m\times S_m)} u(h_{t-1},x,s)\varrho_{(h_{t-1},\xi)}(\mathrm{d}(x,s))\colon \varrho_{(h_{t-1},\xi)}\in \Delta_t(h_{t-1})\} & \tau<t; \\ \Phi(Q_{t+1}^{\tau})(h_{t-1}) & \tau\geq t. \end{cases}$$

The lemma below presents several desirable properties of $Q_t^\tau$.

**Lemma 12.** *For any $t,\tau \geq 1$, $Q_t^\tau$ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous.*

*Proof.* For $t > \tau$, $Q_t^\tau$ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous because of the corresponding properties of $u$ and $\Delta_t$.

For $t \leq \tau$, we can start with $Q_{\tau+1}^\tau$. Repeating the backward induction in Section 6.2.1, $Q_t^\tau$ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous. □

Denote $Q_t^\infty = \cap_{\tau\geq 1} Q_t^\tau$. Recall that $Q_t^\tau$ is the payoff correspondence of correlated strategies when $\tau < t$, and the correspondence obtained by repeating the backward induction from $Q_{\tau+1}^\tau$ when $\tau \geq t$. Given some $t$, when $\tau$ is sufficiently large, it is expected that $Q_{\tau+1}^\tau$ should be close to the payoff correspondence of all the mixed strategies, as the game is continuous at infinity. Given the correspondence $Q_{\tau+1}^\tau$, players play some equilibrium strategies in each backward induction step. As a result, it is expected that $Q_t^\tau$ would be close to the actual equilibrium payoff correspondence $E_t$ for sufficiently large $\tau$. The following three lemmas show that $Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1}) = E_t(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$.[46]

**Lemma 13.** *1. The correspondence $Q_t^\infty$ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous.*

*2. For any $t \geq 1$, $Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$.*

*Proof.* (1) It is obvious that $Q_t^\infty$ is bounded. By the definition of $Q_t^\tau$, for all $h_{t-1} \in H_{t-1}$, $Q_t^{\tau_1}(h_{t-1}) \subseteq Q_t^{\tau_2}(h_{t-1})$ for $\tau_1 \geq \tau_2$. Since $Q_t^\tau$ is nonempty and compact valued, $Q_t^\infty = \cap_{\tau\geq 1} Q_t^\tau$ is nonempty and compact valued. By Lemma 2 (2), $\cap_{\tau\geq 1} Q_t^\tau$ is measurable, which implies that $Q_t^\infty$ is measurable.

Since $Q_t^\tau$ is upper hemicontinuous for any $\tau$, by Lemma 2 (7), it has a closed graph for each $\tau$, which implies that $Q_t^\infty$ has a closed graph. Referring to Lemma 2 (7) again, $Q_t^\infty$ is upper hemicontinuous.

(2) For any $\tau \geq 1$ and $h_{t-1} \in H_{t-1}$, $\Phi(Q_{t+1}^\infty)(h_{t-1}) \subseteq \Phi(Q_{t+1}^\tau)(h_{t-1}) \subseteq Q_t^\tau(h_{t-1})$, and hence $\Phi(Q_{t+1}^\infty)(h_{t-1}) \subseteq Q_t^\infty(h_{t-1})$.

Let $\{1, 2, \ldots \infty\}$ be a countable compact space endowed with the following metric: $d(k,m) = |\frac{1}{k} - \frac{1}{m}|$ for $1 \leq k,m \leq \infty$. The sequence $\{Q_{t+1}^\tau\}_{1\leq\tau\leq\infty}$

[46] The proofs of Lemmas 13 and 15 follow the standard argument with various modifications; see, for example, Harris (1990), Harris, Reny and Robson (1995) and Mariotti (2000).

can be regarded as a correspondence $Q_{t+1}$ from $H_t \times \{1, 2, \ldots, \infty\}$ to $\mathbb{R}^n$, which is measurable, nonempty and compact valued, and upper hemicontinuous on $H_t \times \{1, 2, \ldots, \infty\}$. The step of backward induction in Section 6.2.1 shows that $\Phi(Q_{t+1})$ is measurable, nonempty and compact valued, and upper hemicontinuous on $H_t \times \{1, 2, \ldots, \infty\}$. For $h_{t-1} \in H_{t-1}$ and $a \in Q_t^\infty(h_{t-1})$, by its definition, $a \in Q_t^\tau(h_{t-1}) = \Phi(Q_{t+1}^\tau)(h_{t-1})$ for $\tau \geq t$. Thus, $a \in \Phi(Q_{t+1}^\infty)(h_{t-1})$.

As a result, $Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$. □

Though the definition of $Q_t^\tau$ involves correlated strategies for $\tau < t$, the following lemma shows that one can work with mixed strategies in terms of equilibrium payoffs. In Lemma 13, it is shown that $Q_t^\infty = \Phi(Q_{t+1}^\infty)$ for any $t \geq 1$. Then one can apply the forward induction recursively to obtain the mixed strategies $\{f_{ki}\}_{i \in I}$ and a selection $c_k$ of $Q_k^\infty$ for each $k \geq 1$ such that $f_{ki}$ is an equilibrium and $c_k$ is the corresponding equilibrium payoff given the payoff function $c_{k+1}$. Lemma 14 shows that $\{f_{ki}\}_{k \geq 1, i \in I}$ is indeed an equilibrium and $c_k$ is the equilibrium payoff of the game in stage $k$. The key here is that since $Q_k^\infty \subseteq Q_k^\tau = \Phi^{\tau-k+1}(Q_{\tau+1}^\tau)$ for $k \leq \tau$, one can obtain $c_k$ via the strategies $\{f_{ki}\}_{k \leq k' \leq \tau, i \in I}$ between stages $k$ and $\tau$, and the correlated strategies $\{\xi_{k'}\}_{k' > \tau}$ after stage $\tau$. Because of the assumption of the continuity at infinity, the latter payoff converges to $c_k$ as $\tau \to \infty$. To check the equilibrium property, one can rely on the same asymptotic argument to check the payoff for any deviation. Then we show that $c_k$ is an equilibrium payoff without using the correlated strategies.

**Lemma 14.** *If $c_t$ is a measurable selection of $\Phi(Q_{t+1}^\infty)$, then $c_t(h_{t-1})$ is a subgame-perfect equilibrium payoff vector for any $h_{t-1} \in H_{t-1}$.*

*Proof.* Without loss of generality, we only prove the case $t = 1$.

Suppose that $c_1$ is a measurable selection of $\Phi(Q_2^\infty)$. Apply Proposition 3 recursively to obtain Borel measurable mappings $\{f_{ki}\}_{i \in I}$ for $k \geq 1$. That is, for any $k \geq 1$, there exists a Borel measurable selection $c_k$ of $Q_k^\infty$ such that for all $h_{k-1} \in H_{k-1}$,

1. $f_k(h_{k-1})$ is a Nash equilibrium in the subgame $h_{k-1}$, where the action space is $A_{ki}(h_{k-1})$ for player $i \in I$, and the payoff function is given by

$$\int_{S_k} c_{k+1}(h_{k-1}, \cdot, s_k) f_{k0}(\mathrm{d}s_k | h_{k-1}).$$

2. 

$$c_k(h_{k-1}) = \int_{A_k(h_{k-1})} \int_{S_k} c_{k+1}(h_{k-1}, x_k, s_k) f_{k0}(\mathrm{d}s_k | h_{k-1}) f_k(\mathrm{d}x_k | h_{k-1}).$$

We need to show that $c_1(h_0)$ is a subgame-perfect equilibrium payoff vector for all $h_0 \in H_0$.

Step 1. We show that for any $k \geq 1$ and all $h_{k-1} \in H_{k-1}$,

$$c_k(h_{k-1}) = \int_{\prod_{m\geq k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f)}(\mathrm{d}(x, s)).$$

Fix a positive integer $M > k$. By Lemma 13, $c_k(h_{k-1}) \in Q_k^\infty(h_{k-1}) = \cap_{\tau\geq 1} Q_k^\tau(h_{k-1})$ for all $h_{k-1} \in H_{k-1}$. Since $Q_k^\tau = \Phi^{\tau-k+1}(Q_{\tau+1}^\tau)$ for $k \leq \tau$, $c_k(h_{k-1}) \in \cap_{\tau\geq k} \Phi^{\tau-k+1}(Q_{\tau+1}^\tau)(h_{k-1}) \subseteq \Phi^{M-k+1}(Q_{M+1}^M)(h_{k-1})$ for all $h_{k-1} \in H_{k-1}$. Thus, there exists a Borel measurable selection $w$ of $Q_{M+1}^M$ and some $\xi \in \Upsilon$ such that for all $h_{M-1} \in H_{M-1}$,

i. $f_M(h_{M-1})$ is a Nash equilibrium in the subgame $h_{M-1}$, where the action space is $A_{Mi}(h_{M-1})$ for player $i \in I$, and the payoff function is given by

$$\int_{S_M} w(h_{M-1}, \cdot, s_M) f_{M0}(\mathrm{d}s_M | h_{M-1});$$

ii.

$$c_M(h_{M-1}) = \int_{A_M(h_{M-1})} \int_{S_M} w(h_{M-1}, x_M, s_M) f_{M0}(\mathrm{d}s_M | h_{M-1}) f_M(\mathrm{d}x_M | h_{M-1});$$

iii. $w(h_M) = \int_{\prod_{m\geq M+1}(X_m \times S_m)} u(h_M, x, s) \varrho_{(h_M, \xi)}(\mathrm{d}(x, s))$.

Then for $h_{k-1} \in H_{k-1}$,

$$c_k(h_{k-1}) = \int_{\prod_{m\geq k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f^M)}(\mathrm{d}(x, s)),$$

where $f_k^M$ is $f_k$ if $k \leq M$, and $\xi_k$ if $k \geq M+1$. Since the game is continuous,

$$\int_{\prod_{m\geq k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f^M)}(\mathrm{d}(x, s))$$

converges to

$$\int_{\prod_{m\geq k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f)}(\mathrm{d}(x, s))$$

when $M$ goes to infinity. Thus, for all $h_{k-1} \in H_{k-1}$,

$$c_k(h_{t-1}) = \int_{\prod_{m\geq k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f)}(\mathrm{d}(x, s)). \tag{2}$$

Step 2. Below, we show that $\{f_{ki}\}_{i\in I}$ is a subgame-perfect equilibrium.

Fix a player $i$ and a strategy $g_i = \{g_{ki}\}_{k\geq 1}$. For each $k \geq 1$, define a new strategy $\tilde{f}_i^k$ as follows: $\tilde{f}_i^k = (g_{1i}, \ldots, g_{ki}, f_{(k+1)i}, f_{(k+2)i}, \ldots)$. That is, we simply replace the initial $k$ stages of $f_i$ by $g_i$. Denote $\tilde{f}^k = (\tilde{f}_i^k, f_{k(-i)})$.

Fix $k \geq 1$. For any $h_k = (x^k, s^k)$, we have

$$\begin{aligned}
&\int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s)\varrho_{(h_k,f)}(\mathrm{d}(x,s)) \\
=& \int_{A_{k+1}(h_k)}\int_{S_{k+1}} c_{(k+2)i}(h_k, x_{k+1}, s_{k+1}) f_{(k+1)0}(\mathrm{d}s_{k+1}|h_k) f_{k+1}(\mathrm{d}x_{k+1}|h_k) \\
\geq& \int_{A_{k+1}(h_k)}\int_{S_{k+1}} c_{(k+2)i}(h_k, x_{k+1}, s_{k+1}) f_{(k+1)0}(\mathrm{d}s_{k+1}|h_k) \left(f_{(k+1)(-i)} \otimes g_{(k+1)i}\right)(\mathrm{d}x_{k+1}|h_k) \\
=& \int_{A_{k+1}(h_k)}\int_{S_{k+1}}\int_{A_{k+2}(h_k,x_{k+1},s_{k+1})}\int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\
&\quad f_{(k+2)0}(\mathrm{d}s_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes f_{(k+2)i}(\mathrm{d}x_{k+2}|h_k, x_{k+1}, s_{k+1}) \\
&\quad f_{(k+1)0}(\mathrm{d}s_{k+1}|h_k) f_{(k+1)(-i)} \otimes g_{(k+1)i}(\mathrm{d}x_{k+1}|h_k) \\
\geq& \int_{A_{k+1}(h_k)}\int_{S_{k+1}}\int_{A_{k+2}(h_k,x_{k+1},s_{k+1})}\int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\
&\quad f_{(k+2)0}(\mathrm{d}s_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes g_{(k+2)i}(\mathrm{d}x_{k+2}|h_k, x_{k+1}, s_{k+1}) \\
&\quad f_{(k+1)0}(\mathrm{d}s_{k+1}|h_k) f_{(k+1)(-i)} \otimes g_{(k+1)i}(\mathrm{d}x_{k+1}|h_k) \\
=& \int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s)\varrho_{(h_k,\tilde{f}^{k+2})}(\mathrm{d}(x,s)).
\end{aligned}$$

The first and the last equalities follow from Equation (2) in the end of step 1. The second equality is due to (ii) in step 1. The first inequality is based on (i) in step 1. The second inequality holds since by the choice of $h_k$ and (i) in step 1, for $f_{(k+1)0}(h_k)$-almost all $s_{k+1} \in S_{k+1}$ and all $x_{k+1} \in X_{k+1}$, we have

$$\begin{aligned}
&\int_{A_{k+2}(h_k,x_{k+1},s_{k+1})}\int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\
&f_{(k+2)0}(\mathrm{d}s_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes f_{(k+2)i}(\mathrm{d}x_{k+2}|h_k, x_{k+1}, s_{k+1}) \\
\geq& \int_{A_{k+2}(h_k,x_{k+1},s_{k+1})}\int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\
&f_{(k+2)0}(\mathrm{d}s_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes g_{(k+2)i}(\mathrm{d}x_{k+2}|h_k, x_{k+1}, s_{k+1}).
\end{aligned}$$

Repeating the above argument, one can show that

$$\int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s)\varrho_{(h_k,f)}(\mathrm{d}(x,s))$$

$$\geq \int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k,x,s)\varrho_{(h_k,\tilde{f}^{\tilde{M}+1})}(\mathrm{d}(x,s))$$

for any $\tilde{M} > k$. Since

$$\int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k,x,s)\varrho_{(h_k,\tilde{f}^{\tilde{M}+1})}(\mathrm{d}(x,s))$$

converges to

$$\int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k,x,s)\varrho_{(h_k,(g_i,f_{-i}))}(\mathrm{d}(x,s))$$

as $\tilde{M}$ goes to infinity, we have

$$\int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k,x,s)\varrho_{(h_k,f)}(\mathrm{d}(x,s))$$
$$\geq \int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k,x,s)\varrho_{(h_k,(g_i,f_{-i}))}(\mathrm{d}(x,s)).$$

Therefore, $\{f_{ki}\}_{i\in I}$ is a subgame-perfect equilibrium. □

By Lemmas 10 and 13, the correspondence $\Phi(Q_{t+1}^{\infty})$ is measurable, nonempty and compact valued. By Lemma 2 (3), it has a measurable selection. Then the equilibrium existence result in Theorem 1 follows from the above lemma.

For $t \geq 1$ and $h_{t-1} \in H_{t-1}$, recall that $E_t(h_{t-1})$ is the set of payoff vectors of subgame-perfect equilibria in the subgame $h_{t-1}$. The following lemma shows that $E_t(h_{t-1})$ is the same as $Q_t^{\infty}(h_{t-1})$.

**Lemma 15.** *For any $t \geq 1$, $E_t(h_{t-1}) = Q_t^{\infty}(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$.*

*Proof.* (1) We will first prove the following claim: for any $t$ and $\tau$, if $E_{t+1}(h_t) \subseteq Q_{t+1}^{\tau}(h_t)$ for all $h_t \in H_t$, then $E_t(h_{t-1}) \subseteq Q_t^{\tau}(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$. We only need to consider the case that $t \leq \tau$.

By the construction of $\Phi(Q_{t+1}^{\tau})$ in Section 6.2.1, for any $c_t$ and $h_{t-1} = (x^{t-1}, s^{t-1}) \in H_{t-1}$, if

1. $c_t = \int_{A_t(h_{t-1})}\int_{S_t} q_{t+1}(h_{t-1},x_t,s_t)f_{t0}(\mathrm{d}s_t|h_{t-1})\alpha(\mathrm{d}x_t)$, where $q_{t+1}(h_{t-1},\cdot)$ is measurable and $q_{t+1}(h_{t-1},x_t,s_t) \in Q_{t+1}^{\tau}(h_{t-1},x_t,s_t)$ for all $s_t \in S_t$ and $x_t \in A_t(h_{t-1})$;
2. $\alpha \in \otimes_{i\in I}\mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame $h_{t-1}$ with payoff $\int_{S_t} q_{t+1}(h_{t-1},\cdot,s_t)f_{t0}(\mathrm{d}s_t|h_{t-1})$ and action space $\prod_{i\in I} A_{ti}(h_{t-1})$,

then $c_t \in \Phi(Q_{t+1}^{\tau})(h_{t-1})$.

Fix a subgame $h_{t-1} = (x^{t-1}, s^{t-1})$. Pick a point $c_t \in E_t(h_{t-1})$. There exists a strategy profile $f$ such that $f$ is a subgame-perfect equilibrium in the subgame

$h_{t-1}$ and the payoff is $c_t$. Let $c_{t+1}(h_{t-1}, x_t, s_t)$ be the payoff vector induced by $\{f_{ti}\}_{i \in I}$ in the subgame $(h_{t-1}, x_t, s_t) \in \mathrm{Gr}(A_t) \times S_t$. Then we have

1. $c_t = \int_{A_t(h_{t-1})} \int_{S_t} c_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}) f_t(\mathrm{d}x_t|h_{t-1})$;
2. $f_t(\cdot|h_{t-1})$ is a Nash equilibrium in the subgame $h_{t-1}$ with action space $A_t(h_{t-1})$ and payoff $\int_{S_t} c_{t+1}(h_{t-1}, \cdot, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1})$.

Since $f$ is a subgame-perfect equilibrium in the subgame $h_{t-1}$, $c_{t+1}(h_{t-1}, x_t, s_t) \in E_{t+1}(h_{t-1}, x_t, s_t) \subseteq Q^\tau_{t+1}(h_{t-1}, x_t, s_t)$ for all $s_t \in S_t$ and $x_t \in A_t(h_{t-1})$, which implies that $c_t \in \Phi(Q^\tau_{t+1})(h_{t-1}) = Q^\tau_t(h_{t-1})$.

Therefore, $E_t(h_{t-1}) \subseteq Q^\tau_t(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$.

(2) For any $t > \tau$, $E_t \subseteq Q^\tau_t$. If $t \leq \tau$, we can start with $E_{\tau+1} \subseteq Q^\tau_{\tau+1}$ and repeat the argument in (1), then we can show that $E_t(h_{t-1}) \subseteq Q^\tau_t(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$. Thus, $E_t(h_{t-1}) \subseteq Q^\infty_t(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$.

Suppose that $c_t$ is a measurable selection from $\Phi(Q^\infty_{t+1})$. Apply Proposition 3 recursively to obtain Borel measurable mappings $\{f_{ki}\}_{i \in I}$ for $k \geq t$. By Lemma 14, $c_t(h_{t-1})$ is a subgame-perfect equilibrium payoff vector for all $h_{t-1} \in H_{t-1}$. Consequently, $\Phi(Q^\infty_{t+1})(h_{t-1}) \subseteq E_t(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$.

By Lemma 13, $E_t(h_{t-1}) = Q^\infty_t(h_{t-1}) = \Phi(Q^\infty_{t+1})(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$. □

Therefore, we complete the proof of Theorem 1.

## 6.3 Proof of Theorem 2

Step 1. Backward induction.

For any $t \geq 1$, suppose that the correspondence $Q_{t+1}$ from $H_t$ to $\mathbb{R}^n$ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous on $X^t$.

If player $j$ is the active player in stage $t$, then we assume that $S_t = \{\acute{s}_t\}$. Thus, $P_t(h_{t-1}, x_t) = Q_{t+1}(h_{t-1}, x_t, \acute{s}_t)$, which is nonempty and compact valued, and upper hemicontinuous. Note that $P_t$ may not be convex valued. Then define the correspondence $\Phi_t$ from $H_{t-1}$ to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \triangle(X_t)$ as $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if

1. $v = p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x^*_{tj})$ such that $p_t(h_{t-1}, \cdot)$ is a measurable selection of $P_t(h_{t-1}, \cdot)$;
2. $x^*_{tj} \in A_{tj}(h_{t-1})$ is a maximization point of player $j$ given the payoff function $p_{tj}(h_{t-1}, A_{t(-j)}(h_{t-1}), \cdot)$ and the action space $A_{tj}(h_{t-1})$, $\alpha_i = \delta_{A_{ti}(h_{t-1})}$ for $i \neq j$ and $\alpha_j = \delta_{x^*_{tj}}$;
3. $\mu = \delta_{p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x^*_{tj})}$.

