NUS RMI Working Paper Series – No. 2019-05

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22 July, 2019

NUS Risk Management Institute
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Non-Concave Utility Maximization without the Concavification Principle

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The problems of non-concave utility maximization appear in many areas of finance and economics, such as in behavior economics, incentive schemes, aspiration utility, and goal-reaching problems. Existing literature solves these problems using the concavification principle. We provide a framework for solving non-concave utility maximization problems, where the concavification principle may not hold and the utility functions can be discontinuous. In particular, we find that adding bounded portfolio constraints, which makes the concavification principle invalid, can significantly affect economic insights in the existing literature. Theoretically, we give a new definition of viscosity solution and show that a monotone, stable, and consistent finite difference scheme converges to the solution of the utility maximization problem.

Key words: Non-Concave Utility, Portfolio Constraints, Discontinuous Viscosity Solution

1. Introduction

Although in traditional utility maximization problems the objective functions are concave, there is also a large literature on non-concave utility maximization in economics and finance. Examples include the S-shaped utility function in behavior economics (e.g., Kahneman and Tversky (1979), Berkelaar, Kouwenberg and Post (2004), and Jin and Zhou (2008)), the goal-reaching problem (e.g. Browne (1999a, 2000) and Spivak and Cvitanić (1999)), delegated portfolio choices with non-concave compensation schemes (e.g. Carpenter (2000), Basak, Pavlova and Shapiro (2007), and He and Kou (2018)), and the aspiration utility maximization (e.g. Diecidue and van de Ven (2008) and Lee, Zapatero and Giga (2018)). Almost all models in the above literature rely on the concavification principle, namely, replacing a non-concave utility by its concave envelope, and thus reducing the non-concave utility maximization problem to a concave one.

In this paper we attempt to provide a general framework for solving non-concave utility maximization problems where the concavification principle may not hold. In particular, we find that adding bounded portfolio constraints, which makes the concavification principle invalid, can significantly affect economic insights in the existing literature. To achieve this, theoretically, we show that a monotone, stable, and consistent finite difference scheme converges to the solution of the utility maximization problem, via a new definition of viscosity solution.

1.1. Motivation

We begin our discussion by looking at Figure 1, which plots the portfolio weights from 6 models in the literature of nonconcave utility maximization. One can immediately see high leverage ratios, from 300% to 4000%, even infinite. This naturally leads us to consider borrowing constraints.

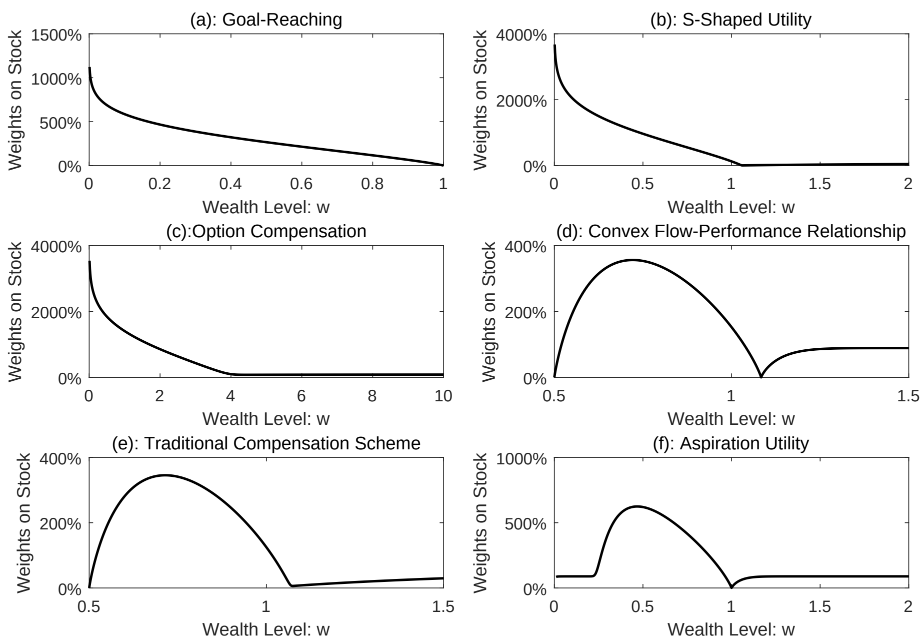

Figure 1 The time 0 unconstrained optimal fraction of total wealth invested in the stock against wealth level. It can be seen that the optimal fraction is likely very high or even infinite. The six sub-figures (a)-(f) correspond to the models of Browne (1999a), Berkelaar, Kouwenberg and Post (2004), Carpenter (2000), Basak, Pavlova and Shapiro (2007), He and Kou (2018), and Lee, Zapatero and Giga (2018), respectively. A compulsory liquidation at $w = 0.5$ is imposed for the models of Basak, Pavlova and Shapiro (2007) and He and Kou (2018).

However, surprisingly, when the borrowing constraints are imposed, the optimal strategies (as we will prove later) all involve short-selling to the highest extents. More precisely, in Figure 2 we study the same models as in Figure 1 using the same parameters, except that the constraint of $[-500\%, 300\%]$ (i.e. 300% borrowing constraint and 500% short-selling constraint) is imposed; the figure shows that the optimal strategies may borrow or short-sell as much as permitted, as the portfolio weights in all 6 sub-figures reach both limits of -500% and 300%. Indeed, if one relaxes the constraint to be $[-1000\%, 300\%]$, the optimal strategies for all 6 sub-figures reach both limits of -1000% and 300%. This should be contrasted with the optimal strategies without borrowing constraints, which do not involve short-selling at all.

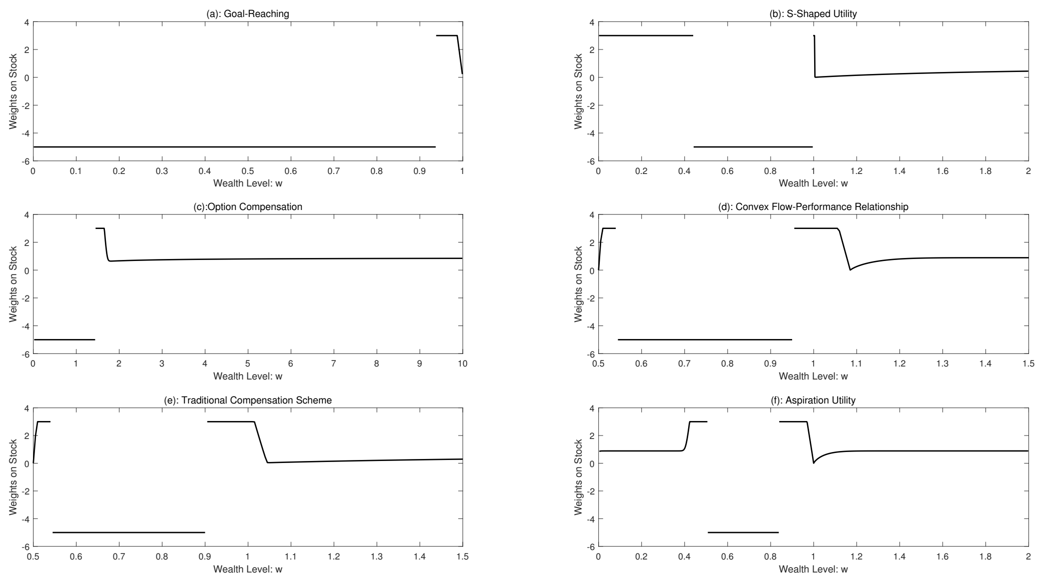

Figure 2 The time 0 optimal fraction of total wealth invested in the stock against wealth level under a less restricted portfolio constraint (300% borrowing and 500% short-selling constraints, i.e. $d = -5$ and $u = 3$). It can be seen that the optimal strategies may use as much leverage or short-sale as permitted. The six sub-figures (a)-(f) correspond to the models of Browne (1999a), Berkelaar, Kouwenberg and Post (2004), Carpenter (2000), Basak, Pavlova and Shapiro (2007), He and Kou (2018), and Lee, Zapatero and Giga (2018), respectively. A compulsory liquidation at $w = 0.5$ is imposed for the models of Basak, Pavlova and Shapiro (2007) and He and Kou (2018).

This motivates us to study two-side constraints and also to treat things carefully, as intuition can easily go wrong in the case of nonconcave utilities. Technically, the presence of the constraints makes the concavification principle invalid, because the value functions are not globally concave before maturity; see, e.g., Figures 3 and 6.

1.2. Our Contribution

The contribution of this paper is twofold.

- We find that adding bounded portfolio constraints, which makes the concavification principle invalid, can significantly affect economic and financial insights in the existing literature. See Table 1.

Table 1 Economic and Financial Insights

Effects of Portfolio Constraints : General Findings	
1. In general the concavification principle no longer holds, because the value function is not globally concave before maturity; see Figures 3 and 6.	
2. Investors may not be myopic with respective to portfolio constraints in the sense that they may act before portfolio constraints being binding; see the upper panel of Figures 5 and 8.	
3. Investors may gamble by short-selling (borrowing) a stock even with positive (negative) risk premium; see the lower panel of Figures 5 and 8.	
Effects of Portfolio Constraints : Model-Specific Findings	
Original Models	Effects of Portfolio Constraints Found in This Paper
Browne (1999a)	The optimal goal-reaching strategy is no longer equivalent to the replicating strategy of a digital option; see Figure 9.
Berkelaar, Kouwenberg and Post (2004)	Loss averse investors may further reduce stock investment; see Figure 10.
Carpenter (2000)	Convex incentives may reduce stock investment for wider wealth levels; see Figure 11.
Basak, Pavlova and Shapiro (2007)	The costs of misaligned incentives associated with delegated portfolio management are alleviated; see Table 4.
He and Kou (2018)	A larger performance fee is needed to make both fund managers' and investors' utilities better off when adopting the first-loss scheme in place of the traditional scheme; see Table 5.
Lee, Zapatero and Giga (2018)	A narrower set of skewness is generated; see Figure 12.

• Theoretically, in view of the difficulties due to the non-concave value functions, we attempt to do three things: (1) We introduce a new definition of viscosity solution; see Section 3.1. (2) Based on the new definition, we establish the comparison principle (Theorem 3.1), which is used to prove (in Theorem 3.2) that the value function of the non-concave utility maximization problem is the unique viscosity solution (in terms of the new definition) of the Hamilton-Jacobi-Bellman (HJB) equation. (3) We then show (in Theorem 3.3) that a monotone, stable, and consistent finite difference scheme converges to the solution of the utility maximization problem.

1.3. Literature Review

DeMiguel et al. (2009) provide a general framework for combining portfolio constraints and estimation for the mean variance investment problem, and show an excellent out-of-sample performance in the presence of estimation error. Karatzas et al. (1991) and Cvitanić and Karatzas (1992) develop a duality method for a concave utility optimization problem with convex portfolio constraints. In particular, for the power utility with portfolio constraints, the optimal investment strategy is myopic in the sense that no additional action would be made before the portfolio constraints are binding (see, e.g. Grossman and Vila (1992) and Vila and Zariphopoulou (1997)). We complement this stream of the existing literature by showing that in the case of non-concave utility maximization, the optimal investment strategy may no longer be myopic, in the sense that actions may be taken before portfolio constraints being binding; see the upper panel of Figures 5 and 8. It is worthwhile pointing out that an optimal investment strategy in incomplete market is often non-myopic with respective to portfolio constraints (see, e.g., Dai, Jin and Liu (2011) for CRRA utility maximization with transaction costs and Dai et al. (2018) for dynamic mean-variance analysis in a general incomplete market), but the non-myopic phenomenon arising from non-concave utility maximization occurs even in a complete market.

In terms of the economic and financial insights, we complement the existing literature of non-concave utility maximization, e.g. Basak, Pavlova and Shapiro (2007), Berkelaar, Kouwenberg and Post (2004), Browne (1999a), Carpenter (2000), He and Kou (2018), and Lee, Zapatero and Giga (2018), by adding constraints, which leads to different insights.

Theoretically, there are at least two methods to solve utility maximization for concave utility functions with portfolio constraints. One is the martingale duality method; see, e.g., Karatzas et al. (1991) and Cvitanić and Karatzas (1992). Using the martingale duality method, Haugh, Kogan and Wang (2006) derive an effective way to check whether a given control policy can lead to a good approximation to the optimal control policy. Due to the nonconcave utility functions, in general it is difficult to use the martingale duality method; one exception is Spivak and Cvitanić (1999), who study the goal problem without constraints via the martingale duality method. That is the main reason that we study the second method, which is based on partial differential equations.

Due to the presence of non-concavity, portfolio constraints, and possible discontinuities in the objective functions, the value function may be globally non-concave and singular at the terminal time. This poses a great challenge for the partial differential equation method. For example, it is not easy to obtain the comparison principle and the uniqueness of viscosity solution to the HJB equation, because the standard comparison principle for the HJB equations in Crandall, Ishii and Lions (1992), which guarantees the uniqueness of viscosity solution, requires the continuity of the viscosity solution.

Facing these challenges, we introduce a new definition of viscosity solution for discontinuous value functions. The new definition satisfies some asymptotic conditions at the terminal time; see (18) or (21). Then we show that comparison principle holds for the new definition of the viscosity solutions. Interestingly, the asymptotic behavior implies that if the time horizon is too short, investors may gamble by using as much leverage (borrowing or short-selling) as permitted when their target is yet to be reached.

The finite difference method has been widely used to solve the HJB equations arising from continuous time portfolio optimization problems (see, e.g., Barles and Souganidis (1991), Fleming

Table 2 Comparison between Our Paper and Related Theoretical Papers

Non-concave Portfolio Optimization with Portfolio Constraints	
Bian, Chen and Xu (2019)	One side portfolio constraints for which the concavification principle holds
This paper	General portfolio constraints for which the concavification principle may not hold
Viscosity Solutions	
Crandall, Ishii and Lions (1992)	Continuous viscosity solutions
This paper	Discontinuous viscosity solutions
Convergence of Finite Difference Schemes Arising from Finance	
Barles and Souganidies (1991), Forsyth and Labahn (2007), Wang and Forsyth (2008)	Continuous value functions
This paper	Value functions are allowed to be discontinuous

and Soner (2006), Forsyth and Labahn (2007), and Wang and Forsyth (2008)). Using the new definition of viscosity solution and the new comparison principle, we show that a monotone, stable, and consistent finite difference scheme is still applicable and convergent even for discontinuous utility with portfolio constraints. We then employ such a finite difference scheme to conduct an extensive numerical analysis.