If $P_t$ is nonempty, convex and compact valued, and upper hemicontinuous, then we can use Lemma 10, the main result of Simon and Zame (1990), to prove the nonemptiness, compactness, and upper hemicontinuity of $\Phi_t$. In Simon and Zame (1990), the only step they need the convexity of $P_t$ for the proof of their main theorem is Lemma 2 therein. However, the one-player pure-strategy version of their Lemma 2, stated in the following paragraph, directly follows from the upper hemicontinuity of $P_t$ without requiring the convexity.

> Let $Z$ be a compact metric space, and $\{z_n\}_{n \geq 0} \subseteq Z$. Let $P\colon Z \to \mathbb{R}_+$ be a bounded, upper hemicontinuous correspondence with nonempty and compact values. For each $n \geq 1$, let $q_n$ be a Borel measurable selection of $P$ such that $q_n(z_n) = d_n$. If $z_n$ converges to $z_0$ and $d_n$ converges to some $d_0$, then $d_0 \in P(z_0)$.

Repeat the argument in the proof of the main theorem of Simon and Zame (1990), one can show that $\Phi_t$ is nonempty and compact valued, and upper hemicontinuous.

Next, we consider the case that Nature moves in stage $t$. That is, there is no active player in $I$ moving in this stage and $A_t(h_{t-1})$ is a singleton set. Suppose that the correspondence $Q_{t+1}$ from $H_t$ to $\mathbb{R}^n$ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous. Let

$$P_t(h_{t-1}, x_t) = \int_{S_t} Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t | h_{t-1}),$$

where $A_t(h_{t-1}) = \{x_t\}$. Since $f_{t0}(\cdot | h_{t-1})$ is atomless, as in Section 6.2.1, $P_t$ is nonempty, convex and compact valued, and upper hemicontinuous. The rest of the step remains the same as in Section 6.2.1.

In summary, $\Phi_t$ is nonempty and compact valued, and upper hemicontinuous.

Steps 2 and 3. Forward induction and the infinite horizon case.

These two steps are the same as that in Section 6.2, except the corresponding notations need to be changed to be consistent with the perfect information environment whenever necessary.

**Remark 4.** *Theorem 2 remains to be true if the state transitions either are atomless, or have the support inside a fixed finite set irrespective of the history at a particular stage. In the backward induction step, at the stage $t$ that Nature is active and concentrates inside a fixed finite set $\{s_{t1}, \ldots, s_{tK}\}$, we have*

$$P_t(h_{t-1}, x_t) = \sum_{s_{tk} \in \{s_{t1}, \ldots, s_{tK}\}} Q_{t+1}(h_{t-1}, x_t, s_{tk}) f_{t0}(\{s_{tk}\} | h_{t-1}),$$

*where $A_t(h_{t-1}) = \{x_t\}$. Note that $P_t$ is also nonempty and compact valued, and upper hemicontinuous. The proof is the same in other cases. Similarly, Theorem 4* *still holds if the state transitions either satisfy the ARM condition, or have the support inside a fixed finite set irrespective of the history at a particular stage.*

## 6.4 Proof of Proposition 1

The proof is essentially the combination of the proofs in Sections 6.2 and 6.3. That is, when there is only one active player, we refer to the argument in Section 6.3. When there are more than one active players or Nature is the only player who moves, we modify the argument in Section 6.2.

Step 1. Backward induction.

For any $t \geq 1$, suppose that the correspondence $Q_{t+1}$ from $H_t$ to $\mathbb{R}^n$ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous on $X^t$.

If $N_t = 1$, then $S_t = \{\acute{s}_t\}$. Thus, $P_t(h_{t-1}, x_t) = Q_{t+1}(h_{t-1}, x_t, \acute{s}_t)$, which is nonempty and compact valued, and upper hemicontinuous. Then define the correspondence $\Phi_t$ from $H_{t-1}$ to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \triangle(X_t)$ as $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if

1. $v = p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x^*_{tj})$ such that $p_t(h_{t-1}, \cdot)$ is a measurable selection of $P_t(h_{t-1}, \cdot)$;
2. $x^*_{tj} \in A_{tj}(h_{t-1})$ is a maximization point of player $j$ given the payoff function $p_{tj}(h_{t-1}, A_{t(-j)}(h_{t-1}), \cdot)$ and the action space $A_{tj}(h_{t-1})$, $\alpha_i = \delta_{A_{ti}(h_{t-1})}$ for $i \neq j$ and $\alpha_j = \delta_{x^*_{tj}}$;
3. $\mu = \delta_{p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x^*_{tj})}$.

As discussed in Section 6.3, $\Phi_t$ is nonempty and compact valued, and upper hemicontinuous.

When $N_t = 0$, for any $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$,

$$P_t(h_{t-1}, x_t) = \int_{A_{t0}(h_{t-1}, x_t)} Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t | h_{t-1}, x_t).$$

Let $\mathrm{co}Q_{t+1}(h_{t-1}, x_t, s_t)$ be the convex hull of $Q_{t+1}(h_{t-1}, x_t, s_t)$. Because $Q_{t+1}$ is bounded, nonempty and compact valued, $\mathrm{co}Q_{t+1}$ is bounded, nonempty, convex and compact valued. By Lemma 2 (8), $\mathrm{co}Q_{t+1}$ is upper hemicontinuous.

Note that $f_{t0}(\cdot | h_{t-1}, x_t)$ is atomless and $Q_{t+1}$ is nonempty and compact valued. We have

$$P_t(h_{t-1}, x_t) = \int_{A_{t0}(h_{t-1}, x_t)} \mathrm{co}Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t | h_{t-1}, x_t).$$

By Lemma 7, $P_t$ is bounded, nonempty, convex and compact valued, and upper hemicontinuous. Then by Lemma 10, one can conclude that $\Phi_t$ is bounded, nonempty and compact valued, and upper hemicontinuous.

Steps 2 and 3. Forward induction and the infinite horizon case.

These two steps are the same as that in Section 6.2. The only change is to modify the notations correspondingly.

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# Dynamic Games with (Almost) Perfect Information: Appendix B

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In Section B.1, we shall present the model of measurable dynamic games with partially perfect information and show the existence of subgame-perfect equilibria in Proposition B.1. It covers the results in Theorem 3 (Theorem 4) for dynamic games with almost perfect information (perfect information), and in discounted stochastic games.

In Section B.2, we present Lemmas B.1-B.6 as the mathematical preparations for proving Theorem 3. We present in Section B.3 a new equilibrium existence result for discontinuous games with stochastic endogenous sharing rules. The proof of Theorem 3 is given in Section B.4. The proof of Proposition B.1 is provided in Section B.5, which covers Theorem 4 as a special case. One can skip Sections B.2 and B.3 first, and refer to the technical results in these two sections whenever necessary.

## B.1 Measurable dynamic games with partially perfect information

In this section, we will generalize the model of measurable dynamic games in three directions. The ARM condition is partially relaxed such that (1) perfect information may be allowed in some stages, (2) the state transitions could have a weakly continuous component in all other stages, and (3) the state transition in any period can depend on the action profile in the current stage as well as on the previous history. The first change allows us to combine the models of dynamic games with perfect and almost perfect information. The second generalization implies that the state transitions need not be

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norm continuous on the Banach space of finite measures. The last modification covers the model of stochastic games as a special case.

The changes are described below.

1. The state space is a product space of two Polish spaces; that is, $S_t = \hat{S}_t \times \tilde{S}_t$ for each $t \geq 1$.

2. For each $i \in I$, the action correspondence $A_{ti}$ from $H_{t-1}$ to $X_{ti}$ is measurable, nonempty and compact valued, and sectionally continuous on $X^{t-1} \times \hat{S}^{t-1}$. The additional component of Nature is given by a measurable, nonempty and closed valued correspondence $\hat{A}_{t0}$ from $\mathrm{Gr}(A_t)$ to $\hat{S}^t$, which is sectionally continuous on $X^t \times \hat{S}^{t-1}$. Then $H_t = \mathrm{Gr}(\hat{A}_{t0}) \times \tilde{S}_t$, and $H_\infty$ is the subset of $X^\infty \times S^\infty$ such that $(x, s) \in H_\infty$ if $(x^t, s^t) \in H_t$ for any $t \geq 0$.

3. The choice of Nature depends not only on the history $h_{t-1}$, but also on the action profile $x_t$ in the current stage. The state transition $f_{t0}(h_{t-1}, x_t) = \hat{f}_{t0}(h_{t-1}, x_t) \diamond \tilde{f}_{t0}(h_{t-1}, x_t)$, where $\hat{f}_{t0}$ is a transition probability from $\mathrm{Gr}(A_t)$ to $\mathcal{M}(\hat{S}_t)$ such that $\hat{f}_{t0}(\hat{A}_{t0}(h_{t-1}, x_t)|h_{t-1}, x_t) = 1$ for all $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$, and $\tilde{f}_{t0}$ is a transition probability from $\mathrm{Gr}(\hat{A}_{t0})$ to $\mathcal{M}(\tilde{S}_t)$.

4. For each $i \in I$, the payoff function $u_i$ is a Borel measurable mapping from $H_\infty$ to $\mathbb{R}_{++}$, which is sectionally continuous on $X^\infty \times \hat{S}^\infty$.

As in Subsection 3.3, we allow the possibility for the players to have perfect information in some stages. For $t \geq 1$, let

$$N_t = \begin{cases} 1, & \text{if } f_{t0}(h_{t-1}, x_t) \equiv \delta_{s_t} \text{ for some } s_t \text{ and} \\ & |\{i \in I \colon A_{ti} \text{ is not point valued}\}| = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if $N_t = 1$ for some stage $t$, then the player who is active in the period $t$ is the only active player and has perfect information.

We will drop the ARM condition in those periods with only one active player, and weaken the ARM condition in other periods.

**Assumption B.1** (ARM′). *1. For any $t \geq 1$ with $N_t = 1$, $S_t$ is a singleton set $\{\acute{s}_t\}$ and $\lambda_t = \delta_{\acute{s}_t}$.*

*2. For each $t \geq 1$ with $N_t = 0$, $\hat{f}_{t0}$ is sectionally continuous on $X^t \times \hat{S}^{t-1}$, where the range space $\mathcal{M}(\hat{S}_t)$ is endowed with topology of weak convergence of measures on*

$\hat{S}_t$. *The probability measure* $\tilde{f}_{t0}(\cdot|h_{t-1}, x_t, \hat{s}_t)$ *is absolutely continuous with respect to an atomless Borel probability measure* $\lambda_t$ *on* $\tilde{S}_t$ *for all* $(h_{t-1}, x_t, \hat{s}_t) \in Gr(\hat{A}_{t0})$, *and* $\varphi_{t0}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t)$ *is the corresponding density.*[1]

3. *The mapping* $\varphi_{t0}$ *is Borel measurable and sectionally continuous on* $X^t \times \hat{S}^t$, *and integrably bounded in the sense that there is a* $\lambda_t$*-integrable function* $\phi_t \colon \tilde{S}_t \to \mathbb{R}_+$ *such that* $\varphi_{t0}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t,) \leq \phi_t(\tilde{s}_t)$ *for any* $(h_{t-1}, x_t, \hat{s}_t)$.

The following result shows that the existence result is still true in this more general setting.

**Proposition B.1.** *If an infinite-horizon dynamic game as described above satisfies the ARM′ condition and is continuous at infinity, then it possesses a subgame-perfect equilibrium* $f$. *In particular, for* $j \in I$ *and* $t \geq 1$ *such that* $N_t = 1$ *and player* $j$ *is the only active player in this period,* $f_{tj}$ *can be deterministic. Furthermore, the equilibrium payoff correspondence* $E_t$ *is nonempty and compact valued, and essentially sectionally upper hemicontinuous on* $X^{t-1} \times \hat{S}^{t-1}$.

**Remark B.1.** *The result above also implies a new existence result of subgame-perfect equilibria for stochastic games. In the existence result of [6], the state transitions are assumed to be norm continuous with respect to the actions in the previous stage. They did not assume the ARM condition. On the contrary, our Proposition B.1 allows the state transitions to have a weakly continuous component.*

## B.2 Technical preparations

The following lemma shows that the space of nonempty compact subsets of a Polish space is still Polish under the Hausdorff metric topology.

**Lemma B.1.** *Suppose that* $X$ *is a Polish space and* $\mathcal{K}$ *is the set of all nonempty compact subsets of* $X$ *endowed with the Hausdorff metric topology. Then* $\mathcal{K}$ *is a Polish space.*

*Proof.* By Theorem 3.88 (2) of [1], $\mathcal{K}$ is complete. In addition, Corollary 3.90 and Theorem 3.91 of [1] imply that $\mathcal{K}$ is separable. Thus, $\mathcal{K}$ is a Polish space. □

The following result presents a variant of Lemma 5 in terms of transition correspondences.

[1] In this section, a property is said to hold for $\lambda^t$-almost all $h_t \in H_t$ if it is satisfied for $\lambda^t$-almost all $\tilde{s}^t \in \tilde{S}^t$ and all $(x^t, \hat{s}^t) \in H_t(\tilde{s}^t)$.

**Lemma B.2.** *Let $X$ and $Y$ be Polish spaces, and $Z$ a compact subset of $\mathbb{R}^l_+$. Let $G$ be a measurable, nonempty and compact valued correspondence from $X$ to $\mathcal{M}(Y)$. Suppose that $F$ is a measurable, nonempty, convex and compact valued correspondence from $X \times Y$ to $Z$. Define a correspondence $\Pi$ from $X$ to $Z$ as follows:*

$$\Pi(x) = \{ \int_Y f(x,y)g(\mathrm{d}y|x) \colon g \text{ is a Borel measurable selection of } G, \\ f \text{ is a Borel measurable selection of } F\}.$$

*If $F$ is sectionally continuous on $Y$, then*

1. *the correspondence $\tilde{F} \colon X \times \mathcal{M}(Y) \to Z$ as $\tilde{F}(x,\nu) = \int_Y F(x,y)\nu(\mathrm{d}y)$ is sectionally continuous on $\mathcal{M}(Y)$; and*

2. *$\Pi$ is a measurable, nonempty and compact valued correspondence.*

3. *If $F$ and $G$ are both continuous, then $\Pi$ is continuous.*

*Proof.* (1) For any fixed $x \in X$, the upper hemicontinuity of $\tilde{F}(x,\cdot)$ follows from Lemma 7.

Next, we shall show the lower hemicontinuity. Fix any $x \in X$. Suppose that $\{\nu_j\}_{j\geq 0}$ is a sequence in $\mathcal{M}(Y)$ such that $\nu_j \to \nu_0$ as $j \to \infty$. Pick an arbitrary point $z_0 \in \tilde{F}(x,\nu_0)$. Then there exists a Borel measurable selection $f$ of $F(x,\cdot)$ such that $z_0 = \int_Y f(y)\nu_0(\mathrm{d}y)$.

By Lemma 3 (Lusin's theorem), for each $k \geq 1$, there exists a compact subset $D_k \subseteq Y$ such that $f$ is continuous on $D_k$ and $\nu_0(Y \setminus D_k) < \frac{1}{3kM}$, where $M > 0$ is the bound of $Z$. Define a correspondence $F_k \colon Y \to Z$ as follows:

$$F_k(y) = \begin{cases} \{f(y)\}, & y \in D_k; \\ F(x,y), & y \in Y \setminus D_k. \end{cases}$$

Then $F_k$ is nonempty, convex and compact valued, and lower hemicontinuous. By Theorem 3.22 in [1], $Y$ is paracompact. Then by Lemma 3 (Michael's selection theorem), $F_k$ has a continuous selection $f_k$.

For each $k$, since $\nu_j \to \nu_0$ and $f_k$ is bounded and continuous, $\int_Y f_k(y)\nu_j(\mathrm{d}y) \to \int_Y f_k(y)\nu_0(\mathrm{d}y)$ as $j \to \infty$. Thus, there exists a subsequence $\{\nu_{j_k}\}$ such that $\{j_k\}$ is an increasing sequence, and for each $k \geq 1$,

$$\left\| \int_Y f_k(y)\nu_{j_k}(\mathrm{d}y) - \int_Y f_k(y)\nu_0(\mathrm{d}y) \right\| < \frac{1}{3k},$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^l$.

Since $f_k$ coincides with $f$ on $D_k$, $\nu_0(Y \setminus D_k) < \frac{1}{3kM}$, and $Z$ is bounded by $M$,

$$\left\| \int_Y f_k(y)\nu_0(\mathrm{d}y) - \int_Y f(y)\nu_0(\mathrm{d}y) \right\| < \frac{2}{3k}.$$

Thus,

$$\left\| \int_Y f_k(y)\nu_{j_k}(\mathrm{d}y) - \int_Y f(y)\nu_0(\mathrm{d}y) \right\| < \frac{1}{k}.$$

Let $z_{j_k} = \int_Y f_k(y)\nu_{j_k}(\mathrm{d}y)$ for each $k$. Then $z_{j_k} \in \tilde{F}(x, \nu_{j_k})$ and $z_{j_k} \to z_0$ as $k \to \infty$. By Lemma 1, $\tilde{F}(x, \cdot)$ is lower hemicontinuous.

(2) Since $G$ is measurable and compact valued, there exists a sequence of Borel measurable selections $\{g_k\}_{k\geq 1}$ of $G$ such that $G(x) = \overline{\{g_1(x), g_2(x), \ldots\}}$ for any $x \in X$ by Lemma 2 (5). For each $k \geq 1$, define a correspondence $\Pi^k$ from $X$ to $Z$ by letting $\Pi^k(x) = \tilde{F}(x, g_k(x)) = \int_Y F(x, y) g_k(\mathrm{d}y|x)$. Since $F$ is convex valued, so is $\Pi^k$. By Lemma 5, $\Pi^k$ is also measurable, nonempty and compact valued.

Fix any $x \in X$. It is clear that $\Pi(x) = \tilde{F}(x, G(x))$ is nonempty valued. Since $G(x)$ is compact, and $\tilde{F}(x, \cdot)$ is compact valued and continuous, $\Pi(x)$ is compact by Lemma 2. Thus, $\overline{\bigcup_{k\geq 1} \Pi^k(x)} \subseteq \Pi(x)$.

Fix any $x \in X$ and $z \in \Pi(x)$. There exists a point $\nu \in G(x)$ such that $z \in \tilde{F}(x, \nu)$. Since $\{g_k(x)\}_{k\geq 1}$ is dense in $G(x)$, it has a subsequence $\{g_{k_m}(x)\}$ such that $g_{k_m}(x) \to \nu$. As $\tilde{F}(x, \cdot)$ is continuous, $\tilde{F}(x, g_{k_m}(x)) \to \tilde{F}(x, \nu)$. That is,

$$z \in \overline{\bigcup_{k\geq 1} \tilde{F}(x, g_k(x))} = \overline{\bigcup_{k\geq 1} \Pi^k(x)}.$$

Therefore, $\overline{\bigcup_{k\geq 1} \Pi^k(x)} = \Pi(x)$ for any $x \in X$. Lemma 2 (1) and (2) imply that $\Pi$ is measurable.

(3) Define a correspondence $\hat{F} \colon \mathcal{M}(X \times Y) \to Z$ as follows:

$$\hat{F}(\tau) = \left\{ \int_{X\times Y} f(x, y)\tau(\mathrm{d}(x, y)) \colon f \text{ is a Borel measurable selection of } F \right\}.$$

By (1), $\hat{F}$ is continuous. Define a correspondence $\hat{G} \colon X \to \mathcal{M}(X \times Y)$ as $\hat{G}(x) = \{\delta_x \otimes \nu : \nu \in G(x)\}$. Since $\hat{G}$ and $\hat{F}$ are both nonempty valued, $\Pi(x) = \hat{F}(\hat{G}(x))$ is nonempty. As

$\hat{G}$ is compact valued and $\hat{F}$ is continuous, $\Pi$ is compact valued by Lemma 2. As $\hat{G}$ and $\hat{F}$ are both continuous, $\Pi$ is continuous by Lemma 1 (7). □

The following lemma shows that a measurable and sectionally continuous correspondence on a product space is approximately continuous on the product space.