Bian, Chen and Xu (2019) investigate the non-concave utility optimization problem with one-side portfolio constraints. They find that the concavification principle still holds and the standard comparison principle remains valid. We complement their results by studying the two-side portfolio constraints and discontinuous utility functions. In our study, the concavifiction principle may no longer hold, the viscosity solution may be singular, and the classical comparison principle may not hold. Table 2 offers a comparison of the current paper and the related literature on the theoretical side.

The remainder of the paper is organized as follows. Section 2 presents the model setup and several non-concave utility functions involved in the non-concave portfolio optimization problem. Section 3 is devoted to theoretical analysis. An extensive numerical analysis is given in Section 4. We conclude in Section 5. The new definition of the viscosity solution and a new numerical algorithm are presented in Appendix. All proofs and additional numerical results are relegated to E-Companion.

2. Model Formulation

2.1. The Financial Market

We consider a financial market that with one riskless bond with the constant risk-free rate r and one risky stock. The dynamic of the risky stock follows the geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t d\mathcal{B}_t, \tag{1}$$

where $\mathcal{B}_t$ is the standard Brownian motion, and the drift μ and volatility σ are assumed to be constant. Consider a self-financing portfolio strategy that invests Π_t dollars in the stock at time t. Then the value of the fund, $\tilde{W}_t, t \geq 0$, evolves according to

$$d\tilde{W}_t = r\tilde{W}_t dt + \Pi_t[(\mu - r)dt + \sigma d\mathcal{B}_t].$$

Following Goetzmann, Ingersoll and Ross (2003), Hodder and Jackwerth (2007), and He and Kou (2018), we add a liquidation constraint, that is, $\tilde{W}_t \geq Be^{-r(T-t)},\ 0 \leq t \leq T$, for some non-negative constant B.

For convenience, let $\pi_t := \Pi_t / \tilde{W}_t$ be the proportion of wealth invested in the stock, and let $W_t = \tilde{W}_t e^{r(T-t)},\ 0 \leq t \leq T$, be the (forward) wealth at time t. It follows that

$$dW_t = W_t \pi_t (\eta dt + \sigma d\mathcal{B}_t), \tag{2}$$

where $\eta = \mu - r$ is the excess rate of return, and the liquidation constraint is simplified as

$$W_t \geq B,\ 0 \leq t \leq T. \tag{3}$$

2.2. The Non-Concave Utility Optimization Problem

Let $U(\cdot)$ be a utility that an agent will receive at a finite horizon T. The utility function $U(\cdot)$ is not necessarily concave or continuous, but is assumed to satisfy the following assumption throughout the paper.

Assumption 2.1. The utility function $U(w),\ w \geq B$ is nondecreasing, right-continuous, bounded from below, and bounded above by a power utility w^p when w is sufficiently large for some $0 < p < 1$.

The agent attempts to choose an optimal portfolio strategy π to maximize the expected utility of the terminal wealth W_T, namely,

$$\sup_{d \leq \pi \leq u} \mathbb{E}[U(W_T)], \tag{4}$$

where the wealth W follows (2) subject to liquidation constraint (3). Here, d and u are the constant lower and upper portfolio constraints. Without loss of generality, we assume that $d \leq 0$ and $u \geq 0$

throughout the paper.[1] A finite upper bound $u \geq 1$ implies a borrowing constraint that limits the leverage ratio. In particular, $u = 1$ means that no borrowing is permitted. On the other side, a finite lower bound $d \leq 0$ implies the degree of short-sale constraints. In particular, $d = 0$ means that short-sale is prohibited.

2.3. Examples of Non-Concave Utility Functions

We will study some examples of non-concave utility functions in the existing literature.

2.3.1. A Discontinuous Utility of The Goal-Reaching Problem Many portfolio managers are interested in achieving some performance goal, such as beating a stock index. Browne (1999a) studies the optimal investment strategy of a fund manager who aims to maximize the probability of beating a benchmark by a given finite horizon, where the corresponding utility function is expressed as an indicator function

$$U(w) = 1_{w \geq 1}. \tag{5}$$

Here w refers to the value of the fund under management normalized by the benchmark. The utility function indicates that if the fund manager fails to reach the goal, then he will be indifferent to any outcomes incurred. Note that the utility function is discontinuous and non-concave.

2.3.2. The S-shaped Utility of Prospect Theory Kahneman and Tversky (1979) propose the following S-shaped utility function:

$$U(w) = \begin{cases} (w - W_0)^p & \text{for} \quad w > W_0 \\ -\lambda(W_0 - w)^p & \text{for} \quad w \leq W_0, \end{cases} \tag{6}$$

where W_0 is the reference point that distinguishes gains from losses, $0 < p < 1$ measures the degree of risk aversion over gains, and the loss aversion coefficient $\lambda > 1$ indicates that the pain from one dollar loss is higher than the pleasure from one dollar gain. Berkelaar, Kouwenberg and Post (2004) study the optimal investment strategy with the S-shaped utility function (6).

2.3.3. The Delegated Portfolio Choice with Convex Compensation Schemes Delegated fund managers are usually paid by convex compensation schemes, such as an option compensation, management fee proportional to the asset under management which is convex on the performance of the fund, or a performance fee linked to the profit which has limited liability.

Option Compensation. Carpenter (2000) considers a risk averse manager compensated with a call option. The manager's payoff at time T is α shares of call option over the fund that matures at T with strike K, plus a constant base C. Then, the payoff function f_{Car} is given by

$$f_{Car}(w) = \alpha \max\{w - K, 0\} + C, \tag{7}$$

[1] Unbounded portfolio constraints are covered by setting $d = -\infty$ and/or $u = \infty$.

where w is the terminal wealth level of the fund. Assume that the manager is risk averse over his terminal payoff, then the utility function of the manager can be given by

$$U(w) = (f_{Car}(w))^p/p, \tag{8}$$

where $0 < p < 1$. The convex structure of the option payoff makes the utility function U non-concave over the terminal wealth level of the fund.

Convex Flow-Performance Relationship. Basak, Pavlova and Shapiro (2007) study a portfolio choice model that the fund manager's compensation is proportional to the assets under management. The assets under management depend on the performance of the fund: in general, outperformance attracts cash inflow and underperformance leads to money redemptions. For example, the flow rate of the fund, f_{BPS}, can be specified as follows:

$$f_{BPS}(w) = \begin{cases} f_L & \text{for } \ln(w/W_0) < \eta_L \\ f_L + \psi(\ln(w/W_0) - \eta_L) & \text{for } \eta_L \leq \ln(w/W_0) < \eta_H \\ f_H & \text{for } \ln(w/W_0) \leq \eta_H, \end{cases} \tag{9}$$

where W_0 is the initial asset under management,[2] η_L, η_H are the lower and upper performance thresholds, f_L, f_H are the flow rate in case of bad and good performance, and $\psi = (f_H - f_L)/(\eta_H - \eta_L)$ such that the function $f_{BPS}(w)$ is continuous. If the manager is risk averse over the overall value of assets under management at time T, the utility function of the manager can be modelled by

$$U(w) = (w f_{BPS}(w))^p/p, \tag{10}$$

which is non-concave over w due to the convex flow-performance $f_{BPS}(w)$.

Convex Performance Fee Schemes. He and Kou (2018) study the optimal investment strategy of a fund manager with two kinds of performance fee schemes: the traditional scheme and the first-loss scheme. By assuming that a proportion γ of the fund belongs to the manager and the manager can charge a proportion α of the profit, the manager's net profit-or-loss function under the traditional scheme is given by

$$f_T(w) = \begin{cases} (\gamma + \alpha(1-\gamma))(w - W_0) & \text{for } w > W_0 \\ \gamma(w - W_0) & \text{for } 0 \leq w \leq W_0. \end{cases} \tag{11}$$

In contrast, under the first-loss scheme, the manager will firstly use his money in the fund to cover the loss. Thus, the net profit-or-loss structure is changed into

$$f_{FL}(w) = \begin{cases} (\gamma + \alpha(1-\gamma))(w - W_0) & \text{for } w > W_0 \\ w - W_0 & \text{for } (1-\gamma)W_0 \leq w \leq W_0 \\ -\gamma W_0 & \text{for } 0 \leq w \leq (1-\gamma)W_0. \end{cases} \tag{12}$$

[2] Strictly speaking, W_t represents the time t relative performance, i.e. the ratio of the fund value to the benchmark.

Assume the fund manager is risk averse over the profit and risk seeking over the loss, then the utility function of the fund manager is given by

$$U(w) = g(f_T(w)), \text{ or } U(w) = g(f_{FL}(w)), \tag{13}$$

where $g(\cdot)$ is an S-shaped function defined by

$$g(z) = \begin{cases} z^p & \text{for } z > 0 \\ -\lambda(-z)^p & \text{for } z \le 0, \end{cases}$$

with $0 < p < 1$. The non-concave feature of the utility function originates from both the S-shaped function and the performance fee scheme.

2.3.4. Aspiration Utility Lee, Zapatero and Giga (2018) analyze the demand for skewness that results from an aspiration utility similar to Diecidue and van de Ven (2008). In Lee, Zapatero and Giga (2018), the economic agent cares not only about the normal consumption but also about the status which is conveyed through the consumption of non-divisible good, such as a luxury car or a house. So, the utility of the agent will jump when his wealth reaches the level from which he can consume the non-divisible good. The utility function could be given by

$$U(w) = \begin{cases} u_1(w) := \frac{w^p}{p} & \text{if } w < R \\ u_2(w) := c_1\frac{w^p}{p} + c_2 & \text{if } w \ge R, \end{cases} \tag{14}$$

where w is the terminal wealth level, R is the aspiration level, $0 < p < 1$, and $c_1 > 1$ and $c_2 \ge 0$ such that $U(R-) < U(R)$. The utility function jumps at the aspiration level R, thus it is non-concave.

3. Theoretical Analysis

This section is devoted to theoretical analysis for the portfolio optimization problem (4). Denote by $V(t,w)$ the value function of the optimization problem (4) conditional on $W_t = w$. At the terminal time T, the value function equals the utility function by definition, i.e.

$$V(T,w) = U(w), \text{ for all } w \ge B. \tag{15}$$

When $W_t = B$ for some $t < T$, liquidation is necessary, which implies a boundary condition

$$V(t,B) = U(B), \text{ for all } t \le T. \tag{16}$$

The value function formally satisfies the following HJB equation

$$\frac{\partial V(t,w)}{\partial t} + \sup_{d \le \pi \le u}\left\{\frac{1}{2}\pi^2 w^2 \sigma^2 \frac{\partial^2 V(t,w)}{\partial w^2} + \pi w \eta \frac{\partial V(t,w)}{\partial w}\right\} = 0, \tag{17}$$

for $t < T$ and $w > B$; we will justify this rigorously by introducing a new definition of viscosity solution.

3.1. A New Definition of Viscosity Solution

Facing the challenges of non-concavity, portfolio constraints, and possible discontinuity together, we introduce a new definition of viscosity solution to the HJB equation in Appendix A, which adds a special treatment at the terminal time T. To understand the intuition behind the new definition, consider two cases.

(i) The portfolio set $[d,u]$ is bounded. In this case, we later show that the value function satisfies the following asymptotic property (see part (i) of Proposition EC.2.3):

$$\lim_{(t,\zeta)\to(T-,w)} V(t,\zeta) - U(w-) - 2\Phi\left(\frac{\min\{0,\log\zeta/w\}}{L\sigma\sqrt{T-t}}\right)(U(w)-U(w-)) = 0, \tag{18}$$

where $U(w-)$ is the left limit of U at w, $U(B-)=U(B)$, $L=\max\{u,-d\}$, and $\Phi(x)$ is the normal cumulative distribution function. When the utility function $U(\cdot)$ is continuous, the asymptotic condition (18) is simplified as

$$\lim_{(t,\zeta)\to(T-,w)} V(t,\zeta) = U(w) = V(T,w), \tag{19}$$

which implies the continuity of the value function at maturity. However, when the utility function $U(\cdot)$ is discontinuous, e.g., at some w_0, the value function has singularity at $(T-,w_0)$.

To elaborate the singularity, let us take the goal-reaching utility (5) as an example which is discontinuous at $w=1$. Then the asymptotic condition (18) reduces to

$$\lim_{(t,\zeta)\to(T-,1-)} V(t,\zeta) - 2\Phi\left(\frac{\log\zeta}{L\sigma\sqrt{T-t}}\right) = 0. \tag{20}$$

On the left hand side of (20), both terms have singularity at $(T-,1-)$ but their difference vanishes. In fact, the latter term proves to be the value function of an alternative goal-reaching problem that only concerns about the diffusion term of the dynamic process (cf. Lemma EC.2.1).

The intuition behind (20) is the following: if the goal is yet to be reached ($\zeta<1$) for a sufficiently short time to maturity, fund managers are inclined to *use as much leverage or short-selling as permitted to raise the likelihood of achieving the goal* (i.e. $\pi=u$ or $\pi=d$).

(ii) The portfolio set $[d,u]$ is unbounded. In this case, the value function converges to the concave envelope of the utility function, i.e.,

$$\lim_{(t,\zeta)\to(T-,w)} V(t,\zeta) = \hat{U}(w), \tag{21}$$

where $\hat{U}$ is the concave envelope of U; see part (ii) of Proposition EC.2.3 or Bian, Chen and Xu (2019). Thus, for the one-side constraints, the value function is concave and the concavification principle holds. The new definition of the viscosity solution is not needed for this case. Note that for the two-side constraints, the concavification principle may not hold.