**Lemma B.3.** *Let $S$, $X$ and $Y$ be Polish spaces endowed with the Borel $\sigma$-algebras, and $\lambda$ a Borel probability measure on $S$. Denote $\mathcal{S}$ as the completion of the Borel $\sigma$-algebra $\mathcal{B}(S)$ of $S$ under the probability measure $\lambda$. Suppose that $D$ is a $\mathcal{B}(S) \otimes \mathcal{B}(Y)$-measurable subset of $S \times Y$, where $D(s)$ is nonempty and compact for all $s \in S$. Let $A$ be a nonempty and compact valued correspondence from $D$ to $X$, which is sectionally continuous on $Y$ and has a $\mathcal{B}(S \times Y \times X)$-measurable graph. Then*

*(i) $\tilde{A}(s) = Gr(A(s, \cdot))$ is an $\mathcal{S}$-measurable mapping from $S$ to the set of nonempty and compact subsets $\mathcal{K}_{Y \times X}$ of $Y \times X$;*

*(ii) there exist countably many disjoint compact subsets $\{S_m\}_{m \geq 1}$ of $S$ such that (1) $\lambda(\cup_{m \geq 1} S_m) = 1$, and (2) for each $m \geq 1$, $D_m = D \cap (S_m \times Y)$ is compact, and $A$ is nonempty and compact valued, and continuous on each $D_m$.*

*Proof.* (i) $A(s, \cdot)$ is continuous and $D(s)$ is compact, $\text{Gr}(A(s, \cdot)) \subseteq Y \times X$ is compact by Lemma 2. Thus, $\tilde{A}$ is nonempty and compact valued. Since $A$ has a measurable graph, $\tilde{A}$ is an $\mathcal{S}$-measurable mapping from $S$ to the set of nonempty and compact subsets $\mathcal{K}_{Y \times X}$ of $Y \times X$ by Lemma 1 (4).

(ii) Define a correspondence $\tilde{D}$ from $S$ to $Y$ such that $\tilde{D}(s) = \{y \in Y : (s, y) \in D\}$. Then $\tilde{D}$ is nonempty and compact valued. As in (i), $\tilde{D}$ is $\mathcal{S}$-measurable. By Lemma 3 (Lusin's Theorem), there exists a compact subset $S_1 \subseteq S$ such that $\lambda(S \setminus S_1) < \frac{1}{2}$, $\tilde{D}$ and $\tilde{A}$ are continuous functions on $S_1$. By Lemma 1 (3), $\tilde{D}$ and $\tilde{A}$ are continuous correspondences on $S_1$. Let $D_1 = \{(s, y) \in D : s \in S_1, y \in \tilde{D}(s)\}$. Since $S_1$ is compact and $\tilde{D}$ is continuous, $D_1$ is compact (see Lemma 2 (6)).

Following the same procedure, for any $m \geq 1$, there exists a compact subset $S_m \subseteq S$ such that (1) $S_m \cap (\cup_{1 \leq k \leq m-1} S_k) = \emptyset$ and $D_m = D \cap (S_m \times Y)$ is compact, (2) $\lambda(S_m) > 0$ and $\lambda\left(S \setminus (\cup_{1 \leq k \leq m} S_m)\right) < \frac{1}{2m}$, and (3) $A$ is nonempty and compact valued, and continuous on $D_m$. This completes the proof. □

The lemma below states an equivalence property for the weak convergence of Borel probability measures obtained from the product of transition probabilities.

**Lemma B.4.** *Let $S$ and $X$ be Polish spaces, and $\lambda$ a Borel probability measure on $S$. Suppose that $\{S_k\}_{k\geq 1}$ is a sequence of disjoint compact subsets of $S$ such that $\lambda(\cup_{k\geq 1}S_k) = 1$. For each $k$, define a probability measure on $S_k$ as $\lambda_k(D) = \frac{\lambda(D)}{\lambda(S_k)}$ for any measurable subset $D \subseteq S_k$. Let $\{\nu_m\}_{m\geq 0}$ be a sequence of transition probabilities from $S$ to $\mathcal{M}(X)$, and $\tau_m = \lambda \diamond \nu_m$ for any $m \geq 0$. Then $\tau_m$ weakly converges to $\tau_0$ if and only if $\lambda_k \diamond \nu_m$ weakly converges to $\lambda_k \diamond \nu_0$ for each $k \geq 1$.*

*Proof.* First, we assume that $\tau_m$ weakly converges to $\tau_0$. For any closed subset $E \subseteq S_k \times X$, we have $\limsup_{m\to\infty} \tau_m(E) \leq \tau_0(E)$. That is, $\limsup_{m\to\infty} \lambda \diamond \nu_m(E) \leq \lambda \diamond \nu_0(E)$. For any $k$, $\frac{1}{\lambda(S_k)} \limsup_{m\to\infty} \lambda \diamond \nu_m(E) \leq \frac{1}{\lambda(S_k)} \lambda \diamond \nu_0(E)$, which implies that $\limsup_{m\to\infty} \lambda_k \diamond \nu_m(E) \leq \lambda_k \diamond \nu_0(E)$. Thus, $\lambda_k \diamond \nu_m$ weakly converges to $\lambda_k \diamond \nu_0$ for each $k \geq 1$.

Second, we consider the case that $\lambda_k \diamond \nu_m$ weakly converges to $\lambda_k \diamond \nu_0$ for each $k \geq 1$. For any closed subset $E \subseteq S \times X$, let $E_k = E \cap (S_k \times X)$ for each $k \geq 1$. Then $\{E_k\}$ are disjoint closed subsets and $\limsup_{m\to\infty} \lambda_k \diamond \nu_m(E_k) \leq \lambda_k \diamond \nu_0(E_k)$. Since $\lambda_k \diamond \nu_m(E') = \frac{1}{\lambda(S_k)} \lambda \diamond \nu_m(E')$ for any $k$, $m$ and measurable subset $E' \subseteq S_k \times X$, we have that $\limsup_{m\to\infty} \lambda \diamond \nu_m(E_k) \leq \lambda \diamond \nu_0(E_k)$. Thus,

$$\sum_{k\geq 1} \limsup_{m\to\infty} \lambda \diamond \nu_m(E_k) \leq \sum_{k\geq 1} \lambda \diamond \nu_0(E_k) = \lambda \diamond \nu_0(E).$$

Since the limit superior is subadditive, we have

$$\sum_{k\geq 1} \limsup_{m\to\infty} \lambda \diamond \nu_m(E_k) \geq \limsup_{m\to\infty} \sum_{k\geq 1} \lambda \diamond \nu_m(E_k) = \limsup_{m\to\infty} \lambda \diamond \nu_m(E).$$

Therefore, $\limsup_{m\to\infty} \lambda \diamond \nu_m(E) \leq \lambda \diamond \nu_0(E)$, which implies that $\tau_m$ weakly converges to $\tau_0$. □

The following is a key lemma that allows one to drop the continuity condition on the state variables through a reference measure in Theorem 3.

**Lemma B.5.** *Suppose that $X, Y$ and $S$ are Polish spaces and $Z$ is a compact metric space. Let $\lambda$ be a Borel probability measure on $S$, and $A$ a nonempty and compact valued correspondence from $Z \times S$ to $X$ which is sectionally upper hemicontinuous on $Z$ and has a $\mathcal{B}(Z \times S \times X)$-measurable graph. Let $G$ be a nonempty and compact valued, continuous correspondence from $Z$ to $\mathcal{M}(X \times S)$. We assume that for any $z \in Z$ and $\tau \in G(z)$, the marginal of $\tau$ on $S$ is $\lambda$ and $\tau(Gr(A(z, \cdot))) = 1$. Let $F$ be a measurable, nonempty, convex and compact valued correspondence from $Gr(A) \to \mathcal{M}(Y)$ such that $F$*

*is sectionally continuous on $Z \times X$. Define a correspondence $\Pi$ from $Z$ to $\mathcal{M}(X \times S \times Y)$ by letting*

$$\Pi(z) = \{g(z) \diamond f(z, \cdot)\colon g \text{ is a Borel measurable selection of } G,\\ f \text{ is a Borel measurable selection of } F\}.$$

*Then the correspondence $\Pi$ is nonempty and compact valued, and continuous.*

*Proof.* Let $\mathcal{S}$ be the completion of $\mathcal{B}(S)$ under the probability measure $\lambda$. By Lemma B.3, $\tilde{A}(s) = \mathrm{Gr}(A(s, \cdot))$ can be viewed as an $\mathcal{S}$-measurable mapping from $S$ to the set of nonempty and compact subsets $\mathcal{K}_{Z \times X}$ of $Z \times X$. For any $s \in S$, the correspondence $F_s = F(\cdot, s)$ is continuous on $\tilde{A}(s)$. By Lemma 3, there exists a measurable, nonempty and compact valued correspondence $\tilde{F}$ from $Z \times X \times S$ to $\mathcal{M}(Y)$ and a Borel measurable subset $S'$ of $S$ with $\lambda(S') = 1$ such that for each $s \in S'$, $\tilde{F}_s$ is continuous on $Z \times X$, and the restriction of $\tilde{F}_s$ to $\tilde{A}(s)$ is $F_s$.

By Lemma 3 (Lusin's theorem), there exists a compact subset $S_1 \subseteq S'$ such that $\tilde{A}$ is continuous on $S_1$ and $\lambda(S_1) > \frac{1}{2}$. Let $K_1 = \tilde{A}(S_1)$. Then $K_1 \subseteq Z \times X$ is compact.

Let $C(K_1, \mathcal{K}_{\mathcal{M}(Y)})$ be the space of continuous functions from $K_1$ to $\mathcal{K}_{\mathcal{M}(Y)}$, where $\mathcal{K}_{\mathcal{M}(Y)}$ is the set of nonempty and compact subsets of $\mathcal{M}(Y)$. Suppose that the restriction of $\mathcal{S}$ on $S_1$ is $\mathcal{S}_1$. Let $\tilde{F}_1$ be the restriction of $\tilde{F}$ to $K_1 \times S_1$. Then $\tilde{F}_1$ can be viewed as an $\mathcal{S}_1$-measurable function from $S_1$ to $C(K_1, \mathcal{K}_{\mathcal{M}(Y)})$ (see Theorem 4.55 in [1]). Again by Lemma 3 (Lusin's theorem), there exists a compact subset of $S_1$, say itself, such that $\lambda(S_1) > \frac{1}{2}$ and $\tilde{F}_1$ is continuous on $S_1$. As a result, $\tilde{F}_1$ is a continuous correspondence on $\mathrm{Gr}(A) \cap (S_1 \times Z \times X)$, so is $F$. Let $\lambda_1$ be a probability measure on $S_1$ such that $\lambda_1(D) = \frac{\lambda(D)}{\lambda(S_1)}$ for any measurable subset $D \subseteq S_1$.

For any $z \in Z$ and $\tau \in G(z)$, the definition of $G$ implies that there exists a transition probability $\nu$ from $S$ to $X$ such that $\lambda \diamond \nu = \tau$. Define a correspondence $G_1$ from $Z$ to $\mathcal{M}(X \times S)$ as follows: for any $z \in Z$, $G_1(z)$ is the set of all $\tau_1 = \lambda_1 \diamond \nu$ such that $\tau = \lambda \diamond \nu \in G(z)$. It can be easily checked that $G_1$ is also a nonempty and compact valued, and continuous correspondence. Let

$$\Pi_1(z) = \{\tau_1 \diamond f(z, \cdot)\colon \tau_1 = \lambda_1 \diamond \nu \in G_1(z),\\ f \text{ is a Borel measurable selection of } \tilde{F}\}.$$

By Lemma 9, $\Pi_1$ is nonempty and compact valued, and continuous. Furthermore, it is easy to see that for any $z$, $\Pi_1(z)$ coincides with the set

$$\{(\lambda_1 \diamond \nu) \diamond f(z, \cdot) \colon \lambda \diamond \nu \in G(z), f \text{ is a Borel measurable selection of } F\}.$$

Repeat this procedure, one can find a sequence of compact subsets $\{S_t\}$ such that (1) for any $t \geq 1$, $S_t \subseteq S'$, $S_t \cap (S_1 \cup \ldots S_{t-1}) = \emptyset$ and $\lambda(S_1 \cup \ldots \cup S_t) \geq \frac{t}{t+1}$, (2) $F$ is continuous on $\mathrm{Gr}(A) \cap (S_t \times Z \times X)$, $\lambda_t$ is a probability measure on $S_t$ such that $\lambda_t(D) = \frac{\lambda(D)}{\lambda(S_t)}$ for any measurable subset $D \subseteq S_t$, and (3) the correspondence

$$\Pi_t(z) = \{(\lambda_t \diamond \nu) \diamond f(z, \cdot) \colon \lambda \diamond \nu \in G(z), \\ f \text{ is a Borel measurable selection of } F\}.$$

is nonempty and compact valued, and continuous.

Pick a sequence $\{z_k\}$, $\{\nu_k\}$ and $\{f_k\}$ such that $(\lambda \diamond \nu_k) \diamond f_k(z_k, \cdot) \in \Pi(z_k)$, $z_k \to z_0$ and $(\lambda \diamond \nu_k) \diamond f_k(z_k, \cdot)$ weakly converges to some $\kappa$. It is easy to see that $(\lambda_t \diamond \nu_k) \diamond f_k(z_k, \cdot) \in \Pi_t(z_k)$ for each $t$. As $\Pi_1$ is compact valued and continuous, it has a subsequence, say itself, such that $z_k$ converges to some $z_0 \in Z$ and $(\lambda_1 \diamond \nu_k) \diamond f_k(z_k, \cdot)$ weakly converges to some $(\lambda_1 \diamond \mu^1) \diamond f^1(z_0, \cdot) \in \Pi_1(z_0)$. Repeat this procedure, one can get a sequence of $\{\mu^m\}$ and $f^m$. Let $\mu(s) = \mu^m(s)$ and $f(z_0, s, x) = f^m(z_0, s, x)$ for any $x \in A(z_0, s)$ when $s \in S_m$. By Lemma B.4, $(\lambda \diamond \mu) \diamond f(z_0, \cdot) = \kappa$, which implies that $\Pi$ is upper hemicontinuous.

Similarly, the compactness and lower hemicontinuity of $\Pi$ follow from the compactness and lower hemicontinuity of $\Pi_t$ for each $t$. □

The next lemma presents the convergence property for the integrals of a sequence of functions and probability measures.

**Lemma B.6.** *Let $S$ and $X$ be Polish spaces, and $A$ a measurable, nonempty and compact valued correspondence from $S$ to $X$. Suppose that $\lambda$ is a Borel probability measure on $S$ and $\{\nu_n\}_{1 \leq n \leq \infty}$ is a sequence of transition probabilities from $S$ to $\mathcal{M}(X)$ such that $\nu_n(A(s)|s) = 1$ for each $s$ and $n$. For each $n \geq 1$, let $\tau_n = \lambda \diamond \nu_n$. Assume that the sequence $\{\tau_n\}$ of Borel probability measures on $S \times X$ converges weakly to a Borel probability measure $\tau_\infty$ on $S \times X$. Let $\{g_n\}_{1 \leq n \leq \infty}$ be a sequence of functions satisfying the following three properties.*

1. *For each $n$ between 1 and $\infty$, $g_n \colon S \times X \to \mathbb{R}_+$ is measurable and sectionally continuous on $X$.*

2. *For any $s \in S$ and any sequence $x_n \to x_\infty$ in $X$, $g_n(s, x_n) \to g_\infty(s, x_\infty)$ as $n \to \infty$.*

3. *The sequence $\{g_n\}_{1 \le n \le \infty}$ is integrably bounded in the sense that there exists a $\lambda$-integrable function $\psi \colon S \to \mathbb{R}_+$ such that for any $n$, $s$ and $x$, $g_n(s, x) \le \psi(s)$.*

*Then we have*

$$\int_{S \times X} g_n(s, x) \tau_n(\mathrm{d}(s, x)) \to \int_{S \times X} g_\infty(s, x) \tau_\infty(\mathrm{d}(s, x)).$$

*Proof.* By Theorem 2.1.3 in [2], for any integrably bounded function $g \colon S \times X \to \mathbb{R}_+$ which is sectionally continuous on $X$, we have

$$\int_{S \times X} g(s, x) \tau_n(\mathrm{d}(s, x)) \to \int_{S \times X} g(s, x) \tau_\infty(\mathrm{d}(s, x)). \tag{1}$$

Let $\{y_n\}_{1 \le n \le \infty}$ be a sequence such that $y_n = \frac{1}{n}$ and $y_\infty = 0$. Then $y_n \to y_\infty$. Define a mapping $\tilde{g}$ from $S \times X \times \{y_1, \dots, y_\infty\}$ such that $\tilde{g}(s, x, y_n) = g_n(s, x)$. Then $\tilde{g}$ is measurable on $S$ and continuous on $X \times \{y_1, \dots, y_\infty\}$. Define a correspondence $G$ from $S$ to $X \times \{y_1, \dots, y_\infty\} \times \mathbb{R}_+$ such that

$$G(s) = \{(x, y_n, c) \colon c \in \tilde{g}(s, x, y_n), x \in A(s), 1 \le n \le \infty\}.$$

For any $s$, $A(s) \times \{y_1, \dots, y_\infty\}$ is compact and $\tilde{g}(s, \cdot, \cdot)$ is continuous. By Lemma 2 (6), $G(s)$ is compact. By Lemma 1 (2), $G$ can be viewed as a measurable mapping from $S$ to the space of nonempty compact subsets of $X \times \{y_1, \dots, y_\infty\} \times \mathbb{R}_+$. Similarly, $A$ can be viewed as a measurable mapping from $S$ to the space of nonempty compact subsets of $X$.

Fix an arbitrary $\epsilon > 0$. By Lemma 3 (Lusin's theorem), there exists a compact subset $S_1 \subseteq S$ such that $A$ and $G$ are continuous on $S_1$ and $\lambda(S \setminus S_1) < \epsilon$. Without loss of generality, we can assume that $\lambda(S \setminus S_1)$ is sufficiently small such that $\int_{S \setminus S_1} \psi(s) \lambda(\mathrm{d}s) < \frac{\epsilon}{6}$. Thus, for any $n$,

$$\int_{(S \setminus S_1) \times X} \psi(s) \tau_n(\mathrm{d}(s, x)) = \int_{(S \setminus S_1)} \psi(s) \nu_n(X) \lambda(\mathrm{d}s) < \frac{\epsilon}{6}.$$

By Lemma 2 (6), the set $E = \{(s, x) \colon s \in S_1, x \in A(s)\}$ is compact. Since $G$ is continuous on $S_1$, $\tilde{g}$ is continuous on $E \times \{y_1, \dots, y_\infty\}$. Since $E \times \{y_1, \dots, y_\infty\}$ is compact, $\tilde{g}$ is uniformly continuous on $E \times \{y_1, \dots, y_\infty\}$. Thus, there exists a positive integer $N_1 > 0$ such that for any $n \ge N_1$, $|g_n(s, x) - g_\infty(s, x)| < \frac{\epsilon}{3}$ for any $(s, x) \in E$.

By Equation (1), there exists a positive integer $N_2$ such that for any $n \geq N_2$,

$$\left| \int_{S\times X} g_\infty(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{S\times X} g_\infty(s,x)\tau_\infty(\mathrm{d}(s,x)) \right| < \frac{\epsilon}{3}.$$

Let $N_0 = \max\{N_1, N_2\}$. For any $n \geq N_0$,

$$\begin{aligned}
&\left| \int_{S\times X} g_n(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{S\times X} g_\infty(s,x)\tau_\infty(\mathrm{d}(s,x)) \right| \\
\leq &\left| \int_{S\times X} g_n(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{S\times X} g_\infty(s,x)\tau_n(\mathrm{d}(s,x)) \right| \\
+ &\left| \int_{S\times X} g_\infty(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{S\times X} g_\infty(s,x)\tau_\infty(\mathrm{d}(s,x)) \right| \\
\leq &\left| \int_{S_1\times X} g_n(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{S_1\times X} g_\infty(s,x)\tau_n(\mathrm{d}(s,x)) \right| \\
+ &\left| \int_{(S\setminus S_1)\times X} g_n(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{(S\setminus S_1)\times X} g_\infty(s,x)\tau_n(\mathrm{d}(s,x)) \right| \\
+ &\left| \int_{S\times X} g_\infty(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{S\times X} g_\infty(s,x)\tau_\infty(\mathrm{d}(s,x)) \right| \\
\leq &\int_E |g_n(s,x) - g_\infty(s,x)|\, \tau_n(\mathrm{d}(s,x)) + 2 \cdot \int_{(S\setminus S_1)\times X} \psi(s)\tau_n(\mathrm{d}(s,x)) \\
+ &\left| \int_{S\times X} g_\infty(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{S\times X} g_\infty(s,x)\tau_\infty(\mathrm{d}(s,x)) \right| \\
< &\frac{\epsilon}{3} + 2 \cdot \frac{\epsilon}{6} + \frac{\epsilon}{3} \\
= &\epsilon.
\end{aligned}$$

This completes the proof. □

## B.3 Discontinuous games with endogenous stochastic sharing rules

It was proved in [7] that a Nash equilibrium exists in discontinuous games with endogenous sharing rules. In particular, they considered a static game with a payoff correspondence $P$ that is bounded, nonempty, convex and compact valued, and upper hemicontinuous. They showed that there exists a Borel measurable selection $p$ of the payoff correspondence, namely the endogenous sharing rule, and a mixed strategy profile $\alpha$ such that $\alpha$ is a Nash equilibrium when players take $p$ as the payoff function (see Lemma 10).