3.2. Comparison Principle, Uniqueness of Viscosity Solution, and Convergence of a Numerical Algorithm

The singularity of the value function at the terminal time brings a great challenge on comparison principle and uniqueness of viscosity solution to the HJB equation. This is because the standard comparison principle, which guarantees the uniqueness of viscosity solution, requires the continuity of viscosity solution. The following theorem shows that a new comparison principle holds for the new definition of the viscosity solutions, i.e. viscosity solutions satisfying the asymptotic property (18) or (21).

THEOREM 3.1 **(Comparison Principle)**. *(i) Assume the portfolio set* $[d,u]$ *is bounded. Let* $\bar{v}$ *and* $\underline{v}$ *be a viscosity subsolution and supersolution of the HJB equation* (17)*, respectively, with the boundary condition* (16) *and the asymptotic condition* (18)*. Suppose* $\bar{v}$ *and* $\underline{v}$ *are bounded in absolute value by* $C_1 w^p + C_2$*, for some* $0 < p < 1$*,* $C_1, C_2 > 0$*. Then* $\bar{v} \leq \underline{v}$ *for all* $w \geq B$ *and* $0 < t < T$*.*
(ii) Assume the portfolio set $[d,u]$ *is unbounded. Let* $\bar{v}$ *and* $\underline{v}$ *be a viscosity subsolution and supersolution of the HJB equation* (17)*, respectively, with the boundary condition* (16) *and the asymptotic condition* (21)*. Suppose* $\bar{v}$ *and* $\underline{v}$ *are bounded in absolute value by* $C_1 w^p + C_2$*, for some* $0 < p < 1$*,* $C_1, C_2 > 0$*. Then* $\bar{v} \leq \underline{v}$ *for all* $w \geq B$ *and* $0 < t < T$*.*

Based on the new comparison principle, we can prove the following theorem that the value function is the unique (new) viscosity solution of the HJB equation.

THEOREM 3.2 **(Uniqueness and a Link to the Value Function)**. *(i) When the portfolio set* $[d,u]$ *is bounded, the value function* V *of* (4) *is the unique viscosity solution of the HJB equation* (17) *with the boundary condition* (16) *and the asymptotic condition* (18)*. Besides,* V *is continuous, except at* (T,w) *where* w *denotes discontinuous point of* $U(\cdot)$*.*
(ii) When the portfolio set $[d,u]$ *is unbounded, the value function* V *of* (4) *is the unique viscosity solution of the HJB equation* (17) *with the boundary condition* (16) *and the asymptotic condition* (21)*. Besides,* $V(t,w)$ *is continuous for* $t < T$*.*

Since analytical solutions are usually unavailable for non-concave portfolio optimization with portfolio constraints, we resort to the finite difference method to numerically solve for the value function and the optimal portfolios. The following theorem gives the convergence of monotone, stable, and consistent finite difference schemes.

THEOREM 3.3 **(Convergence of a Numerical Algorithm)**. *The discrete solution of a monotone, stable, and consistent finite difference scheme for the HJB equation* (17) *with the boundary condition* (16) *and the asymptotic condition* (18) *(or* (21)*) converges to the value function as the discretization size tends to zero.*

An example of the finite difference scheme described in the above theorem is a fully implicit finite difference scheme with upwind treatment for the first-order derivatives (see, e.g., IX.3.13 of Fleming and Soner (2006)). In our numerical experiments, according to Wang and Forsyth (2008), we employ a different finite difference scheme satisfying the monotonicity, stability, and consistency, where central differencing is used as much as possible to improve the accuracy while the monotonicity is guaranteed. The detail of the monotone scheme is presented in Appendix B.

The proof of Theorem 3.3 relies on the method of viscosity solution by Barles and Souganidis (1991), where the comparison principle for viscosity solutions, as given in Theorem 3.1, plays a critical role.

3.3. Additional Remarks

Theorem 3.2 characterizes the value function $V(t,w)$ as the unique viscosity solution of the HJB equation (17). The value function is continuous at the liquidation boundary $w=B$, that is,

$$\lim_{(t',\zeta)\to(t,B)} V(t',\zeta)=U(B)=V(t,B).$$

At the terminal time T, there are three cases.

(1). If the portfolio set $[d,u]$ is bounded and the utility function U is continuous, part (i) of Theorem 3.2 shows that the value function is continuous at the terminal condition, that is,

$$\lim_{(t,\zeta)\to(T-,w)} V(t,\zeta)=U(w)=V(T,w).$$

(2). If the portfolio set $[d,u]$ is unbounded, part (ii) of Theorem 3.2 shows[3] that the value function converges to the concave envelope of the utility function (see the right panel of Figures 4 and 7) and thus is in general discontinuous, that is,

$$\lim_{(t,\zeta)\to(T-,w)} V(t,\zeta)=\hat{U}(w)\neq U(w) \text{ in general.}$$

(3). If the portfolio set $[d,u]$ is bounded and the utility function U is discontinuous, part (i) of Theorem 3.2 shows that the value function is discontinuous at (T,w) where w denotes discontinuous point of $U(\cdot)$. Different from case (2), the left limit of the value function even does not exist (see the left panel of Figures 4 and 7), that is,

$$\lim_{(t,\zeta)\to(T-,w)} V(t,\zeta) \text{ does not exist,}$$

where w denotes discontinuous point of $U(\cdot)$.

Part (ii) of Theorem 3.1 and part (ii) of Theorem 3.2 are related to continuous viscosity solutions and have been obtained in existing literature (e.g., Bian, Chen and Xu (2019)). For completeness,

[3] See also Bian, Chen and Xu (2019).

we list them here. Note that a one-side portfolio constraint is unbounded. According to part (ii) of Theorem 3.2, the corresponding non-concave utility U can be replaced by its concave envelope $\hat{U}(w)$. Thus, the non-concave portfolio optimization problem with one-side portfolio constraint studied in Bian, Chen and Xu (2019) is essentially a concave one and is different from the one with two-side portfolio constraints (i.e. bounded portfolio set).

4. Numerical Analysis

In this section, we employ the numerical scheme in Appendix B to conduct an extensive numerical analysis. We will provide numerical evidences of the findings as given in Table 1.

4.1. General Findings

To demonstrate the general findings as given in Table 1, we employ the goal-reaching model and the aspiration utility maximization model, by incorporating bounded portfolio constraints. Both models have discontinuous utility functions. The cases without portfolio constraints have been studied by Browne (1999b) and Lee, Zapatero and Giga (2018), respectively. Numerical evidences for other models with portfolio constraints are similar and are given in E-Companion.

4.1.1. The Goal-Reaching Problem with Portfolio Constraints Take the goal-reaching problem as in (5). We first consider the no-borrowing and no-short-sale constraints, i.e., $[d,u] = [0,1]$. The default parameter values come from Browne (1999b): $\mu = 0.15$, $r = 0.07$, $\sigma = 0.3$, and $B = 0$ (no liquidation).

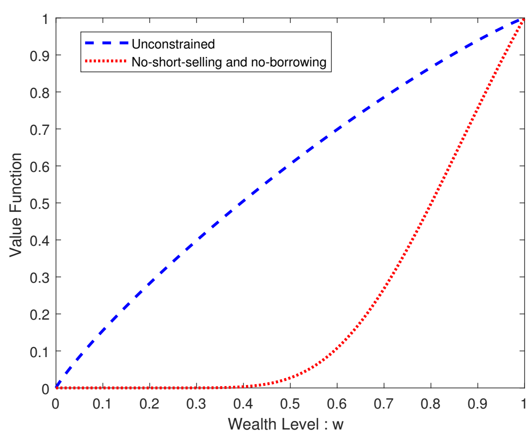

Figure 3 A comparison between the constrained value function and the unconstrained value function associated with the goal-reaching problem in Browne (1999a). It can be seen that the value function is globally concave in the unconstrained case but is not concave in the constrained case. The dotted (dashed) line represents the time 0 value function against wealth level in the constrained (no constrained) case. The parameters are: $r = 0.07$, $\mu = 0.15$, $\sigma = 0.3$, $W_0 = 1$, $T = 1$, $B = 0$, and $[d,u] = [0,1]$.

In Figure 3, we plot the time 0 value functions against wealth level for the constrained case (dotted line) and unconstrained case (dashed line), respectively. It can be seen that the value function is globally concave in the unconstrained case, but is not concave in the constrained case. This indicates that a non-concave optimization problem with (bounded) portfolio constraints cannot be reduced to a concave optimization problem in general. It is worthwhile pointing out that short selling is never optimal even in the unconstrained case (cf. Figure 1a). Hence, if only the no-short-selling constraint is imposed, the corresponding value function must be the same as the unconstrained value function that is concave, consistent with part (ii) of Theorem 3.2 (see also Bian, Chen and Xu (2019)).

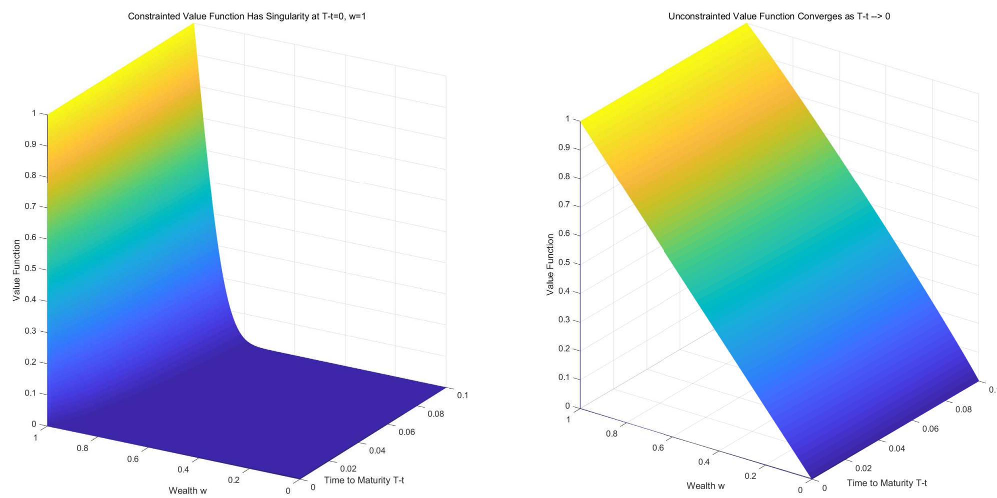

Figure 4　The constrained(left panel) and unconstrained (right panel) value functions associated with the goal-reaching problem in Browne (1999a). The constrained value function is discontinuous at $w=1$ and $t=T-$, while the unconstrained value function converges to the concave envelope of the utility in the goal-reaching problem as $t\to T-$. The parameters are: $r=0.07$, $\mu=0.15$, $\sigma=0.3$, $W_0=1$, $T=1$, $B=0$, $[d,u]=[0,1]$ and $L=\max\{-d,u\}=1$.

In Figure 4, we give a 3-D plot of the value function against the wealth and the time to maturity for the constrained case (left panel) and unconstrained case (right panel), respectively. It can be seen that the value function in the constrained case has a singularity at $w=1$ and $t=T-$, consistent with part (i) of Theorem 3.2 (see also the asymptotic condition (18) or (20)), while the unconstrained (or one-side portfolio constrained) value function converges to the concave envelope of the utility in the goal-reaching problem as $t\to T-$, consistent with part (ii) of Theorem 3.2 (see the asymptotic condition (21) or Bian, Chen and Xu (2019)).

In Figure 5, we plot the time 0 optimal fraction of total wealth invested in the stock π^* against wealth level for the constrained case (dotted line) and the Browne's unconstrained case (dashed

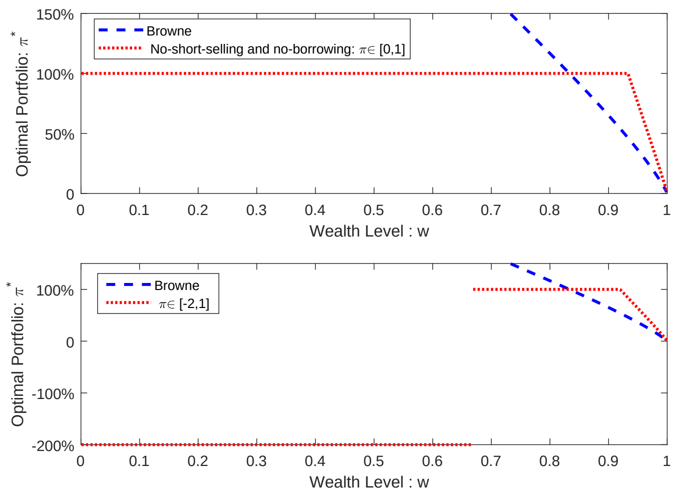

Figure 5 A comparison between the constrained strategy and the Browne's unconstrained strategy. The upper panel indicates that the constrained investors are non-myopic with respect to portfolio constraints, such that an early action is taken before portfolio constraints being binding. The lower panel indicates that given a relatively large loss, short-selling is likely optimal even with a positive risk premium, provided that a large short-selling ratio ($d=-2$) is permitted. The dotted line is the time 0 optimal fraction of total wealth invested in the stock π^* against wealth level for the constrained case, where the portfolio constraint is $\pi\in[0,1]$ (upper panel) and $\pi\in[-2,1]$ (lower panel), respectively. The dashed line stands for the Browne's unconstrained case (some part that exceeds the scope of the figure is not displayed). Parameter values: $r=0.07,\ \mu=0.15,\ \sigma=0.3,\ T=1$**, and** $B=0$**.**

line). The portfolio constraints are $\pi\in[0,1]$ for the upper panel and $\pi\in[-2,1]$ for the lower panel, respectively. For lower wealth level, the Browne's strategy requires large leverage ratio that exceeds the scope of the figure and is not displayed (see Figure 1a for a complete picture). Observe that our constrained optimal strategy is not myopic with respect to portfolio constraints. For example, the upper panel of Figure 5 shows that our constrained optimal portfolio (with portfolio constraint $\pi\in[0,1]$) takes more weight on the stock than the Browne strategy does when the wealth is close to the target level $w=1$. This is because fund managers who face portfolio constraints would like to raise risk exposure in advance to compensate for the potential binding of portfolio constraints.