In this section, we shall consider discontinuous games with endogenous stochastic sharing rules. That is, we allow the payoff correspondence to depend on some state variable in a measurable way as follows:

1. let $S$ be a Borel subset of a Polish space, $Y$ a Polish space, and $\lambda$ a Borel probability measure on $S$;
2. $D$ is a $\mathcal{B}(S) \otimes \mathcal{B}(Y)$-measurable subset of $S \times Y$, where $D(s)$ is compact for all $s \in S$ and $\lambda(\{s \in S \colon D(s) \neq \emptyset\}) > 0$;
3. $X = \prod_{1 \le i \le n} X_i$, where each $X_i$ is a Polish space;
4. for each $i$, $A_i$ is a measurable, nonempty and compact valued correspondence from $D$ to $X_i$, which is sectionally continuous on $Y$;
5. $A = \prod_{1 \le i \le n} A_i$ and $E = \mathrm{Gr}(A)$;
6. $P$ is a bounded, measurable, nonempty, convex and compact valued correspondence from $E$ to $\mathbb{R}^n$ which is essentially sectionally upper hemicontinuous on $Y \times X$.

A stochastic sharing rule at $(s, y) \in D$ is a Borel measurable selection of the correspondence $P(s, y, \cdot)$; i.e., a Borel measurable function $p \colon A(s, y) \to \mathbb{R}^n$ such that $p(x) \in P(s, y, x)$ for all $x \in A(s, y)$. Given $(s, y) \in D$, $P(s, y, \cdot)$ represents the set of all possible payoff profiles, and a sharing rule $p$ is a particular choice of the payoff profile.

Now we shall prove the following proposition.

**Proposition B.2.** *There exists a $\mathcal{B}(D)$-measurable, nonempty and compact valued correspondence $\Phi$ from $D$ to $\mathbb{R}^n \times \mathcal{M}(X) \times \triangle(X)$ such that $\Phi$ is essentially sectionally upper hemicontinuous on $Y$, and for $\lambda$-almost all $s \in S$ with $D(s) \neq \emptyset$ and $y \in D(s)$, $\Phi(s, y)$ is the set of points $(v, \alpha, \mu)$ that*

1. *$v = \int_X p(s, y, x)\alpha(\mathrm{d}x)$ such that $p(s, y, \cdot)$ is a Borel measurable selection of $P(s, y, \cdot)$;*[2]
2. *$\alpha \in \otimes_{i \in I} \mathcal{M}(A_i(s, y))$ is a Nash equilibrium in the subgame $(s, y)$ with payoff profile $p(s, y, \cdot)$, and action space $A_i(s, y)$ for each player $i$;*
3. *$\mu = p(s, y, \cdot) \circ \alpha$.*[3]

[2]Note that we require $p(s, y, \cdot)$ to be measurable for each $(s, y)$, but $p$ may not be jointly measurable.
[3]The finite measure $\mu = p(s, y, \cdot) \circ \alpha$ if $\mu(B) = \int_B p(s, y, x)\alpha(\mathrm{d}x)$ for any Borel subset $B \subseteq X$.

*In addition, denote the restriction of $\Phi$ on the first component $\mathbb{R}^n$ as $\Phi|_{\mathbb{R}^n}$, which is a correspondence from $D$ to $\mathbb{R}^n$. Then $\Phi|_{\mathbb{R}^n}$ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $Y$.*

*Proof.* There exists a Borel subset $\hat{S} \subseteq S$ with $\lambda(\hat{S}) = 1$ such that $D(s) \neq \emptyset$ for each $s \in \hat{S}$, and $P$ is sectionally upper hemicontinuous on $Y$ when it is restricted on $D \cap (\hat{S} \times Y)$. Without loss of generality, we assume that $\hat{S} = S$.

Suppose that $\mathcal{S}$ is the completion of $\mathcal{B}(S)$ under the probability measure $\lambda$. Let $\mathcal{D}$ and $\mathscr{E}$ be the restrictions of $\mathcal{S} \otimes \mathcal{B}(Y)$ and $\mathcal{S} \otimes \mathcal{B}(Y) \otimes \mathcal{B}(X)$ on $D$ and $E$, respectively.

Define a correspondence $\tilde{D}$ from $S$ to $Y$ such that $\tilde{D}(s) = \{y \in Y : (s, y) \in D\}$. Then $\tilde{D}$ is nonempty and compact valued. By Lemma 1 (4), $\tilde{D}$ is $\mathcal{S}$-measurable.

Since $\tilde{D}(s)$ is compact and $A(s, \cdot)$ is upper hemicontinuous for any $s \in S$, $E(s)$ is compact by Lemma 2 (6). Define a correspondence $\Gamma$ from $S$ to $Y \times X \times \mathbb{R}^n$ as $\Gamma(s) = \mathrm{Gr}(P(s, \cdot, \cdot))$. For all $s$, $P(s, \cdot, \cdot)$ is bounded, upper hemicontinuous and compact valued on $E(s)$, hence it has a compact graph. As a result, $\Gamma$ is compact valued. By Lemma 1 (1), $P$ has an $\mathcal{S} \otimes \mathcal{B}(Y \times X \times \mathbb{R}^n)$-measurable graph. Since $\mathrm{Gr}(\Gamma) = \mathrm{Gr}(P)$, $\mathrm{Gr}(\Gamma)$ is $\mathcal{S} \otimes \mathcal{B}(Y \times X \times \mathbb{R}^n)$-measurable. Due to Lemma 1 (4), the correspondence $\Gamma$ is $\mathcal{S}$-measurable. We can view $\Gamma$ as a function from $S$ into the space $\mathcal{K}$ of nonempty compact subsets of $Y \times X \times \mathbb{R}^n$. By Lemma B.1, $\mathcal{K}$ is a Polish space endowed with the Hausdorff metric topology. Then by Lemma 1 (2), $\Gamma$ is an $\mathcal{S}$-measurable function from $S$ to $\mathcal{K}$. One can also define a correspondence $\tilde{A}_i$ from $S$ to $Y \times X$ as $\tilde{A}_i(s) = \mathrm{Gr}(A_i(s, \cdot))$. It is easy to show that $\tilde{A}_i$ can be viewed as an $\mathcal{S}$-measurable function from $S$ to the space of nonempty compact subsets of $Y \times X$, which is endowed with the Hausdorff metric topology. By a similar argument, $\tilde{D}$ can be viewed as an $\mathcal{S}$-measurable function from $S$ to the space of nonempty compact subsets of $Y$.

By Lemma 3 (Lusin's Theorem), there exists a compact subset $S_1 \subseteq S$ such that $\lambda(S \setminus S_1) < \frac{1}{2}$, $\Gamma$, $\tilde{D}$ and $\{\tilde{A}_i\}_{1 \le i \le n}$ are continuous functions on $S_1$. By Lemma 1 (3), $\Gamma$, $\tilde{D}$ and $\tilde{A}_i$ are continuous correspondences on $S_1$. Let $D_1 = \{(s, y) \in D : s \in S_1, y \in \tilde{D}(s)\}$. Since $S_1$ is compact and $\tilde{D}$ is continuous, $D_1$ is compact (see Lemma 2 (6)). Similarly, $E_1 = E \cap (S_1 \times Y \times X)$ is also compact. Thus, $P$ is an upper hemicontinuous correspondence on $E_1$. Define a correspondence $\Phi_1$ from $D_1$ to $\mathbb{R}^n \times \mathcal{M}(X) \times \triangle(X)$ as in Lemma 10, then it is nonempty and compact valued, and upper hemicontinuous on $D_1$.

Following the same procedure, for any $m \geq 1$, there exists a compact subset $S_m \subseteq S$ such that (1) $S_m \cap (\cup_{1 \le k \le m-1} S_k) = \emptyset$ and $D_m = D \cap (S_m \times Y)$ is compact, (2) $\lambda(S_m) > 0$ and $\lambda\left(S \setminus (\cup_{1 \le k \le m} S_m)\right) < \frac{1}{2m}$, and (3) there is a nonempty and compact valued, upper hemicontinuous correspondence $\Phi_m$ from $D_m$ to $\mathbb{R}^n \times \mathcal{M}(X) \times \triangle(X)$, which satisfies conditions (1)-(3) in Lemma 10. Thus, we have countably many disjoint sets $\{S_m\}_{m\geq 1}$ such that (1) $\lambda(\cup_{m\geq 1} S_m) = 1$, (2) $\Phi_m$ is nonempty and compact valued, and upper hemicontinuous on each $D_m$, $m \geq 1$.

Since $A_i$ is a $\mathcal{B}(S)\otimes\mathcal{B}(Y)$-measurable, nonempty and compact valued correspondence, it has a Borel measurable selection $a_i$ by Lemma 2 (3). Fix a Borel measurable selection $p$ of $P$ (such a selection exists also due to Lemma 2 (3)). Define a mapping $(v_0, \alpha_0, \mu_0)$ from $D$ to $\mathbb{R}^n \times \mathcal{M}(X) \times \triangle(X)$ such that (1) $\alpha_i(s,y) = \delta_{a_i(s,y)}$ and $\alpha_0(s,y) = \otimes_{i\in I}\alpha_i(s,y)$; (2) $v_0(s,y) = p(s,y,a_1(s,y). \dots, a_n(s,y))$ and (3) $\mu_0(s,y) = p(s,y,\cdot) \circ \alpha_0$. Let $D_0 = D \setminus (\cup_{m\geq 1} D_m)$ and $\Phi_0(s,y) = \{(v_0(s,y), \alpha_0(s,y), \mu_0(s,y))\}$ for $(s,y) \in D_0$. Then, $\Phi_0$ is $\mathcal{B}(S) \otimes \mathcal{B}(Y)$-measurable, nonempty and compact valued.

Let $\Phi(s,y) = \Phi_m(s,y)$ if $(s,y) \in D_m$ for some $m \geq 0$. Then, $\Phi(s,y)$ satisfies conditions (1)-(3) if $(s,y) \in D_m$ for $m \geq 1$. That is, $\Phi$ is $\mathcal{B}(D)$-measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $Y$, and satisfies conditions (1)-(3) for $\lambda$-almost all $s \in S$.

Then consider $\Phi|_{\mathbb{R}^n}$, which is the restriction of $\Phi$ on the first component $\mathbb{R}^n$. Let $\Phi_m|_{\mathbb{R}^n}$ be the restriction of $\Phi_m$ on the first component $\mathbb{R}^n$ with the domain $D_m$ for each $m \geq 0$. It is obvious that $\Phi_0|_{\mathbb{R}^n}$ is measurable, nonempty and compact valued. For each $m \geq 1$, $D_m$ is compact and $\Phi_m$ is upper hemicontinuous and compact valued. By Lemma 2 (6), $\mathrm{Gr}(\Phi_m)$ is compact. Thus, $\mathrm{Gr}(\Phi_m|_{\mathbb{R}^n})$ is also compact. By Lemma 2 (4), $\Phi_m|_{\mathbb{R}^n}$ is measurable. In addition, $\Phi_m|_{\mathbb{R}^n}$ is nonempty and compact valued, and upper hemicontinuous on $D_m$. Notice that $\Phi|_{\mathbb{R}^n}(s,y) = \Phi_m|_{\mathbb{R}^n}(s,y)$ if $(s,y) \in D_m$ for some $m \geq 0$. Thus, $\Phi|_{\mathbb{R}^n}$ is measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $Y$.

The proof is complete. □

## B.4 Proof of Theorem 3

### B.4.1 Backward induction

For any $t \geq 1$, suppose that the correspondence $Q_{t+1}$ from $H_t$ to $\mathbb{R}^n$ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t$. For any $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$, let

$$P_t(h_{t-1}, x_t) = \int_{S_t} Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t | h_{t-1})$$

$$= \int_{S_t} Q_{t+1}(h_{t-1}, x_t, s_t)\varphi_{t0}(h_{t-1}, s_t)\lambda_t(\mathrm{d}s_t).$$

It is obvious that the correspondence $P_t$ is measurable and nonempty valued. Since $Q_{t+1}$ is bounded, $P_t$ is bounded. For $\lambda^t$-almost all $s^t \in S^t$, $Q_{t+1}(\cdot, s^t)$ is bounded and upper hemicontinuous on $H_t(s^t)$, and $\varphi_{t0}(s^t, \cdot)$ is continuous on $\mathrm{Gr}(A_0^t)(s^t)$. As $\varphi_{t0}$ is integrably bounded, $P_t(s^{t-1}, \cdot)$ is also upper hemicontinuous on $\mathrm{Gr}(A^t)(s^{t-1})$ for $\lambda^{t-1}$-almost all $s^{t-1} \in S^{t-1}$ (see Lemma 4); that is, the correspondence $P_t$ is essentially sectionally upper hemicontinuous on $X^t$. Again by Lemma 4, $P_t$ is convex and compact valued since $\lambda_t$ is an atomless probability measure. That is, $P_t \colon \mathrm{Gr}(A^t) \to \mathbb{R}^n$ is a bounded, measurable, nonempty, convex and compact valued correspondence which is essentially sectionally upper hemicontinuous on $X^t$.

By Proposition B.2, there exists a bounded, measurable, nonempty and compact valued correspondence $\Phi_t$ from $H_{t-1}$ to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \triangle(X_t)$ such that $\Phi_t$ is essentially sectionally upper hemicontinuous on $X^{t-1}$, and for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$, $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if

1. $v = \int_{A_t(h_{t-1})} p_t(h_{t-1}, x)\alpha(\mathrm{d}x)$ such that $p_t(h_{t-1}, \cdot)$ is a Borel measurable selection of $P_t(h_{t-1}, \cdot)$;
2. $\alpha \in \otimes_{i\in I}\mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame $h_{t-1}$ with payoff $p_t(h_{t-1}, \cdot)$ and action space $\prod_{i\in I} A_{ti}(h_{t-1})$;
3. $\mu = p_t(h_{t-1}, \cdot) \circ \alpha$.

Denote the restriction of $\Phi_t$ on the first component $\mathbb{R}^n$ as $\Phi(Q_{t+1})$, which is a correspondence from $H_{t-1}$ to $\mathbb{R}^n$. By Proposition B.2, $\Phi(Q_{t+1})$ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^{t-1}$.

### B.4.2 Forward induction

The following proposition presents the result on the step of forward induction.

**Proposition B.3.** *For any $t \geq 1$ and any Borel measurable selection $q_t$ of $\Phi(Q_{t+1})$, there exists a Borel measurable selection $q_{t+1}$ of $Q_{t+1}$ and a Borel measurable mapping $f_t \colon H_{t-1} \to \otimes_{i\in I}\mathcal{M}(X_{ti})$ such that for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$,*

1. *$f_t(h_{t-1}) \in \otimes_{i\in I}\mathcal{M}(A_{ti}(h_{t-1}))$;*

2. $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}) f_t(\mathrm{d}x_t|h_{t-1})$;

3. *$f_t(\cdot|h_{t-1})$ is a Nash equilibrium in the subgame $h_{t-1}$ with action spaces $A_{ti}(h_{t-1}), i \in I$ and the payoff functions*

$$\int_{S_t} q_{t+1}(h_{t-1}, \cdot, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}).$$

*Proof.* We divide the proof into three steps. In step 1, we show that there exist Borel measurable mappings $f_t \colon H_{t-1} \to \otimes_{i \in I} \mathcal{M}(X_{ti})$ and $\mu_t \colon H_{t-1} \to \triangle(X_t)$ such that $(q_t, f_t, \mu_t)$ is a selection of $\Phi_t$. In step 2, we obtain a Borel measurable selection $g_t$ of $P_t$ such that for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$,

1. $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} g_t(h_{t-1}, x) f_t(\mathrm{d}x|h_{t-1})$;

2. $f_t(h_{t-1})$ is a Nash equilibrium in the subgame $h_{t-1}$ with payoff $g_t(h_{t-1}, \cdot)$ and action space $A_t(h_{t-1})$;

In step 3, we show that there exists a Borel measurable selection $q_{t+1}$ of $Q_{t+1}$ such that for all $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$,

$$g_t(h_{t-1}, x_t) = \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}).$$

Combining Steps 1-3, the proof is complete.

Step 1. Let $\Psi_t \colon \mathrm{Gr}(\Phi_t(Q_{t+1})) \to \mathcal{M}(X_t) \times \triangle(X_t)$ be

$$\Psi_t(h_{t-1}, v) = \{(\alpha, \mu) \colon (v, \alpha, \mu) \in \Phi_t(h_{t-1})\}.$$

Recall the construction of $\Phi_t$ and the proof of Proposition B.2, $H_{t-1}$ can be divided into countably many Borel subsets $\{H_{t-1}^m\}_{m \geq 0}$ such that

1. $H_{t-1} = \cup_{m \geq 0} H_{t-1}^m$ and $\frac{\lambda^{t-1}(\cup_{m \geq 1} \mathrm{proj}_{S^{t-1}}(H_{t-1}^m))}{\lambda^{t-1}(\mathrm{proj}_{S^{t-1}}(H_{t-1}))} = 1$, where $\mathrm{proj}_{S^{t-1}}(H_{t-1}^m)$ and $\mathrm{proj}_{S^{t-1}}(H_{t-1})$ are projections of $H_{t-1}^m$ and $H_{t-1}$ on $S^{t-1}$;

2. for $m \geq 1$, $H_{t-1}^m$ is compact, $\Phi_t$ is upper hemicontinuous on $H_{t-1}^m$, and $P_t$ is upper hemicontinuous on

$$\{(h_{t-1}, x_t) \colon h_{t-1} \in H_{t-1}^m, x_t \in A_t(h_{t-1})\};$$

3. there exists a Borel measurable mapping $(v_0, \alpha_0, \mu_0)$ from $H_{t-1}^0$ to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \triangle(X_t)$ such that $\Phi_t(h_{t-1}) \equiv \{(v_0(h_{t-1}), \alpha_0(h_{t-1}), \mu_0(h_{t-1}))\}$ for any $h_{t-1} \in H_{t-1}^0$.

Denote the restriction of $\Phi_t$ on $H_{t-1}^m$ as $\Phi_t^m$. For $m \geq 1$, $\mathrm{Gr}(\Phi_t^m)$ is compact, and hence the correspondence $\Psi_t^m(h_{t-1}, v) = \{(\alpha, \mu) \colon (v, \alpha, \mu) \in \Phi_t^m(h_{t-1})\}$ has a compact graph. For $m \geq 1$, $\Psi_t^m$ is measurable by Lemma 2 (4), and has a Borel measurable selection $\psi_t^m$ due to Lemma 2 (3). Define $\psi_t^0(h_{t-1}, v_0(h_{t-1})) = (\alpha_0(h_{t-1}), \mu_0(h_{t-1}))$ for $h_{t-1} \in H_{t-1}^0$. For $(h_{t-1}, v) \in \mathrm{Gr}(\Phi(Q_{t+1}))$, let $\psi_t(h_{t-1}, v) = \psi_t^m(h_{t-1}, v)$ if $h_{t-1} \in H_{t-1}^m$. Then $\psi_t$ is a Borel measurable selection of $\Psi_t$.

Given a Borel measurable selection $q_t$ of $\Phi(Q_{t+1})$, let

$$\phi_t(h_{t-1}) = (q_t(h_{t-1}), \psi_t(h_{t-1}, q_t(h_{t-1}))).$$

Then $\phi_t$ is a Borel measurable selection of $\Phi_t$. Denote $\tilde{H}_{t-1} = \cup_{m \geq 1} H_{t-1}^m$. By the construction of $\Phi_t$, there exists Borel measurable mappings $f_t \colon H_{t-1} \to \otimes_{i \in I} \mathcal{M}(X_{ti})$ and $\mu_t \colon H_{t-1} \to \triangle(X_t)$ such that for all $h_{t-1} \in \tilde{H}_{t-1}$,

1. $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} p_t(h_{t-1}, x) f_t(\mathrm{d}x | h_{t-1})$ such that $p_t(h_{t-1}, \cdot)$ is a Borel measurable selection of $P_t(h_{t-1}, \cdot)$;

2. $f_t(h_{t-1}) \in \otimes_{i \in I} \mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame $h_{t-1}$ with payoff $p_t(h_{t-1}, \cdot)$ and action space $\prod_{i \in I} A_{ti}(h_{t-1})$;

3. $\mu_t(\cdot | h_{t-1}) = p_t(h_{t-1}, \cdot) \circ f_t(\cdot | h_{t-1})$.

Step 2. Since $P_t$ is upper hemicontinuous on $\{(h_{t-1}, x_t) \colon h_{t-1} \in H_{t-1}^m, x_t \in A_t(h_{t-1})\}$, due to Lemma 6, there exists a Borel measurable mapping $g^m$ such that (1) $g^m(h_{t-1}, x_t) \in P_t(h_{t-1}, x_t)$ for any $h_{t-1} \in H_{t-1}^m$ and $x_t \in A_t(h_{t-1})$, and (2) $g^m(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot | h_{t-1})$-almost all $x_t$. Fix an arbitrary Borel measurable selection $g'$ of $P_t$. Define a Borel measurable mapping from $\mathrm{Gr}(A_t)$ to $\mathbb{R}^n$ as

$$g(h_{t-1}, x_t) = \begin{cases} g^m(h_{t-1}, x_t) & \text{if } h_{t-1} \in H_{t-1}^m \text{ for } m \geq 1; \\ g'(h_{t-1}, x_t) & \text{otherwise.} \end{cases}$$

Then $g$ is a Borel measurable selection of $P_t$.

In a subgame $h_{t-1} \in \tilde{H}_{t-1}$, let

$$B_{ti}(h_{t-1}) = \{y_i \in A_{ti}(h_{t-1}) \colon$$

$$\int_{A_{t(-i)}(h_{t-1})} g_i(h_{t-1}, y_i, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) > \int_{A_t(h_{t-1})} p_{ti}(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1})\}.$$

Since $g(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all $x_t$,

$$\int_{A_t(h_{t-1})} g(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1}) = \int_{A_t(h_{t-1})} p_t(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1}).$$

Thus, $B_{ti}$ is a measurable correspondence from $\tilde{H}_{t-1}$ to $A_{ti}(h_{t-1})$. Let $B_{ti}^c(h_{t-1}) = A_{ti}(h_{t-1}) \setminus B_{ti}(h_{t-1})$ for each $h_{t-1} \in H_{t-1}$. Then $B_{ti}^c$ is a measurable and closed valued correspondence, which has a Borel measurable graph by Lemma 1. As a result, $B_{ti}$ also has a Borel measurable graph. As $f_t(h_{t-1})$ is a Nash equilibrium in the subgame $h_{t-1} \in \tilde{H}_{t-1}$ with payoff $p_t(h_{t-1}, \cdot)$, $f_{ti}(B_{ti}(h_{t-1})|h_{t-1}) = 0$.

Denote $\beta_i(h_{t-1}, x_t) = \min P_{ti}(h_{t-1}, x_t)$, where $P_{ti}(h_{t-1}, x_t)$ is the projection of $P_t(h_{t-1}, x_t)$ on the $i$-th dimension. Then the correspondence $P_{ti}$ is measurable and compact valued, and $\beta_i$ is Borel measurable. Let $\Lambda_i(h_{t-1}, x_t) = \{\beta_i(h_{t-1}, x_t)\} \times [0, \gamma]^{n-1}$, where $\gamma > 0$ is the upper bound of $P_t$. Denote $\Lambda_i'(h_{t-1}, x_t) = \Lambda_i(h_{t-1}, x_t) \cap P_t(h_{t-1}, x_t)$. Then $\Lambda_i'$ is a measurable and compact valued correspondence, and hence has a Borel measurable selection $\beta_i'$. Note that $\beta_i'$ is a Borel measurable selection of $P_t$. Let

$$g_t(h_{t-1}, x_t) =$$

$$\begin{cases} \beta_i'(h_{t-1}, x_t) & \text{if } h_{t-1} \in \tilde{H}_{t-1}, x_{ti} \in B_{ti}(h_{t-1}) \text{ and } x_{tj} \notin B_{tj}(h_{t-1}), \forall j \neq i; \\ g(h_{t-1}, x_t) & \text{otherwise.} \end{cases}$$

Notice that

$$\begin{aligned} &\{(h_{t-1}, x_t) \in \mathrm{Gr}(A_t) \colon h_{t-1} \in \tilde{H}_{t-1}, x_{ti} \in B_{ti}(h_{t-1}) \text{ and } x_{tj} \notin B_{tj}(h_{t-1}), \forall j \neq i;\} \\ = {}& \mathrm{Gr}(A_t) \cap \cup_{i \in I} \left( (\mathrm{Gr}(B_{ti}) \times \prod_{j \neq i} X_{tj}) \setminus (\cup_{j \neq i} (\mathrm{Gr}(B_{tj}) \times \prod_{k \neq j} X_{tk})) \right), \end{aligned}$$

which is a Borel set. As a result, $g_t$ is a Borel measurable selection of $P_t$. Moreover, $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for all $h_{t-1} \in \tilde{H}_{t-1}$ and $f_t(\cdot|h_{t-1})$-almost all $x_t$.

Fix a subgame $h_{t-1} \in \tilde{H}_{t-1}$. We will show that $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $g_t(h_{t-1}, \cdot)$ in the subgame $h_{t-1}$. Suppose that player $i$ deviates to some action $\tilde{x}_{ti}$.

If $\tilde{x}_{ti} \in B_{ti}(h_{t-1})$, then player $i$'s expected payoff is

$$\begin{aligned}
&\int_{A_{t(-i)}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
=& \int_{\prod_{j \neq i} B^c_{tj}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
=& \int_{\prod_{j \neq i} B^c_{tj}(h_{t-1})} \beta_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
\leq& \int_{\prod_{j \neq i} B^c_{tj}(h_{t-1})} p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
=& \int_{A_{t(-i)}(h_{t-1})} p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
\leq& \int_{A_t(h_{t-1})} p_{ti}(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1}) \\
=& \int_{A_t(h_{t-1})} g_{ti}(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1}).
\end{aligned}$$

The first and the third equalities hold since $f_{tj}(B_{tj}(h_{t-1})|h_{t-1}) = 0$ for each $j$, and hence $f_{t(-i)}(\prod_{j \neq i} B^c_{tj}(h_{t-1})|h_{t-1}) = f_{t(-i)}(A_{t(-i)}(h_{t-1})|h_{t-1})$. The second equality and the first inequality are due to the fact that $g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = \beta_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = \min P_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) \leq p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)})$ for $x_{t(-i)} \in \prod_{j \neq i} B^c_{tj}(h_{t-1})$. The second inequality holds since $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $p_t(h_{t-1}, \cdot)$ in the subgame $h_{t-1}$. The fourth equality follows from the fact that $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all $x_t$.

If $\tilde{x}_{ti} \notin B_{ti}(h_{t-1})$, then player $i$'s expected payoff is

$$\begin{aligned}
&\int_{A_{t(-i)}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
=& \int_{\prod_{j \neq i} B^c_{tj}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
=& \int_{\prod_{j \neq i} B^c_{tj}(h_{t-1})} g_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
=& \int_{A_{t(-i)}(h_{t-1})} g_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
\leq& \int_{A_t(h_{t-1})} p_{ti}(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1}) \\
=& \int_{A_t(h_{t-1})} g_{ti}(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1}).
\end{aligned}$$

The first and the third equalities hold since

$$f_{t(-i)}\left(\prod_{j\neq i} B_{tj}^c(h_{t-1})|h_{t-1}\right) = f_{t(-i)}(A_{t(-i)}(h_{t-1})|h_{t-1}).$$

The second equality is due to the fact that $g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = g_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)})$ for $x_{t(-i)} \in \prod_{j\neq i} B_{tj}^c(h_{t-1})$. The first inequality follows from the definition of $B_{ti}$, and the fourth equality holds since $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all $x_t$.

Thus, player $i$ cannot improve his payoff in the subgame $h_t$ by a unilateral change in his strategy for any $i \in I$, which implies that $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $g_t(h_{t-1}, \cdot)$ in the subgame $h_{t-1}$.

Step 3. For any $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$,

$$P_t(h_{t-1}, x_t) = \int_{S_t} Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}).$$

By Lemma 5, there exists a Borel measurable mapping $q$ from $\mathrm{Gr}(P_t) \times S_t$ to $\mathbb{R}^n$ such that

1. $q(h_{t-1}, x_t, e, s_t) \in Q_{t+1}(h_{t-1}, x_t, s_t)$ for any $(h_{t-1}, x_t, e, s_t) \in \mathrm{Gr}(P_t) \times S_t$;
2. $e = \int_{S_t} q(h_{t-1}, x_t, e, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1})$ for any $(h_{t-1}, x_t, e) \in \mathrm{Gr}(P_t)$, where $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$.

Let

$$q_{t+1}(h_{t-1}, x_t, s_t) = q(h_{t-1}, x_t, g_t(h_{t-1}, x_t), s_t)$$

for any $(h_{t-1}, x_t, s_t) \in H_t$. Then $q_{t+1}$ is a Borel measurable selection of $Q_{t+1}$.

For $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$,

$$\begin{aligned} g_t(h_{t-1}, x_t) &= \int_{S_t} q(h_{t-1}, x_t, g_t(h_{t-1}, x_t), s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}) \\ &= \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}). \end{aligned}$$

Therefore, we have a Borel measurable selection $q_{t+1}$ of $Q_{t+1}$, and a Borel measurable mapping $f_t \colon H_{t-1} \to \otimes_{i\in I}\mathcal{M}(X_{ti})$ such that for all $h_{t-1} \in \tilde{H}_{t-1}$, properties (1)-(3) are satisfied. The proof is complete. □

If a dynamic game has only $T$ stages for some positive integer $T \geq 1$, then let $Q_{T+1}(h_T) = \{u(h_T)\}$ for any $h_T \in H_T$, and $Q_t = \Phi(Q_{t+1})$ for $1 \leq t \leq T-1$. We can start with the backward induction from the last period and stop at the initial period, then run the forward induction from the initial period to the last period. Thus, the following result is immediate.

**Proposition B.4.** *Any finite-horizon dynamic game with the ARM condition has a subgame-perfect equilibrium.*

### B.4.3 Infinite horizon case

Pick a sequence $\xi = (\xi_1, \xi_2, \ldots)$ such that (1) $\xi_m$ is a transition probability from $H_{m-1}$ to $\mathcal{M}(X_m)$ for any $m \geq 1$, and (2) $\xi_m(A_m(h_{m-1})|h_{m-1}) = 1$ for any $m \geq 1$ and $h_{m-1} \in H_{m-1}$. Denote the set of all such $\xi$ as $\Upsilon$.

Fix any $t \geq 1$, define correspondences $\Xi_t^t$ and $\Delta_t^t$ as follows: in the subgame $h_{t-1}$,

$$\Xi_t^t(h_{t-1}) = \mathcal{M}(A_t(h_{t-1})) \otimes \lambda_t,$$

and

$$\Delta_t^t(h_{t-1}) = \mathcal{M}(A_t(h_{t-1})) \otimes f_{t0}(h_{t-1}).$$

For any $m_1 > t$, suppose that the correspondences $\Xi_t^{m_1-1}$ and $\Delta_t^{m_1-1}$ have been defined. Then we can define correspondences $\Xi_t^{m_1}\colon H_{t-1} \to \mathcal{M}\left(\prod_{t \leq m \leq m_1}(X_m \times S_m)\right)$ and $\Delta_t^{m_1}\colon H_{t-1} \to \mathcal{M}\left(\prod_{t \leq m \leq m_1}(X_m \times S_m)\right)$ as follows:

$$\begin{aligned}\Xi_t^{m_1}(h_{t-1}) = &\{g(h_{t-1}) \diamond (\xi_{m_1}(h_{t-1}, \cdot) \otimes \lambda_{m_1})\colon \\ &g \text{ is a Borel measurable selection of } \Xi_t^{m_1-1}, \\ &\xi_{m_1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_1})\},\end{aligned}$$

and

$$\begin{aligned}\Delta_t^{m_1}(h_{t-1}) = &\{g(h_{t-1}) \diamond (\xi_{m_1}(h_{t-1}, \cdot) \otimes f_{m_1 0}(h_{t-1}, \cdot))\colon \\ &g \text{ is a Borel measurable selection of } \Delta_t^{m_1-1}, \\ &\xi_{m_1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_1})\},\end{aligned}$$

where $\mathcal{M}(A_{m_1})$ is regarded as a correspondence from $H_{m_1-1}$ to the space of Borel probability measures on $X_{m_1}$. For any $m_1 \geq t$, let $\rho_{(h_{t-1},\xi)}^{m_1} \in \Xi_t^{m_1}$ be the probability measure on $\prod_{t \leq m \leq m_1}(X_m \times S_m)$ induced by $\{\lambda_m\}_{t \leq m \leq m_1}$ and $\{\xi_m\}_{t \leq m \leq m_1}$, and $\varrho_{(h_{t-1},\xi)}^{m_1} \in$

$\Delta_t^{m_1}$ be the probability measure on $\prod_{t\le m\le m_1}(X_m\times S_m)$ induced by $\{f_{m0}\}_{t\le m\le m_1}$ and $\{\xi_m\}_{t\le m\le m_1}$. Then, $\Xi_t^{m_1}(h_{t-1})$ is the set of all such $\rho^{m_1}_{(h_{t-1},\xi)}$, while $\Delta_t^{m_1}(h_{t-1})$ is the set of all such $\varrho^{m_1}_{(h_{t-1},\xi)}$. Note that $\varrho^{m_1}_{(h_{t-1},\xi)}\in\Delta_t^{m_1}(h_{t-1})$ if and only if $\rho^{m_1}_{(h_{t-1},\xi)}\in\Xi_t^{m_1}(h_{t-1})$. Both $\varrho^{m_1}_{(h_{t-1},\xi)}$ and $\rho^{m_1}_{(h_{t-1},\xi)}$ can be regarded as probability measures on $H_{m_1}(h_{t-1})$.

Similarly, let $\rho_{(h_{t-1},\xi)}$ be the probability measure on $\prod_{m\ge t}(X_m\times S_m)$ induced by $\{\lambda_m\}_{m\ge t}$ and $\{\xi_m\}_{m\ge t}$, and $\varrho_{(h_{t-1},\xi)}$ the probability measure on $\prod_{m\ge t}(X_m\times S_m)$ induced by $\{f_{m0}\}_{m\ge t}$ and $\{\xi_m\}_{m\ge t}$. Denote the correspondence

$$\Xi_t\colon H_{t-1}\to\mathcal{M}(\prod_{m\ge t}(X_m\times S_m))$$

as the set of all such $\rho_{(h_{t-1},\xi)}$, and

$$\Delta_t\colon H_{t-1}\to\mathcal{M}(\prod_{m\ge t}(X_m\times S_m))$$

as the set of all such $\varrho_{(h_{t-1},\xi)}$.

The following lemma demonstrates the relationship between $\varrho^{m_1}_{(h_{t-1},\xi)}$ and $\rho^{m_1}_{(h_{t-1},\xi)}$.

**Lemma B.7.** *For any $m_1\ge t$ and $h_{t-1}\in H_{t-1}$,*

$$\varrho^{m_1}_{(h_{t-1},\xi)}=\left(\prod_{t\le m\le m_1}\varphi_{m0}(h_{t-1},\cdot)\right)\circ\rho^{m_1}_{(h_{t-1},\xi)}.$$ [4]

*Proof.* Fix $\xi\in\Upsilon$, and Borel subsets $C_m\subseteq X_m$ and $D_m\subseteq S_m$ for $m\ge t$. First, we have

$$\begin{aligned}&\varrho^t_{(h_{t-1},\xi)}(C_t\times D_t)=\xi_t(C_t|h_{t-1})\cdot f_{t0}(D_t|h_{t-1})\\&=\int_{X_t\times S_t}\mathbf{1}_{C_t\times D_t}(x_t,s_t)\varphi_{t0}(h_{t-1},s_t)(\xi_t(h_{t-1})\otimes\lambda_t)(\mathrm{d}(x_t,s_t)),\end{aligned}$$

which implies that $\varrho^t_{(h_{t-1},\xi)}=\varphi_{t0}(h_{t-1},\cdot)\circ\rho^t_{(h_{t-1},\xi)}$.[5]

---

[4]For $m\ge t\ge 1$ and $h_{t-1}\in H_{t-1}$, the function $\varphi_{m0}(h_{t-1},\cdot)$ is defined on $H_{m-1}(h_{t-1})\times S_m$, which is measurable and sectionally continuous on $\prod_{t\le k\le m-1}X_k$. By Lemma 3, $\varphi_{m0}(h_{t-1},\cdot)$ can be extended to be a measurable function $\acute{\varphi}_{m0}(h_{t-1},\cdot)$ on the product space $\left(\prod_{t\le k\le m-1}X_k\right)\times\left(\prod_{t\le k\le m}S_k\right)$, which is also sectionally continuous on $\prod_{t\le k\le m-1}X_k$. Given any $\xi\in\Upsilon$, since $\rho^m_{(h_{t-1},\xi)}$ concentrates on $H_m(h_{t-1})$, $\varphi_{m0}(h_{t-1},\cdot)\circ\rho^m_{(h_{t-1},\xi)}=\acute{\varphi}_{m0}(h_{t-1},\cdot)\circ\rho^m_{(h_{t-1},\xi)}$. For notational simplicity, we still use $\varphi_{m0}(h_{t-1},\cdot)$, instead of $\acute{\varphi}_{m0}(h_{t-1},\cdot)$, to denote the above extension. Similarly, we can work with a suitable extension of the payoff function $u$ as needed.

[5]For a set $A$ in a space $X$, $\mathbf{1}_A$ is the indicator function of $A$, which is one on $A$ and zero on $X\setminus A$.

Suppose that $\varrho^{m_2}_{(h_{t-1},\xi)} = \left(\prod_{t \le m \le m_2} \varphi_{m0}(h_{t-1},\cdot)\right) \circ \rho^{m_2}_{(h_{t-1},\xi)}$ for some $m_2 \ge t$. Then

$$\begin{aligned}
&\varrho^{m_2+1}_{(h_{t-1},\xi)}\left(\prod_{t \le m \le m_2+1} (C_m \times D_m)\right) \\
&= \varrho^{m_2}_{(h_{t-1},\xi)} \diamond (\xi_{m_2+1}(h_{t-1},\cdot) \otimes f_{(m_2+1)0}(h_{t-1},\cdot)) \left(\prod_{t \le m \le m_2+1} (C_m \times D_m)\right) \\
&= \int_{\prod_{t\le m\le m_2}(X_m\times S_m)} \int_{X_{m_2+1}\times S_{m_2+1}} \mathbf{1}_{\prod_{t\le m\le m_2+1}(C_m\times D_m)}(x_t,\dots,x_{m_2+1},s_t,\dots,s_{m_2+1}) \cdot \\
&\quad \xi_{m_2+1} \otimes f_{(m_2+1)0}(\mathrm{d}(x_{m_2+1},s_{m_2+1})|h_{t-1},x_t,\dots,x_{m_2},s_t,\dots,s_{m_2}) \\
&\quad \varrho^{m_2}_{(h_{t-1},\xi)}(\mathrm{d}(x_t,\dots,x_{m_2},s_t,\dots,s_{m_2})|h_{t-1}) \\
&= \int_{\prod_{t\le m\le m_2}(X_m\times S_m)} \int_{S_{m_2+1}} \int_{X_{m_2+1}} \mathbf{1}_{\prod_{t\le m\le m_2+1}(C_m\times D_m)}(x_t,\dots,x_{m_2+1},s_t,\dots,s_{m_2+1}) \cdot \\
&\quad \varphi_{(m_2+1)0}(h_{t-1},x_t,\dots,x_{m_2},s_t,\dots,s_{m_2+1}) \xi_{m_2+1}(\mathrm{d}x_{m_2+1}|h_{t-1},x_t,\dots,x_{m_2},s_t,\dots,s_{m_2}) \\
&\quad \lambda_{(m_2+1)0}(\mathrm{d}s_{m_2+1}) \prod_{t\le m\le m_2} \varphi_{m0}(h_{t-1},x_t,\dots,x_{m-1},s_t,\dots,s_m) \\
&\quad \rho^{m_2}_{(h_{t-1},\xi)}(\mathrm{d}(x_t,\dots,x_{m_2},s_t,\dots,s_{m_2})|h_{t-1}) \\
&= \int_{\prod_{t\le m\le m_2+1}(X_m\times S_m)} \mathbf{1}_{\prod_{t\le m\le m_2+1}(C_m\times D_m)}(x_t,\dots,x_{m_2+1},s_t,\dots,s_{m_2+1}) \cdot \\
&\quad \prod_{t\le m\le m_2+1} \varphi_{m0}(h_{t-1},x_t,\dots,x_{m-1},s_t,\dots,s_m) \rho^{m_2+1}_{(h_{t-1},\xi)}(\mathrm{d}(x_t,\dots,x_{m_2},s_t,\dots,s_{m_2})|h_{t-1}),
\end{aligned}$$

which implies that

$$\varrho^{m_2+1}_{(h_{t-1},\xi)} = \left(\prod_{t\le m\le m_2+1} \varphi_{m0}(h_{t-1},\cdot)\right) \circ \rho^{m_2+1}_{(h_{t-1},\xi)}.$$

The proof is thus complete. □

The next lemma shows that the correspondences $\Delta^{m_1}_t$ and $\Delta_t$ are nonempty and compact valued, and sectionally continuous.