The lower panel of Figure 5 presents a surprising result which is, however, consistent with the implication of (18): given the positive risk premium $\mu-r=0.08$, the constrained optimal strategy is to short sell the stock when the current wealth level is away from the target level $w=1$. This is because in this case a restricted short-sale is permitted, which induces fund managers to gamble by taking the largest short-selling ratio $\pi=-2$, even in the presence of positive risk premium, as the current wealth level is far below the target level $w=1$.

Table 3 presents the length of time needed for a strategy to beat the benchmark strategy (All Cash or All Stock) by 10% with a probability of 95% or 99% for the unconstrained case and the constrained case (no-short-selling and no-borrowing, i.e. $\pi\in[0,1]$), respectively. The "All Cash"

Time (in years) need to beat All Cash by 10% with high probability				
	Unconstrained Case	Constrained Case: $\pi \in [0,1]$		
Probability	Browne	Myopic Browne	Our Strategy	Kelly
95%	1.3	8.9	7.4	10.8
99%	13.8	26.9	22.6	49.9

Time (in years) need to beat All Stock by 10% with high probability				
	Unconstrained Case	Constrained Case: $\pi \in [0,1]$		
Probability	Browne	Myopic Browne	Our Strategy	Kelly
95%	86.3	201.5	131.8	693.5
99%	884.2	1193.1	943.6	3190.7

Table 3 The length of time needed to beat the benchmark strategy (All Cash or All Stock) by 10% with a probability of 95% or 99%. Note that the constrained optimal strategy outperforms both the Kelly's strategy and the myopic Browne's strategy. "All Cash" ("All Stock") means a strategy putting all money in the riskless asset (stock). "Browne" refers to the optimal strategy for the unconstrained case studied in Browne (1999a). "Our Strategy" refers to the optimal strategy for the constrained case (no-short-selling and no-borrowing, i.e. $\pi \in [0,1]$). "Myopic Browne" refers to the strategy that follows the "Browne" strategy before the constraints are binding. "Kelly" refers to the Kelly strategy, namely $\pi^* = (\mu - r)/\sigma^2 = 88.9\%$. Default parameter values: $r = 0.07,\ \mu = 0.15,\ \sigma = 0.3,\ T = 1,\ B = 0$.

("All Stock") strategy means a strategy putting all money in the riskless asset (stock). "Browne" refers to the Browne's optimal strategy for the unconstrained case studied in Browne (1999a). "Our Strategy" refers to the optimal strategy for the constrained case. "Myopic Browne" refers to the strategy that follows the Browne's strategy before the constraints are binding. "Kelly" refers to the Kelly's strategy, namely $\pi^* = (\mu - r)/\sigma^2 = 88.9\%$. Note that the Kelly's strategy does not incur short-selling or borrowing for the given parameter values.

From Table 3, we can see that for the unconstrained case, the Browne's strategy is better than the Kelly's strategy. However, as showed in Figure 1a, the Browne's strategy may incur an unlimited leverage ratio. Under the no-borrowing and no-short-selling constraint, our constrained optimal strategy outperforms both the Kelly's strategy and the myopic Browne's strategy. For example, to beat the All Cash benchmark by 10% with a 95% probability, our strategy needs 7.4 years, compared to 10.8 years needed by the Kelly's strategy and 8.9 years by the myopic Browne's strategy. This is consistent with our previous result that the constrained optimal strategy is not myopic.

4.1.2. The Aspiration Utility Maximization with Portfolio Constraints Now we study how portfolio constraints affect the portfolio choice under the aspiration utility given in (14). The unconstrained case with a discrete-time setting has been discussed in Lee, Zapatero and Giga (2018). The default parameter values are set as following: $c_1 = 1.2$, $c_2 = 0$, $p = 0.5$, $R = 1$, $\mu = 0.07$, $r = 0.03$, $\sigma = 0.3$, $W_0 = 1$ and $B = 0$.

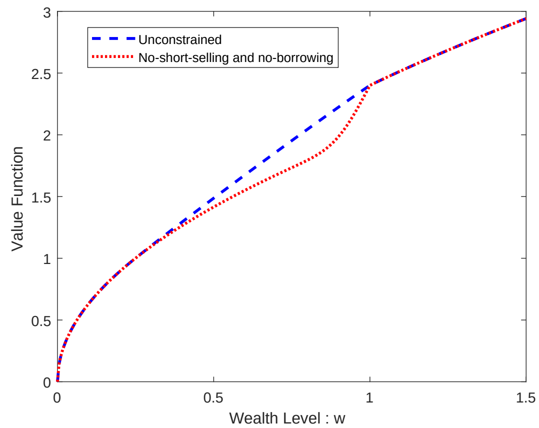

Figure 6 A comparison between our constrained value function and the unconstrained value function associated with the portfolio optimization problem in Lee, Zapatero and Giga (2018). Observe that the value function is globally concave in the unconstrained case but is not concave in the constrained case. The dotted line is the time 0 value functions with portfolio constraints against wealth level, where the portfolio constraint is $\pi \in [0,1]$. The dashed line stands for the value function without portfolio constraints. The parameters are: $c_1 = 1.2$, $c_2 = 0$, $p = 0.5$, $R = 1$, $\mu = 0.07$, $r = 0.03$, $\sigma = 0.3$, $W_0 = 1$ and $B = 0$.

In Figure 6, we plot the time 0 value function against wealth level for the constrained case (dotted line) and unconstrained case (dashed line), respectively. As in the goal-reaching problem, the value function is globally concave in the unconstrained case but is not concave in the constrained case.

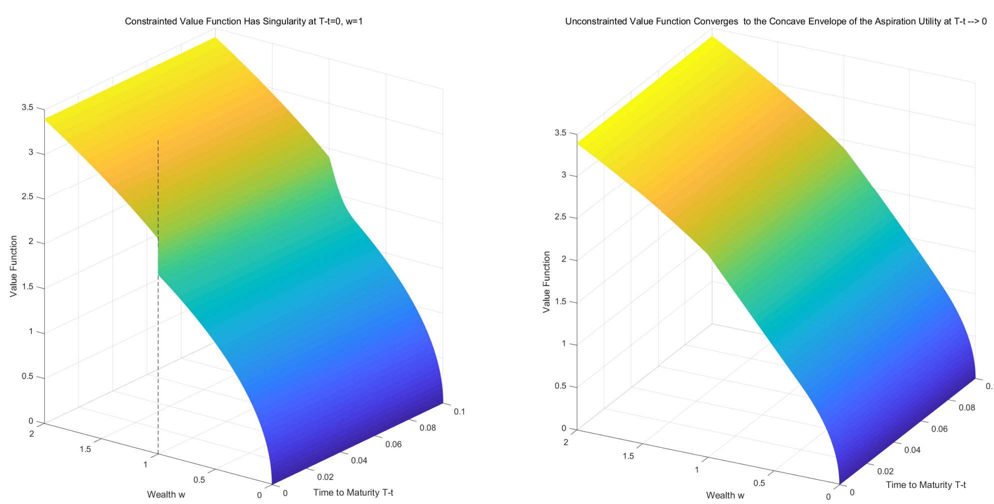

Figure 7 The constrained(left panel) and unconstrained (right panel) value functions associated with the non-concave utility optimization problem discussed in Lee, Zapatero and Giga (2018). The constrained value function is discontinuous at $w = R-$ and $t = T-$, while the unconstrained value function converges to the concave envelope of the aspiration utility at $t \to T-$. The parameter values are: $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $p = 0.5$, $c_1 = 1.2$, $c_2 = 0$, $R = 1$, $B = 0$, $T = 1/12$, $W_0 = 1$, $[d,u] = [0,1]$ and $L = \max\{-d,u\} = 1$.