**Lemma B.8.** *1. For any $t \ge 1$, the correspondence $\Delta^{m_1}_t$ is nonempty and compact valued, and sectionally continuous on $X^{t-1}$ for any $m_1 \ge t$.*

*2. For any $t \ge 1$, the correspondence $\Delta_t$ is nonempty and compact valued, and sectionally continuous on $X^{t-1}$.*

*Proof.* (1) We first show that the correspondence $\Xi^{m_1}_t$ is nonempty and compact valued, and sectionally continuous on $X^{t-1}$ for any $m_1 \geq t$.

Consider the case $m_1 = t \geq 1$, where

$$\Xi_t^t(h_{t-1}) = \mathcal{M}(A_t(h_{t-1})) \otimes \lambda_t.$$

Since $A_{ti}$ is nonempty and compact valued, and sectionally continuous on $X^{t-1}$, $\Xi_t^t$ is nonempty and compact valued, and sectionally continuous on $X^{t-1}$.

Now suppose that $\Xi_t^{m_2}$ is nonempty and compact valued, and sectionally continuous on $X^{t-1}$ for some $m_2 \geq t \geq 1$. Notice that

$$\begin{aligned}\Xi_t^{m_2+1}(h_{t-1}) = &\{g(h_{t-1}) \diamond (\xi_{m_2+1}(h_{t-1}, \cdot) \otimes \lambda_{(m_2+1)}) \colon \\ &g \text{ is a Borel measurable selection of } \Xi_t^{m_2}, \\ &\xi_{m_2+1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_2+1})\}.\end{aligned}$$

First, we claim that $H_t(s_0, s_1, \ldots, s_t)$ is compact for any $(s_0, s_1, \ldots, s_t) \in S^t$. We prove this claim by induction.

1. Notice that $H_0(s_0) = X_0$ for any $s_0 \in S_0$, which is compact.
2. Suppose that $H_{m'}(s_0, s_1, \ldots, s_{m'})$ is compact for some $0 \leq m' \leq t-1$ and any $(s_0, s_1, \ldots, s_{m'}) \in S^{m'}$.
3. Since $A_{m'+1}(\cdot, s_0, s_1, \ldots, s_{m'})$ is continuous and compact valued, it has a compact graph by Lemma 2 (6), which is $H_{m'+1}(s_0, s_1, \ldots, s_{m'+1})$ for any $(s_0, s_1, \ldots, s_{m'+1}) \in S^{m'+1}$.

Thus, we prove the claim.

Define a correspondence $A_t^t$ from $H_{t-1} \times S_t$ to $X_t$ as $A_t^t(h_{t-1}, s_t) = A_t(h_{t-1})$. Then $A_t^t$ is nonempty and compact valued, sectionally continuous on $X_{t-1}$, and has a $\mathcal{B}(X^t \times S^t)$-measurable graph. Since the graph of $A_t^t(\cdot, s_0, s_1, \ldots, s_t)$ is $H_t(s_0, s_1, \ldots, s_t)$ and $H_t(s_0, s_1, \ldots, s_t)$ is compact, $A_t^t(\cdot, s_0, s_1, \ldots, s_t)$ has a compact graph. For any $h_{t-1} \in H_{t-1}$ and $\tau \in \Xi_t^t(h_{t-1})$, the marginal of $\tau$ on $S_t$ is $\lambda_t$ and $\tau(\mathrm{Gr}(A_t^t(h_{t-1}, \cdot))) = 1$.

For any $m_1 > t$, suppose that the correspondence

$$A_t^{m_1-1} \colon H_{t-1} \times \prod_{t \leq m \leq m_1-1} S_m \to \prod_{t \leq m \leq m_1-1} X_m$$

has been defined such that

1. it is nonempty and compact valued, sectionally upper hemicontinuous on $X_{t-1}$, and has a $\mathcal{B}(X^{m_1-1} \times S^{m_1-1})$-measurable graph;

2. for any $(s_0, s_1, \ldots s_{m_1-1})$, $A_t^{m_1-1}(\cdot, s_0, s_1, \ldots s_{m_1-1})$ has a compact graph;

3. for any $h_{t-1} \in H_{t-1}$ and $\tau \in \Xi_t^{m_1-1}(h_{t-1})$, the marginal of $\tau$ on $\prod_{t \le m \le m_1-1} S_m$ is $\otimes_{t \le m \le m_1-1} \lambda_m$ and $\tau(\mathrm{Gr}(A_t^{m_1-1}(h_{t-1}, \cdot))) = 1$.

We define a correspondence $A_t^{m_1} \colon H_{t-1} \times \prod_{t \le m \le m_1} S_m \to \prod_{t \le m \le m_1} X_m$ as follows:

$$
\begin{aligned}
A_t^{m_1}(h_{t-1}, s_t, \ldots, s_{m_1}) = \{ & (x_t, \ldots, x_{m_1}) \colon \\
& x_{m_1} \in A_{m_1}(h_{t-1}, x_t, \ldots, x_{m_1-1}, s_t, \ldots, s_{m_1-1}), \\
& (x_t, \ldots, x_{m_1-1}) \in A_t^{m_1-1}(h_{t-1}, s_t, \ldots, s_{m_1-1}) \}.
\end{aligned}
$$

It is obvious that $A_t^{m_1}$ is nonempty valued. For any $(s_0, s_1, \ldots, s_{m_1})$, since $A_t^{m_1-1}(\cdot, s_0, s_1, \ldots s_{m_1-1})$ has a compact graph and $A_{m_1}(\cdot, s_0, s_1, \ldots, s_{m_1-1})$ is continuous and compact valued, $A_t^{m_1}(\cdot, s_0, s_1, \ldots s_{m_1})$ has a compact graph by Lemma 2 (6), which implies that $A_t^{m_1}$ is compact valued and sectionally upper hemicontinuous on $X_{t-1}$. In addition, $\mathrm{Gr}(A_t^{m_1}) = \mathrm{Gr}(A_{m_1}) \times S_{m_1}$, which is $\mathcal{B}(X^{m_1} \times S^{m_1})$-measurable. For any $h_{t-1} \in H_{t-1}$ and $\tau \in \Xi_t^{m_1}(h_{t-1})$, it is obvious that the marginal of $\tau$ on $\prod_{t \le m \le m_1} S_m$ is $\otimes_{t \le m \le m_1} \lambda_m$ and $\tau(\mathrm{Gr}(A_t^{m_1}(h_{t-1}, \cdot))) = 1$.

By Lemma B.5, $\Xi_t^{m_2+1}$ is nonempty and compact valued, and sectionally continuous on $X^{t-1}$.

Now we show that the correspondence $\Delta_t^{m_1}$ is nonempty and compact valued, and sectionally continuous on $X^{t-1}$ for any $m_1 \ge t$.

Given $s^{t-1}$ and a sequence $\{x_0^k, x_1^k, \ldots, x_{t-1}^k\} \in H_{t-1}(s^{t-1})$ for $1 \le k \le \infty$. Let $h_{t-1}^k = (s^{t-1}, (x_0^k, x_1^k, \ldots, x_{t-1}^k))$. It is obvious that $\Delta_t^{m_1}$ is nonempty valued, we first show that $\Delta_t^{m_1}$ is sectionally upper hemicontinuous on $X^{t-1}$. Suppose that $\varrho_{(h_{t-1}^k, \xi^k)}^{m_1} \in \Delta_t^{m_1}(h_{t-1}^k)$ for $1 \le k < \infty$ and $(x_0^k, x_1^k, \ldots, x_{t-1}^k) \to (x_0^\infty, x_1^\infty, \ldots, x_{t-1}^\infty)$, we need to show that there exists some $\xi^\infty$ such that a subsequence of $\varrho_{(h_{t-1}^k, \xi^k)}^{m_1}$ weakly converges to $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1}$ and $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1} \in \Delta_t^{m_1}(h_{t-1}^\infty)$.

Since $\Xi_t^{m_1}$ is sectionally upper hemicontinuous on $X^{t-1}$, there exists some $\xi^\infty$ such that a subsequence of $\rho_{(h_{t-1}^k, \xi^k)}^{m_1}$, say itself, weakly converges to $\rho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1}$ and $\rho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1} \in \Xi_t^{m_1}(h_{t-1}^\infty)$. Then $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1} \in \Delta_t^{m_1}(h_{t-1}^\infty)$.

For any bounded continuous function $\psi$ on $\prod_{t \le m \le m_1}(X_m \times S_m)$, let

$$\chi_k(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}) =$$

$$\psi(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}) \cdot \prod_{t \le m \le m_1} \varphi_{m0}(h_{t-1}^k, x_t, \ldots, x_{m-1}, s_t, \ldots, s_m).$$

Then $\{\chi_k\}$ is a sequence of functions satisfying the following three properties.

1. For each $k$, $\chi_k$ is jointly measurable and sectionally continuous on $\prod_{t \le m \le m_1} X_m$.
2. For any $(s_t, \ldots, s_{m_1})$ and any sequence $(x_t^k, \ldots, x_{m_1}^k) \to (x_t^\infty, \ldots, x_{m_1}^\infty)$ in $\prod_{t \le m \le m_1} X_m$, $\chi_k(x_t^k, \ldots, x_{m_1}^k, s_t, \ldots, s_{m_1}) \to \chi_\infty(x_t^\infty, \ldots, x_{m_1}^\infty, s_t, \ldots, s_{m_1})$ as $k \to \infty$.
3. The sequence $\{\chi_k\}_{1 \le k \le \infty}$ is integrably bounded in the sense that there exists a function $\chi' \colon \prod_{t \le m \le m_1} S_m \to \mathbb{R}_+$ such that $\chi'$ is $\otimes_{t \le m \le m_1} \lambda_m$-integrable and for any $k$ and $(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1})$, $\chi_k(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}) \le \chi'(s_t, \ldots, s_{m_1})$.

By Lemma B.6, as $k \to \infty$,

$$\int_{\prod_{t \le m \le m_1}(X_m \times S_m)} \chi_k(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}) \rho^{m_1}_{(h_{t-1}^k, \xi^k)}(\mathrm{d}(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}))$$

$$\to \int_{\prod_{t \le m \le m_1}(X_m \times S_m)} \chi_\infty(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}) \rho^{m_1}_{(h_{t-1}^\infty, \xi^\infty)}(\mathrm{d}(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1})).$$

Then by Lemma B.7,

$$\int_{\prod_{t \le m \le m_1}(X_m \times S_m)} \psi(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}) \varrho^{m_1}_{(h_{t-1}^k, \xi^k)}(\mathrm{d}(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}))$$

$$\to \int_{\prod_{t \le m \le m_1}(X_m \times S_m)} \psi(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}) \varrho^{m_1}_{(h_{t-1}^\infty, \xi^\infty)}(\mathrm{d}(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1})),$$

which implies that $\varrho^{m_1}_{(h_{t-1}^k, \xi^k)}$ weakly converges to $\varrho^{m_1}_{(h_{t-1}^\infty, \xi^\infty)}$. Therefore, $\Delta_t^{m_1}$ is sectionally upper hemicontinuous on $X^{t-1}$. If one chooses $h_{t-1}^1 = h_{t-1}^2 = \cdots = h_{t-1}^\infty$, then we indeed show that $\Delta_t^{m_1}$ is compact valued.

In the argument above, we indeed proved that if $\rho^{m_1}_{(h_{t-1}^k, \xi^k)}$ weakly converges to $\rho^{m_1}_{(h_{t-1}^\infty, \xi^\infty)}$, then $\varrho^{m_1}_{(h_{t-1}^k, \xi^k)}$ weakly converges to $\varrho^{m_1}_{(h_{t-1}^\infty, \xi^\infty)}$.

The left is to show that $\Delta_t^{m_1}$ is sectionally lower hemicontinuous on $X^{t-1}$. Suppose that $(x_0^k, x_1^k, \ldots, x_{t-1}^k) \to (x_0^\infty, x_1^\infty, \ldots, x_{t-1}^\infty)$ and $\varrho^{m_1}_{(h_{t-1}^\infty, \xi^\infty)} \in \Delta_t^{m_1}(h_{t-1}^\infty)$, we need to show that there exists a subsequence $\{(x_0^{k_m}, x_1^{k_m}, \ldots, x_{t-1}^{k_m})\}$ of $\{(x_0^k, x_1^k, \ldots, x_{t-1}^k)\}$ and $\varrho^{m_1}_{(h^{k_m}_{t-1}, \xi^{k_m})} \in \Delta_t^{m_1}(h^{k_m}_{t-1})$ for each $k_m$ such that $\varrho^{m_1}_{(h^{k_m}_{t-1}, \xi^{k_m})}$ weakly converges to $\varrho^{m_1}_{(h^{\infty}_{t-1}, \xi^{\infty})}$.

Since $\varrho^{m_1}_{(h^{\infty}_{t-1}, \xi^{\infty})} \in \Delta_t^{m_1}(h^{\infty}_{t-1})$, we have $\rho^{m_1}_{(h^{\infty}_{t-1}, \xi^{\infty})} \in \Xi_t^{m_1}(h^{\infty}_{t-1})$. Because $\Xi_t^{m_1}$ is sectionally lower hemicontinuous on $X^{t-1}$, there exists a subsequence of $\{(x_0^k, x_1^k, \ldots, x_{t-1}^k)\}$, say itself, and $\rho^{m_1}_{(h^k_{t-1}, \xi^k)} \in \Xi_t^{m_1}(h^k_{t-1})$ for each $k$ such that $\rho^{m_1}_{(h^k_{t-1}, \xi^k)}$ weakly converges to $\rho^{m_1}_{(h^{\infty}_{t-1}, \xi^{\infty})}$. As a result, $\varrho^{m_1}_{(h^k_{t-1}, \xi^k)}$ weakly converges to $\varrho^{m_1}_{(h^{\infty}_{t-1}, \xi^{\infty})}$, which implies that $\Delta_t^{m_1}$ is sectionally lower hemicontinuous on $X^{t-1}$.

Therefore, $\Delta_t^{m_1}$ is nonempty and compact valued, and sectionally continuous on $X^{t-1}$ for any $m_1 \geq t$.

(2) We show that $\Delta_t$ is nonempty and compact valued, and sectionally continuous on $X^{t-1}$.

It is obvious that $\Delta_t$ is nonempty valued, we first prove that it is compact valued.

Given $h_{t-1}$ and a sequence $\{\tau^k\} \subseteq \Delta_t(h_{t-1})$, there exists a sequence of $\{\xi^k\}_{k \geq 1}$ such that $\xi^k = (\xi_1^k, \xi_2^k, \ldots) \in \Upsilon$ and $\tau^k = \varrho_{(h_{t-1}, \xi^k)}$ for each $k$.

By (1), $\Xi_t^t$ is compact. Then there exists a measurable mapping $g_t$ such that (1) $g^t = (\xi_1^1, \ldots, \xi_{t-1}^1, g_t, \xi_{t+1}^1, \ldots) \in \Upsilon$, and (2) a subsequence of $\{\rho^t_{(h_{t-1}, \xi^k)}\}$, say $\{\rho^t_{(h_{t-1}, \xi^{k_{1l}})}\}_{l \geq 1}$, which weakly converges to $\rho^t_{(h_{t-1}, g^t)}$. Note that $\{\xi^k_{t+1}\}$ is a Borel measurable selection of $\mathcal{M}(A_{t+1})$. By Lemma B.5, there is a Borel measurable selection $g_{t+1}$ of $\mathcal{M}(A_{t+1})$ such that there is a subsequence of $\{\rho^{t+1}_{(h_{t-1}, \xi^{k_{1l}})}\}_{l \geq 1}$, say $\{\rho^{t+1}_{(h_{t-1}, \xi^{k_{2l}})}\}_{l \geq 1}$, which weakly converges to $\rho^{t+1}_{(h_{t-1}, g^{t+1})}$, where $g^{t+1} = (\xi_1^1, \ldots, \xi_{t-1}^1, g_t, g_{t+1}, \xi_{t+2}^1, \ldots) \in \Upsilon$.

Repeat this procedure, one can construct a Borel measurable mapping $g$ such that $\rho_{(h_{t-1}, \xi^{k_{11}})}, \rho_{(h_{t-1}, \xi^{k_{22}})}, \rho_{(h_{t-1}, \xi^{k_{33}})}, \ldots$ weakly converges to $\rho_{(h_{t-1}, g)}$. That is, $\rho_{(h_{t-1}, g)}$ is a convergent point of $\{\rho_{(h_{t-1}, \xi^k)}\}$, which implies that $\varrho_{(h_{t-1}, g)}$ is a convergent point of $\{\varrho_{(h_{t-1}, \xi^k)}\}$.

The sectional upper hemicontinuity of $\Delta_t$ follows a similar argument as above. In particular, given $s^{t-1}$ and a sequence $\{x_0^k, x_1^k, \ldots, x_{t-1}^k\} \subseteq H_{t-1}(s^{t-1})$ for $k \geq 0$. Let $h^k_{t-1} = (s^{t-1}, (x_0^k, x_1^k, \ldots, x_{t-1}^k))$. Suppose that $(x_0^k, x_1^k, \ldots, x_{t-1}^k) \to (x_0^0, x_1^0, \ldots, x_{t-1}^0)$. If $\{\tau^k\} \subseteq \Delta_t(h^k_{t-1})$ for $k \geq 1$ and $\tau^k \to \tau^0$, then one can show that $\tau^0 \in \Delta_t(h^0_{t-1})$ by repeating a similar argument as in the proof above.

Finally, we consider the sectional lower hemicontinuity of $\Delta_t$. Suppose that $\tau^0 \in \Delta_t(h^0_{t-1})$. Then there exists some $\xi \in \Upsilon$ such that $\tau^0 = \varrho_{(h^0_{t-1}, \xi)}$. Denote $\tilde{\tau}^m = \varrho^m_{(h^0_{t-1}, \xi)} \in \Delta_t^m(h^0_{t-1})$ for $m \geq t$. As $\Delta_t^m$ is continuous, for each $m$, there exists some $\xi^m \in \Upsilon$ such that $d(\varrho^m_{(h^{k_m}_{t-1}, \xi^m)}, \tilde{\tau}^m) \leq \frac{1}{m}$ for $k_m$ sufficiently large, where $d$ is the Prokhorov metric. Let

$\tau^m = \varrho_{(h_{t-1}^{k_m}, \xi^m)}$. Then $\tau^m$ weakly converges to $\tau^0$, which implies that $\Delta_t$ is sectionally lower hemicontinuous. □

Define a correspondence $Q_t^\tau \colon H_{t-1} \to \mathbb{R}^n_{++}$ as follows:

$$Q_t^\tau(h_{t-1}) =$$

$$\begin{cases} \{\int_{\prod_{m\geq t}(X_m \times S_m)} u(h_{t-1}, x, s) \varrho_{(h_{t-1},\xi)}(\mathrm{d}(x,s)) \colon \varrho_{(h_{t-1},\xi)} \in \Delta_t(h_{t-1})\}; & t > \tau; \\ \Phi(Q_{t+1}^\tau)(h_{t-1}) & t \leq \tau. \end{cases}$$

The lemma below presents several properties of the correspondence $Q_t^\tau$.

**Lemma B.9.** *For any $t, \tau \geq 1$, $Q_t^\tau$ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^{t-1}$.*

*Proof.* We prove the lemma in three steps.

Step 1. Fix $t > \tau$. We will show that $Q_t^\tau$ is bounded, nonempty and compact valued, and sectionally upper hemicontinuous on $X^{t-1}$.

The boundedness and nonemptiness of $Q_t^\tau$ are obvious. We shall prove that $Q_t^\tau$ is sectionally upper hemicontinuous on $X^{t-1}$. Given $s^{t-1}$ and a sequence $\{x_0^k, x_1^k, \ldots, x_{t-1}^k\} \subseteq H_{t-1}(s^{t-1})$ for $k \geq 0$. Let $h_{t-1}^k = (s^{t-1}, (x_0^k, x_1^k, \ldots, x_{t-1}^k))$. Suppose that $a^k \in Q_t^\tau(h_{t-1}^k)$ for $k \geq 1$, $(x_0^k, x_1^k, \ldots, x_{t-1}^k) \to (x_0^0, x_1^0, \ldots, x_{t-1}^0)$ and $a^k \to a^0$, we need to show that $a^0 \in Q_t^\tau(h_{t-1}^0)$.

By the definition, there exists a sequence $\{\xi^k\}_{k\geq 1}$ such that

$$a^k = \int_{\prod_{m\geq t}(X_m \times S_m)} u(h_{t-1}^k, x, s) \varrho_{(h_{t-1}^k, \xi^k)}(\mathrm{d}(x,s)),$$

where $\xi^k = (\xi_1^k, \xi_2^k, \ldots) \in \Upsilon$ for each $k$. As $\Delta_t$ is compact valued and sectionally continuous on $X^{t-1}$, there exist some $\varrho_{(h_{t-1}^0, \xi^0)} \in \Delta_t(h_{t-1}^0)$ and a subsequence of $\varrho_{(h_{t-1}^k, \xi^k)}$, say itself, which weakly converges to $\varrho_{(h_{t-1}^0, \xi^0)}$ for $\xi^0 = (\xi_1^0, \xi_2^0, \ldots) \in \Upsilon$.

We shall show that

$$a^0 = \int_{\prod_{m\geq t}(X_m \times S_m)} u(h_{t-1}^0, x, s) \varrho_{(h_{t-1}^0, \xi^0)}(\mathrm{d}(x,s)).$$

For this aim, we only need to show that for any $\delta > 0$,

$$\left| a^0 - \int_{\prod_{m\geq t}(X_m \times S_m)} u(h_{t-1}^0, x, s) \varrho_{(h_{t-1}^0, \xi^0)}(\mathrm{d}(x,s)) \right| < \delta. \tag{2}$$

Since the game is continuous at infinity, there exists a positive integer $\tilde{M} \geq t$ such that $w^m < \frac{1}{5}\delta$ for any $m > \tilde{M}$.

For each $j > \tilde{M}$, by Lemma 3, there exists a measurable selection $\xi'_j$ of $\mathcal{M}(A_j)$ such that $\xi'_j$ is sectionally continuous on $X^{j-1}$. Let $\mu\colon H_{\tilde{M}} \to \prod_{m>\tilde{M}}(X_m \times S_m)$ be the transition probability which is induced by $(\xi'_{\tilde{M}+1}, \xi'_{\tilde{M}+2}, \ldots)$ and $\{f_{(\tilde{M}+1)0}, f_{(\tilde{M}+2)0}, \ldots\}$. By Lemma 9, $\mu$ is measurable and sectionally continuous on $X^{\tilde{M}}$. Let

$$V_{\tilde{M}}(h_{t-1}, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}) =$$

$$\int_{\prod_{m>\tilde{M}}(X_m \times S_m)} u(h_{t-1}, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}, x, s)\, \mathrm{d}\mu(x, s | h_{t-1}, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}).$$

Then $V_{\tilde{M}}$ is bounded and measurable. In addition, $V_{\tilde{M}}$ is sectionally continuous on $X^{\tilde{M}}$ by Lemma B.6.

For any $k \geq 0$, we have

$$\begin{aligned}
&|\int_{\prod_{m\geq t}(X_m \times S_m)} u(h^k_{t-1}, x, s) \varrho_{(h^k_{t-1}, \xi^k)}(\mathrm{d}(x, s)) \\
&- \int_{\prod_{t\leq m\leq \tilde{M}}(X_m \times S_m)} V_{\tilde{M}}(h^k_{t-1}, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}) \varrho^{\tilde{M}}_{(h^k_{t-1}, \xi^k)}(\mathrm{d}(x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}))| \\
&\leq w^{\tilde{M}+1} \\
&< \frac{1}{5}\delta.
\end{aligned}$$

Since $\varrho_{(h^k_{t-1}, \xi^k)}$ weakly converges to $\varrho_{(h^0_{t-1}, \xi^0)}$ and $\varrho^{\tilde{M}}_{(h^k_{t-1}, \xi^k)}$ is the marginal of $\varrho_{(h^k_{t-1}, \xi^k)}$ on $\prod_{t\leq m\leq \tilde{M}}(X_m \times S_m)$ for any $k \geq 0$, the sequence $\varrho^{\tilde{M}}_{(h^k_{t-1}, \xi^k)}$ also weakly converges to $\varrho^{\tilde{M}}_{(h^0_{t-1}, \xi^0)}$. By Lemma B.6, we have

$$\begin{aligned}
&|\int_{\prod_{t\leq m\leq \tilde{M}}(X_m \times S_m)} V_{\tilde{M}}(h^k_{t-1}, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}) \varrho^{\tilde{M}}_{(h^k_{t-1}, \xi^k)}(\mathrm{d}(x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}})) \\
&- \int_{\prod_{t\leq m\leq \tilde{M}}(X_m \times S_m)} V_{\tilde{M}}(h^0_{t-1}, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}) \varrho^{\tilde{M}}_{(h^0_{t-1}, \xi^0)}(\mathrm{d}(x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}))| \\
&< \frac{1}{5}\delta
\end{aligned}$$

for $k \geq K_1$, where $K_1$ is a sufficiently large positive integer. In addition, there exists a positive integer $K_2$ such that $|a^k - a^0| < \frac{1}{5}\delta$ for $k \geq K_2$.

Fix $k > \max\{K_1, K_2\}$. Combining the inequalities above, we have

$$\begin{aligned}
&\left|\int_{\prod_{m\geq t}(X_m\times S_m)} u(h_{t-1}^0, x, s)\varrho_{(h_{t-1}^0,\xi^0)}(\mathrm{d}(x,s)) - a^0\right| \\
&\leq \Big|\int_{\prod_{m\geq t}(X_m\times S_m)} u(h_{t-1}^0, x, s)\varrho_{(h_{t-1}^0,\xi^0)}(\mathrm{d}(x,s)) \\
&- \int_{\prod_{t\leq m\leq \tilde{M}}(X_m\times S_m)} V_{\tilde{M}}(h_{t-1}^0, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}})\varrho^{\tilde{M}}_{(h_{t-1}^0,\xi^0)}(\mathrm{d}(x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}))\Big| \\
&+ \Big|\int_{\prod_{t\leq m\leq \tilde{M}}(X_m\times S_m)} V_{\tilde{M}}(h_{t-1}^0, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}})\varrho^{\tilde{M}}_{(h_{t-1}^0,\xi^0)}(\mathrm{d}(x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}})) \\
&- \int_{\prod_{t\leq m\leq \tilde{M}}(X_m\times S_m)} V_{\tilde{M}}(h_{t-1}^k, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}})\varrho^{\tilde{M}}_{(h_{t-1}^k,\xi^k)}(\mathrm{d}(x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}))\Big| \\
&+ \Big|\int_{\prod_{t\leq m\leq \tilde{M}}(X_m\times S_m)} V_{\tilde{M}}(h_{t-1}^k, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}})\varrho^{\tilde{M}}_{(h_{t-1}^k,\xi^k)}(\mathrm{d}(x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}})) \\
&- \int_{\prod_{m\geq t}(X_m\times S_m)} u(h_{t-1}^k, x, s)\varrho_{(h_{t-1}^k,\xi^k)}(\mathrm{d}(x,s))\Big| \\
&+ \Big|\int_{\prod_{m\geq t}(X_m\times S_m)} u(h_{t-1}^k, x, s)\varrho_{(h_{t-1}^k,\xi^k)}(\mathrm{d}(x,s)) - a^0\Big| \\
&< \delta.
\end{aligned}$$

Thus, we proved inequality (2), which implies that $Q_t^\tau$ is sectionally upper hemicontinuous on $X^{t-1}$ for $t > \tau$.

Furthermore, to prove that $Q_t^\tau$ is compact valued, we only need to consider the case that $\{x_0^k, x_1^k, \ldots, x_{t-1}^k\} = \{x_0^0, x_1^0, \ldots, x_{t-1}^0\}$ for any $k \geq 0$, and repeat the above proof.

Step 2. Fix $t > \tau$, we will show that $Q_t^\tau$ is measurable.

Fix a sequence $(\xi_1', \xi_2', \ldots)$, where $\xi_j'$ is a selection of $\mathcal{M}(A_j)$ measurable in $s^{j-1}$ and continuous in $x^{j-1}$ for each $j$. For any $M \geq t$, let

$$W_M^M(h_{t-1}, x_t, \ldots, x_M, s_t, \ldots, s_M) =$$

$$\left\{\int_{\prod_{m>M}(X_m\times S_m)} u(h_{t-1}, x_t, \ldots, x_M, s_t, \ldots, s_M, x, s)\varrho_{(h_{t-1},x_t,\ldots,x_M,s_t,\ldots,s_M,\xi')}(\mathrm{d}(x,s))\right\}.$$

By Lemma 9, $\varrho_{(h_{t-1},x_t,\ldots,x_M,s_t,\ldots,s_M,\xi')}$ is measurable from $H_M$ to $\mathcal{M}\left(\prod_{m>M}(X_m \times S_m)\right)$, and sectionally continuous on $X^M$. Thus, $W_M^M$ is bounded, measurable, nonempty, convex and compact valued. By Lemma B.6, $W_M^M$ is sectionally continuous on $X^M$.

Suppose that for some $t \le j \le M$, $W_M^j$ has been defined such that it is bounded, measurable, nonempty, convex and compact valued, and sectionally continuous on $X^j$. Let

$$\begin{aligned}&W_M^{j-1}(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}) = \\ \{&\int_{X_j \times S_j} w_M^j(h_{t-1}, x_t, \ldots, x_j, s_t, \ldots, s_j)\varrho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)}(\mathrm{d}(x_j, s_j)) \colon \\ &\varrho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)} \in \Delta_j^j(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}), \\ &w_M^j \text{ is a Borel measurable selection of } W_M^j\}.\end{aligned}$$

Let $\check{S}_j = S_j$.[6] Since

$$\begin{aligned}&\int_{X_j \times S_j} W_M^j(h_{t-1}, x_t, \ldots, x_j, s_t, \ldots, s_j)\varrho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)}(\mathrm{d}(x_j, s_j)) \\ =&\int_{S_j}\int_{X_j \times \check{S}_j} W_M^j(h_{t-1}, x_t, \ldots, x_j, s_t, \ldots, s_j)\rho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)}(\mathrm{d}(x_j, \check{s}_j)) \\ &\cdot \varphi_{j0}(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_j)\lambda_j(\mathrm{d}s_j),\end{aligned}$$

we have

$$\begin{aligned}&W_M^{j-1}(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}) = \\ \{&\int_{S_j}\int_{X_j \times \check{S}_j} w_M^j(h_{t-1}, x_t, \ldots, x_j, s_t, \ldots, s_j)\rho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)}(\mathrm{d}(x_j, \check{s}_j)) \\ &\cdot \varphi_{j0}(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_j)\lambda_j(\mathrm{d}s_j) \colon \\ &\rho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)} \in \Xi_j^j(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}), \\ &w_M^j \text{ is a Borel measurable selection of } W_M^j\}.\end{aligned}$$

Let

$$\begin{aligned}&\check{W}_M^j(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_j) = \\ \{&\int_{X_j \times \check{S}_j} w_M^j(h_{t-1}, x_t, \ldots, x_j, s_t, \ldots, s_j) \cdot \rho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)}(\mathrm{d}(x_j, \check{s}_j)) \colon \\ &\rho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)} \in \Xi_j^j(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}),\end{aligned}$$

[6] We will need to use Lemma B.2 below, which requires the continuity of the correspondences in terms of the integrated variables. Since $W_M^j$ is only measurable, but not continuous, in $s_j$, we add a dummy variable $\tilde{s}_j$ so that $W_M^j$ is trivially continuous in such a variable.

$w_M^j$ is a Borel measurable selection of $W_M^j\}$.

Since $W_M^j(h_{t-1}, x_t, \ldots, x_j, s_t, \ldots, s_j)$ is continuous in $x_j$ and does not depend on $\check{s}_j$, it is continuous in $(x_j, \check{s}_j)$. In addition, $W_M^j$ is bounded, measurable, nonempty, convex and compact valued. By Lemma B.2, $\check{W}_M^j$ is bounded, measurable, nonempty and compact valued, and sectionally continuous on $X^{j-1}$.

It is easy to see that

$$W_M^{j-1}(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}) =$$

$$\int_{S_j} \check{W}_M^j(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_j)\varphi_{j0}(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_j)\lambda_j(\mathrm{d}s_j).$$

By Lemma 4, it is bounded, measurable, nonempty and compact valued, and sectionally continuous on $X^{j-1}$. By induction, one can show that $W_M^{t-1}$ is bounded, measurable, nonempty and compact valued, and sectionally continuous on $X^{t-1}$.

Let $W^{t-1} = \overline{\cup_{M \geq t} W_M^{t-1}}$. That is, $W^{t-1}$ is the closure of $\cup_{M \geq t} W_M^{t-1}$, which is measurable due to Lemma 2.

First, $W^{t-1} \subseteq Q_t^\tau$ because $W_M^{t-1} \subseteq Q_t^\tau$ for each $M \geq t$ and $Q_t^\tau$ is compact valued. Second, fix $h_{t-1}$ and $q \in Q_t^\tau(h_{t-1})$. Then there exists a mapping $\xi \in \Upsilon$ such that

$$q = \int_{\prod_{m \geq t}(X_m \times S_m)} u(h_{t-1}, x, s)\varrho_{(h_{t-1}, \xi)}(\mathrm{d}(x, s)).$$

For $M \geq t$, let

$$V_M(h_{t-1}, x_t, \ldots, x_M, s_t, \ldots, s_M) =$$

$$\int_{\prod_{m > M}(X_m \times S_m)} u(h_{t-1}, x_t, \ldots, x_M, s_t, \ldots, s_M, x, s)\varrho_{(h_{t-1}, x_t, \ldots, x_M, s_t, \ldots, s_M, \xi)}(x, s)$$

and

$$q_M = \int_{\prod_{t \leq m \leq M}(X_m \times S_m)} V_M(h_{t-1}, x, s)\varrho_{(h_{t-1}, \xi)}^M(\mathrm{d}(x, s)).$$

Hence, $q_M \in W_M^{t-1}$. Because the dynamic game is continuous at infinity, $q_M \to q$, which implies that $q \in W^{t-1}(h_{t-1})$ and $Q_t^\tau \subseteq W^{t-1}$.

Therefore, $W^{t-1} = Q_t^\tau$, and hence $Q_t^\tau$ is measurable for $t > \tau$.

Step 3. For $t \leq \tau$, we can start with $Q_{\tau+1}^\tau$. Repeating the backward induction in Subsection B.4.1, we have that $Q_t^\tau$ is also bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^{t-1}$. □

Denote

$$Q_t^\infty = \begin{cases} Q_t^{t-1}, & \text{if } \cap_{\tau\geq 1} Q_t^\tau = \emptyset; \\ \cap_{\tau\geq 1} Q_t^\tau, & \text{otherwise.} \end{cases}$$

The following three lemmas show that $Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1}) = E_t(h_{t-1})$ for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$.[7]

**Lemma B.10.** *1. The correspondence $Q_t^\infty$ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^{t-1}$.*

*2. For any $t \geq 1$, $Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1})$ for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$.*

*Proof.* (1) It is obvious that $Q_t^\infty$ is bounded. By the definition of $Q_t^\tau$, for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$, $Q_t^{\tau_1}(h_{t-1}) \subseteq Q_t^{\tau_2}(h_{t-1})$ for $\tau_1 \geq \tau_2$. Since $Q_t^\tau$ is nonempty and compact valued, $Q_t^\infty = \cap_{\tau\geq 1} Q_t^\tau$ is nonempty and compact valued for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$. If $\cap_{\tau\geq 1} Q_t^\tau = \emptyset$, then $Q_t^\infty = Q_t^{t-1}$. Thus, $Q_t^\infty(h_{t-1})$ is nonempty and compact valued for all $h_{t-1} \in H_{t-1}$. By Lemma 2 (2), $\cap_{\tau\geq 1} Q_t^\tau$ is measurable, which implies that $Q_t^\infty$ is measurable.

Fix any $s^{t-1} \in S^{t-1}$ such that $Q_t^\tau(\cdot, s^{t-1})$ is upper hemicontinuous on $H_{t-1}(s^{t-1})$ for any $\tau$. By Lemma 2 (7), $Q_t^\tau(\cdot, s^{t-1})$ has a closed graph for each $\tau$, which implies that $Q_t^\infty(\cdot, s^{t-1})$ has a closed graph. Referring to Lemma 2 (7) again, $Q_t^\infty(\cdot, s^{t-1})$ is upper hemicontinuous on $H_{t-1}(s^{t-1})$. Since $Q_t^\tau$ is essentially upper hemicontinuous on $X^{t-1}$ for each $\tau$, $Q_t^\infty$ is essentially upper upper hemicontinuous on $X^{t-1}$.

(2) For any $\tau \geq 1$ and $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$, $\Phi(Q_{t+1}^\infty)(h_{t-1}) \subseteq \Phi(Q_{t+1}^\tau)(h_{t-1}) \subseteq Q_t^\tau(h_{t-1})$, and hence $\Phi(Q_{t+1}^\infty)(h_{t-1}) \subseteq Q_t^\infty(h_{t-1})$.

The space $\{1, 2, \ldots \infty\}$ is a countable compact set endowed with the following metric: $d(k, m) = |\frac{1}{k} - \frac{1}{m}|$ for any $1 \leq k, m \leq \infty$. The sequence $\{Q_{t+1}^\tau\}_{1\leq\tau\leq\infty}$ can be regarded as a correspondence $Q_{t+1}$ from $H_t \times \{1, 2, \ldots, \infty\}$ to $\mathbb{R}^n$, which is measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \{1, 2, \ldots, \infty\}$. The backward induction in Subsection B.4.1 shows that $\Phi(Q_{t+1})$ is measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \{1, 2, \ldots, \infty\}$.

[7] The proofs for Lemmas B.10 and B.12 follow the standard ideas with various modifications; see, for example, [3], [4] and [5].

Since $\Phi(Q_{t+1})$ is essentially sectionally upper hemicontinuous on $X^t \times \{1, 2, \ldots, \infty\}$, there exists a measurable subset $\check{S}^{t-1} \subseteq S^{t-1}$ such that $\lambda^{t-1}(\check{S}^{t-1}) = 1$, and $\Phi(Q_{t+1})(\cdot, \cdot, \check{s}^{t-1})$ is upper hemicontinuous for any $\check{s}^{t-1} \in \check{S}^{t-1}$. Fix $\check{s}^{t-1} \in \check{S}^{t-1}$. For $h_{t-1} = (x^{t-1}, \check{s}^{t-1}) \in H_{t-1}$ and $a \in Q_t^{\infty}(h_{t-1})$, by its definition, $a \in Q_t^{\tau}(h_{t-1}) = \Phi(Q_{t+1}^{\tau})(h_{t-1})$ for $\tau \geq t$. Thus, $a \in \Phi(Q_{t+1}^{\infty})(h_{t-1})$.

In summary, $Q_t^{\infty}(h_{t-1}) = \Phi(Q_{t+1}^{\infty})(h_{t-1})$ for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$. □

Though the definition of $Q_t^{\tau}$ involves correlated strategies for $\tau < t$, the following lemma shows that one can work with mixed strategies in terms of equilibrium payoffs via the combination of backward and forward inductions in multiple steps.

**Lemma B.11.** *If $c_t$ is a measurable selection of $\Phi(Q_{t+1}^{\infty})$, then $c_t(h_{t-1})$ is a subgame-perfect equilibrium payoff vector for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$.*

*Proof.* Without loss of generality, we only prove the case $t = 1$.

Suppose that $c_1$ is a measurable selection of $\Phi(Q_2^{\infty})$. Apply Proposition B.3 recursively to obtain Borel measurable mappings $\{f_{ki}\}_{i \in I}$ for $k \geq 1$. That is, for any $k \geq 1$, there exists a Borel measurable selection $c_k$ of $Q_k^{\infty}$ such that for $\lambda_{k-1}$-almost all $h_{k-1} \in H_{k-1}$,

1. $f_k(h_{k-1})$ is a Nash equilibrium in the subgame $h_{k-1}$, where the action space is $A_{ki}(h_{k-1})$ for player $i \in I$, and the payoff function is given by

$$\int_{S_k} c_{k+1}(h_{k-1}, \cdot, s_k) f_{k0}(\mathrm{d}s_k | h_{k-1}).$$

2. $$c_k(h_{k-1}) = \int_{A_k(h_{k-1})} \int_{S_k} c_{k+1}(h_{k-1}, x_k, s_k) f_{k0}(\mathrm{d}s_k | h_{k-1}) f_k(\mathrm{d}x_k | h_{k-1}).$$

We need to show that $c_1(h_0)$ is a subgame-perfect equilibrium payoff vector for $\lambda_0$-almost all $h_0 \in H_0$.

Step 1. We show that for any $k \geq 1$ and $\lambda_{k-1}$-almost all $h_{k-1} \in H_{k-1}$,

$$c_k(h_{k-1}) = \int_{\prod_{m \geq k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f)}(\mathrm{d}(x, s)).$$

Since the game is continuous at infinity, there exists some positive integer $M > k$ such that $w^M$ is sufficiently small. By Lemma B.10, $c_k(h_{k-1}) \in Q_k^{\infty}(h_{k-1}) = \cap_{\tau \geq 1} Q_k^{\tau}(h_{k-1})$

for $\lambda_{k-1}$-almost all $h_{k-1} \in H_{k-1}$. Since $Q_k^\tau = \Phi^{\tau-k+1}(Q_{\tau+1}^\tau)$ for $k \le \tau$, $c_k(h_{k-1}) \in \cap_{\tau \ge k} \Phi^{\tau-k+1}(Q_{\tau+1}^\tau)(h_{k-1}) \subseteq \Phi^{M-k+1}(Q_{M+1}^M)(h_{k-1})$ for $\lambda_{k-1}$-almost all $h_{k-1} \in H_{k-1}$. Thus, there exists a Borel measurable selection $w$ of $Q_{M+1}^M$ and some $\xi \in \Upsilon$ such that for $\lambda_{M-1}$-almost all $h_{M-1} \in H_{M-1}$,

i. $f_M(h_{M-1})$ is a Nash equilibrium in the subgame $h_{M-1}$, where the action space is $A_{Mi}(h_{M-1})$ for player $i \in I$, and the payoff function is given by

$$\int_{S_M} w(h_{M-1}, \cdot, s_M) f_{M0}(\mathrm{d}s_M | h_{M-1});$$

ii.

$$c_M(h_{M-1}) = \int_{A_M(h_{M-1})} \int_{S_M} w(h_{M-1}, x_M, s_M) f_{M0}(\mathrm{d}s_M | h_{M-1}) f_M(\mathrm{d}x_M | h_{M-1});$$

iii. $w(h_M) = \int_{\prod_{m \ge M+1}(X_m \times S_m)} u(h_M, x, s) \varrho_{(h_M, \xi)}(\mathrm{d}(x, s))$.

Then for $\lambda_{k-1}$-almost all $h_{k-1} \in H_{k-1}$,

$$c_k(h_{k-1}) = \int_{\prod_{m \ge k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f^M)}(\mathrm{d}(x, s)),$$

where $f_k^M$ is $f_k$ if $k \le M$, and $\xi_k$ if $k \ge M+1$. Since the game is continuous at infinity,

$$\int_{\prod_{m \ge k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f^M)}(\mathrm{d}(x, s))$$

converges to

$$\int_{\prod_{m \ge k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f)}(\mathrm{d}(x, s))$$

when $M$ goes to infinity. Thus, for $\lambda_{k-1}$-almost all $h_{k-1} \in H_{k-1}$,

$$c_k(h_{t-1}) = \int_{\prod_{m \ge k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f)}(\mathrm{d}(x, s)). \tag{3}$$

Step 2. Below, we show that $\{f_{ki}\}_{i \in I}$ is a subgame-perfect equilibrium.

Fix a player $i$ and a strategy $g_i = \{g_{ki}\}_{k \ge 1}$. For each $k \ge 1$, define a new strategy $\tilde{f}_i^k$ as follows: $\tilde{f}_i^k = (g_{1i}, \ldots, g_{ki}, f_{(k+1)i}, f_{(k+2)i}, \ldots)$. That is, we simply replace the initial $k$ stages of $f_i$ by $g_i$. Denote $\tilde{f}^k = (\tilde{f}_i^k, f_{-i})$.

Fix $k \geq 1$ and a measurable subset $D^k \subseteq S^k$ such that (1) and (2) of step 1 and Equation (3) hold for all $s_k \in D^k$ and $x^k \in H_k(s^k)$, and $\lambda^k(D^k) = 1$. For each $\tilde{M} > k$, by the Fubini property, there exists a measurable subset $E_k^{\tilde{M}} \subseteq S^k$ such that $\lambda^k(E_k^{\tilde{M}}) = 1$ and $\otimes_{k+1 \leq j \leq \tilde{M}} \lambda_j(D^{\tilde{M}}(s^k)) = 1$ for all $s^k \in E_k^{\tilde{M}}$, where

$$D^{\tilde{M}}(s^k) = \{(s_{k+1}, \ldots, s_{\tilde{M}}) \colon (s^k, s_{k+1}, \ldots, s_{\tilde{M}}) \in D^{\tilde{M}}\}.$$

Let $\hat{D}^k = (\cap_{\tilde{M} > k} E_k^{\tilde{M}}) \cap D^k$. Then $\lambda^k(\hat{D}^k) = 1$.

For any $h_k = (x^k, s^k)$ such that $s^k \in \hat{D}^k$ and $x^k \in H_k(s^k)$, we have

$$\begin{aligned}
&\int_{\prod_{m \geq k+1}(X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, f)}(\mathrm{d}(x, s)) \\
&= \int_{A_{k+1}(h_k)} \int_{S_{k+1}} c_{(k+2)i}(h_k, x_{k+1}, s_{k+1}) f_{(k+1)0}(\mathrm{d}s_{k+1}|h_k) f_{k+1}(\mathrm{d}x_{k+1}|h_k) \\
&\geq \int_{A_{k+1}(h_k)} \int_{S_{k+1}} c_{(k+2)i}(h_k, x_{k+1}, s_{k+1}) f_{(k+1)0}(\mathrm{d}s_{k+1}|h_k) \left(f_{(k+1)(-i)} \otimes g_{(k+1)i}\right)(\mathrm{d}x_{k+1}|h_k) \\
&= \int_{A_{k+1}(h_k)} \int_{S_{k+1}} \int_{A_{k+2}(h_k, x_{k+1}, s_{k+1})} \int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\
&\quad f_{(k+2)0}(\mathrm{d}s_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes f_{(k+2)i}(\mathrm{d}x_{k+2}|h_k, x_{k+1}, s_{k+1}) \\
&\quad f_{(k+1)0}(\mathrm{d}s_{k+1}|h_k) f_{(k+1)(-i)} \otimes g_{(k+1)i}(\mathrm{d}x_{k+1}|h_k) \\
&\geq \int_{A_{k+1}(h_k)} \int_{S_{k+1}} \int_{A_{k+2}(h_k, x_{k+1}, s_{k+1})} \int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\
&\quad f_{(k+2)0}(\mathrm{d}s_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes g_{(k+2)i}(\mathrm{d}x_{k+2}|h_k, x_{k+1}, s_{k+1}) \\
&\quad f_{(k+1)0}(\mathrm{d}s_{k+1}|h_k) f_{(k+1)(-i)} \otimes g_{(k+1)i}(\mathrm{d}x_{k+1}|h_k) \\
&= \int_{\prod_{m \geq k+1}(X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, \tilde{f}^{k+2})}(\mathrm{d}(x, s)).
\end{aligned}$$

The first and the last equalities follow from Equation (3) in the end of step 1. The second equality is due to (2) in step 1. The first inequality is based on (1) in step 1. The second inequality holds by the following arguments:

i. by the choice of $h_k$ and (1) in step 1, for $\lambda_{k+1}$-almost all $s_{k+1} \in S_{k+1}$ and all $x_{k+1} \in X_{k+1}$ such that $(h_k, x_{k+1}, s_{k+1}) \in H_{k+1}$, we have

$$\begin{aligned}
&\int_{A_{k+2}(h_k, x_{k+1}, s_{k+1})} \int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\
&f_{(k+2)0}(\mathrm{d}s_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes f_{(k+2)i}(\mathrm{d}x_{k+2}|h_k, x_{k+1}, s_{k+1})
\end{aligned}$$

$$\geq \int_{A_{k+2}(h_k,x_{k+1},s_{k+1})} \int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2})$$
$$f_{(k+2)0}(\mathrm{d}s_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes g_{(k+2)i}(\mathrm{d}x_{k+2}|h_k, x_{k+1}, s_{k+1});$$

ii. since $f_{(k+1)0}$ is absolutely continuous with respect to $\lambda_{k+1}$, the above inequality also holds for $f_{(k+1)0}(h_k)$-almost all $s_{k+1} \in S_{k+1}$ and all $x_{k+1} \in X_{k+1}$ such that $(h_k, x_{k+1}, s_{k+1}) \in H_{k+1}$.

Repeating the above argument, one can show that

$$\int_{\prod_{m \geq k+1}(X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, f)}(\mathrm{d}(x, s))$$
$$\geq \int_{\prod_{m \geq k+1}(X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, \tilde{f}^{\tilde{M}+1})}(\mathrm{d}(x, s))$$

for any $\tilde{M} > k$. Since

$$\int_{\prod_{m \geq k+1}(X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, \tilde{f}^{\tilde{M}+1})}(\mathrm{d}(x, s))$$

converges to

$$\int_{\prod_{m \geq k+1}(X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, (g_i, f_{-i}))}(\mathrm{d}(x, s))$$

as $\tilde{M}$ goes to infinity, we have

$$\int_{\prod_{m \geq k+1}(X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, f)}(\mathrm{d}(x, s))$$
$$\geq \int_{\prod_{m \geq k+1}(X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, (g_i, f_{-i}))}(\mathrm{d}(x, s)).$$

Therefore, $\{f_{ki}\}_{i \in I}$ is a subgame-perfect equilibrium. □

By Lemma B.10 and Proposition B.2, the correspondence $\Phi(Q^\infty_{t+1})$ is measurable, nonempty and compact valued. By Lemma 2 (3), it has a measurable selection. Then Theorem 3 follows from the above lemma.

For $t \geq 1$ and $h_{t-1} \in H_{t-1}$, recall that $E_t(h_{t-1})$ is the set of payoff vectors of subgame-perfect equilibria in the subgame $h_{t-1}$. The following lemma shows that $E_t(h_{t-1})$ is essentially the same as $Q^\infty_t(h_{t-1})$.

**Lemma B.12.** *For any $t \geq 1$, $E_t(h_{t-1}) = Q^\infty_t(h_{t-1})$ for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$.*

*Proof.* (1) We will first prove the following claim: for any $t$ and $\tau$, if $E_{t+1}(h_t) \subseteq Q^{\tau}_{t+1}(h_t)$ for $\lambda^t$-almost all $h_t \in H_t$, then $E_t(h_{t-1}) \subseteq Q^{\tau}_t(h_{t-1})$ for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$. We only need to consider the case that $t \leq \tau$.

By the construction of $\Phi(Q^{\tau}_{t+1})$ in Subsection B.4.1, there exists a measurable subset $\acute{S}^{t-1} \subseteq S^{t-1}$ with $\lambda^{t-1}(\acute{S}^{t-1}) = 1$ such that for any $c_t$ and $h_{t-1} = (x^{t-1}, \acute{s}^{t-1}) \in H_{t-1}$ with $\acute{s}^{t-1} \in \acute{S}^{t-1}$, if

1. $c_t = \int_{A_t(h_{t-1})} \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}) \alpha(\mathrm{d}x_t)$, where $q_{t+1}(h_{t-1}, \cdot)$ is measurable and $q_{t+1}(h_{t-1}, x_t, s_t) \in Q^{\tau}_{t+1}(h_{t-1}, x_t, s_t)$ for $\lambda_t$-almost all $s_t \in S_t$ and $x_t \in A_t(h_{t-1})$;
2. $\alpha \in \otimes_{i \in I} \mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame $h_{t-1}$ with payoff $\int_{S_t} q_{t+1}(h_{t-1}, \cdot, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1})$ and action space $\prod_{i \in I} A_{ti}(h_{t-1})$,

then $c_t \in \Phi(Q^{\tau}_{t+1})(h_{t-1})$.

Fix a subgame $h_{t-1} = (x^{t-1}, \acute{s}^{t-1})$ such that $\acute{s}^{t-1} \in \acute{S}^{t-1}$. Pick a point $c_t \in E_t(\acute{s}^{t-1})$. There exists a strategy profile $f$ such that $f$ is a subgame-perfect equilibrium in the subgame $h_{t-1}$ and the payoff is $c_t$. Let $c_{t+1}(h_{t-1}, x_t, s_t)$ be the payoff vector induced by $\{f_{ti}\}_{i \in I}$ in the subgame $(h_t, x_t, s_t) \in \mathrm{Gr}(A_t) \times S_t$. Then we have

1. $c_t = \int_{A_t(h_{t-1})} \int_{S_t} c_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}) f_t(\mathrm{d}x_t|h_{t-1})$;
2. $f_t(\cdot|h_{t-1})$ is a Nash equilibrium in the subgame $h_{t-1}$ with action space $A_t(h_{t-1})$ and payoff $\int_{S_t} c_{t+1}(h_{t-1}, \cdot, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1})$.

Since $f$ is a subgame-perfect equilibrium in the subgame $h_{t-1}$, $c_{t+1}(h_{t-1}, x_t, s_t) \in E_{t+1}(h_{t-1}, x_t, s_t) \subseteq Q^{\tau}_{t+1}(h_{t-1}, x_t, s_t)$ for $\lambda_t$-almost all $s_t \in S_t$ and $x_t \in A_t(h_{t-1})$, which implies that $c_t \in \Phi(Q^{\tau}_{t+1})(h_{t-1}) = Q^{\tau}_t(h_{t-1})$.

Therefore, $E_t(h_{t-1}) \subseteq Q^{\tau}_t(h_{t-1})$ for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$.

(2) For any $t > \tau$, $E_t \subseteq Q^{\tau}_t$. If $t \leq \tau$, we can start with $E_{\tau+1} \subseteq Q^{\tau}_{\tau+1}$ and repeat the argument in (1), then we can show that $E_t(h_{t-1}) \subseteq Q^{\tau}_t(h_{t-1})$ for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$. Thus, $E_t(h_{t-1}) \subseteq Q^{\infty}_t(h_{t-1})$ for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$.

(3) Suppose that $c_t$ is a measurable selection from $\Phi(Q^{\infty}_{t+1})$. Apply Proposition B.3 recursively to obtain Borel measurable mappings $\{f_{ki}\}_{i \in I}$ for $k \geq t$. By Lemma B.11, $c_t(h_{t-1})$ is a subgame-perfect equilibrium payoff vector for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$. Consequently, $\Phi(Q^{\infty}_{t+1})(h_{t-1}) \subseteq E_t(h_{t-1})$ for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$.

By Lemma B.10, $E_t(h_{t-1}) = Q^{\infty}_t(h_{t-1}) = \Phi(Q^{\infty}_{t+1})(h_{t-1})$ for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$. □

## B.5 Proof of Proposition B.1

We will highlight the needed changes in comparison with the proofs presented in Subsections B.4.1-B.4.3.

1. Backward induction. We first consider stage $t$ with $N_t = 1$.

If $N_t = 1$, then $S_t = \{\acute{s_t}\}$. Thus, $P_t(h_{t-1}, x_t) = Q_{t+1}(h_{t-1}, x_t, \acute{s_t})$, which is nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^{t-1}$. Notice that $P_t$ may not be convex valued.

We first assume that $P_t$ is upper hemicontinuous. Suppose that $j$ is the player who is active in this period. Consider the correspondence $\Phi_t\colon H_{t-1} \to \mathbb{R}^n \times \mathcal{M}(X_t) \times \triangle(X_t)$ defined as follows: $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if

1. $v = p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x^*_{tj})$ such that $p_t(h_{t-1}, \cdot)$ is a measurable selection of $P_t(h_{t-1}, \cdot)$;[8]

2. $x^*_{tj} \in A_{tj}(h_{t-1})$ is a maximization point of player $j$ given the payoff function $p_{tj}(h_{t-1}, A_{t(-j)}(h_{t-1}), \cdot)$ and the action space $A_{tj}(h_{t-1})$, $\alpha_i = \delta_{A_{ti}(h_{t-1})}$ for $i \neq j$ and $\alpha_j = \delta_{x^*_{tj}}$;

3. $\mu = \delta_{p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x^*_{tj})}$.

This is a single agent problem. We need to show that $\Phi_t$ is nonempty and compact valued, and upper hemicontinuous.

If $P_t$ is nonempty, convex and compact valued, and upper hemicontinuous, then we can use Lemma 10, the main result of [7], to prove the nonemptiness, compactness, and upper hemicontinuity of $\Phi_t$. In [7], the only step they need the convexity of $P_t$ for the proof of their main theorem is Lemma 2 therein. However, the one-player pure-strategy version of their Lemma 2, stated in the following, directly follows from the upper hemicontinuity of $P_t$ without requiring the convexity.

> Let $Z$ be a compact metric space, and $\{z_n\}_{n\geq 0} \subseteq Z$. Let $P\colon Z \to \mathbb{R}_+$ be a bounded, upper hemicontinuous correspondence with nonempty and compact values. For each $n \geq 1$, let $q_n$ be a Borel measurable selection of $P$ such that $q_n(z_n) = d_n$. If $z_n$ converges to $z_0$ and $d_n$ converges to some $d_0$, then $d_0 \in P(z_0)$.

Repeat the argument in the proof of the main theorem of [7], one can show that $\Phi_t$ is nonempty and compact valued, and upper hemicontinuous.

[8] Note that $A_{t(-j)}$ is point valued since all players other than $j$ are inactive.

Then we go back to the case that $P_t$ is nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^{t-1}$. Recall that we proved Proposition B.2 based on Lemma 10. If $P_t$ is essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^{t-1}$, we can show the following result based on a similar argument as in Sections B.3: there exists a bounded, measurable, nonempty and compact valued correspondence $\Phi_t$ from $H_{t-1}$ to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \triangle(X_t)$ such that $\Phi_t$ is essentially sectionally upper hemicontinuous on $X^{t-1} \times \hat{S}^{t-1}$, and for $\lambda^{t-1}$-almost all $h_{t-1} \in H_{t-1}$, $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if

1. $v = p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x_{tj}^*)$ such that $p_t(h_{t-1}, \cdot)$ is a measurable selection of $P_t(h_{t-1}, \cdot)$;
2. $x_{tj}^* \in A_{tj}(h_{t-1})$ is a maximization point of player $j$ given the payoff function $p_{tj}(h_{t-1}, A_{t(-j)}(h_{t-1}), \cdot)$ and the action space $A_{tj}(h_{t-1})$, $\alpha_i = \delta_{A_{ti}(h_{t-1})}$ for $i \neq j$ and $\alpha_j = \delta_{x_{tj}^*}$;
3. $\mu = \delta_{p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x_{tj}^*)}$.

Next we consider the case that $N_t = 0$. Suppose that the correspondence $Q_{t+1}$ from $H_t$ to $\mathbb{R}^n$ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^t$. For any $(h_{t-1}, x_t, \hat{s}_t) \in \mathrm{Gr}(\hat{A}_t)$, let

$$\begin{aligned} R_t(h_{t-1}, x_t, \hat{s}_t) &= \int_{\tilde{S}_t} Q_{t+1}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t) \tilde{f}_{t0}(\mathrm{d}\tilde{s}_t | h_{t-1}, x_t, \hat{s}_t) \\ &= \int_{\tilde{S}_t} Q_{t+1}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t) \varphi_{t0}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t) \lambda_t(\mathrm{d}\tilde{s}_t). \end{aligned}$$

Then following the same argument as in Subsection B.4.1, one can show that $R_t$ is a nonempty, convex and compact valued, and essentially sectionally upper hemicontinuous correspondence on $X^t \times \hat{S}^t$.

For any $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$, let

$$P_t(h_{t-1}, x_t) = \int_{\hat{A}_{t0}(h_{t-1}, x_t)} R_t(h_{t-1}, x_t, \hat{s}_t) \hat{f}_{t0}(\mathrm{d}\hat{s}_t | h_{t-1}, x_t).$$

By Lemma 7, $P_t$ is nonempty, convex and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^{t-1}$. The rest of the step remains the same as in Subsection B.4.1.

2. Forward induction: unchanged.

3. Infinite horizon: we need to slightly modify the definition of $\Xi_t^{m_1}$ for any $m_1 \geq t \geq 1$. Fix any $t \geq 1$. Define a correspondence $\Xi_t^t$ as follows: in the subgame $h_{t-1}$,

$$\Xi_t^t(h_{t-1}) = (\mathcal{M}(A_t(h_{t-1})) \diamond \hat{f}_{t0}(h_{t-1}, \cdot)) \otimes \lambda_t.$$

For any $m_1 > t$, suppose that the correspondence $\Xi_t^{m_1-1}$ has been defined. Then we can define a correspondence $\Xi_t^{m_1} \colon H_{t-1} \to \mathcal{M}\left(\prod_{t \leq m \leq m_1}(X_m \times S_m)\right)$ as follows:

$$\begin{aligned}\Xi_t^{m_1}(h_{t-1}) = &\Big\{ g(h_{t-1}) \diamond \Big( (\xi_{m_1}(h_{t-1}, \cdot) \diamond \hat{f}_{m_1 0}(h_{t-1}, \cdot)) \otimes \lambda_{m_1} \Big) : \\ &g \text{ is a Borel measurable selection of } \Xi_t^{m_1-1}, \\ &\xi_{m_1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_1}) \Big\}.\end{aligned}$$

Then the result in Subsection B.4.3 is true with the above $\Xi_t^{m_1}$.

Consequently, a subgame-perfect equilibrium exists.

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