In Figure 7, for the non-concave utility optimization problem discussed in Lee, Zapatero and Giga (2018) where the utility is discontinuous at the aspiration level R, we give a 3-D plot of the value functions against the wealth and the time to maturity for the constrained case (left panel) and unconstrained case (right panel), respectively. As in the goal-reaching problem, it can be seen that the constrained value function has a singularity at the aspiration level $w = R-$ and the terminal time $t = T-$, consistent with part (i) of Theorem 3.2 (see also the asymptotic condition (18)). As a comparison, the unconstrained (or one-side portfolio constrained) value function converges to the concave envelope of the aspiration utility at $t \to T-$, consistent with part (ii) of Theorem 3.2 (see the asymptotic condition (21) or Bian, Chen and Xu (2019)).

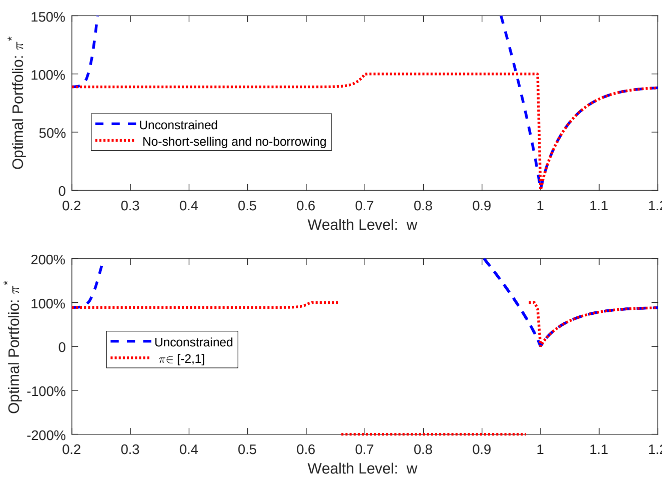

Figure 8 A comparison between our constrained strategy and the unconstrained strategy for the non-concave utility optimization problem discussed in Lee, Zapatero and Giga (2018). The upper panel indicates that the constrained investors are non-myopic with respect to portfolio constraints such that an early action is made before portfolio constraints being binding. The lower panel indicates that given a relatively large loss, short-selling is likely optimal even with a positive risk premium, provided that a large short-selling ratio ($d = -2$) is permitted. The dotted (dashed) line is the time 0 optimal fraction of total wealth invested in the stock π^* against wealth level for the constrained (unconstrained) case. The portfolio constraints in the constrained case are $\pi \in [0,1]$ (upper panel) and $\pi \in [-2,1]$ (lower panel), respectively. The parameter values are: $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $p = 0.5$, $c_1 = 1.2$, $c_2 = 0$, $R = 1$, $B = 0$, $T = 1/12$ and $W_0 = 1$.

In Figure 8, we plot the time 0 optimal fraction of total wealth invested in the stock π^* against wealth level for the constrained case (dotted line) and the unconstrained case (dashed line), respectively. The portfolio constraints are $\pi \in [0,1]$ in the upper panel and $\pi \in [-2,1]$ in the lower panel, respectively. Since the utility is risk averse for extremely low or high wealth levels, the fraction of total wealth invested in the stock tends to the Merton's line as the wealth level approaches infinity or zero. It can be observed that similar to Figure 5, investors are non-myopic with respect to portfolio constraints such that an early action is made before portfolio constraints being binding.

Similar to the lower panel of Figure 5, the lower panel of Figure 8 reveals that as the wealth level moves far away from the aspiration level, short-selling is likely optimal even with a positive risk premium, provided that a large short-selling ratio ($d = -2$) is permitted. The intuition is the same as before: a large short-selling ratio induces investors to gamble.

4.2. Model-Specific Findings

Now we provide numerical evidences for those model-specific findings as given in Table 1.

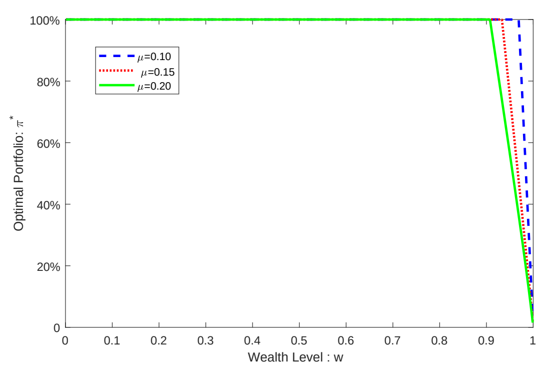

Figure 9 The time 0 optimal fraction of total wealth invested in the stock π^* against wealth level for different stock return μ. Observe that the constrained goal-reaching strategy depends on μ. Thus, it is no longer equivalent to the replicating strategy of some digital option. Borrowing and short-selling are not permitted, i.e. $u = 1$ and $d = 0$. Default parameter values: $r = 0.07, \mu = 0.15, \sigma = 0.3, T = 1, B = 0$.

4.2.1. The Goal-Reaching Problem with Portfolio Constraints Without portfolio constraints, Browne (1999a) proves the equivalence between the unconstrained Browne's strategy and the replication strategy of a specific digital option under the Black-Scholes market.[4] As a result, the unconstrained Browne's strategy must be independent with μ, the drift of the stock, when there is only one stock. However, the constrained optimal strategy usually depends on μ, as revealed by Figure 9. Thus, it is no longer equivalent to the replicating strategy of some digital option.

4.2.2. The S-Shaped Utilities with Portfolio Constraints Now we study how portfolio constraints affect the portfolio choice of a behavioral investor with the S-shaped utility given in (6). The unconstrained case has been discussed in Berkelaar, Kouwenberg and Post (2004). The default parameter values are set as following: $p = 0.5$, $\lambda = 2.25$, $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $W_0 = 1$, $B = 0.5$, and $[d, u] = [0, 1]$.

[4] The result holds for a general complete market setting, see, e.g. the Neyman-Pearson lemma approach of Föllmer and Leukert (1999), the martingale approach of Spivak and Cvitanić (1999) and the quantile approach of He and Zhou (2011).

Assuming no portfolio constraints, Berkelaar, Kouwenberg and Post (2004) show that compared to the Merton strategy, a loss averse investor considerably reduces the weight on stocks around the reference point $W_0 = 1$, which offers an explanation of the equity premium puzzle since the wealth levels of most investors are around $W_0 = 1$. This is confirmed by Figure 10, where the portfolio weight of the unconstrained strategy is below the Merton's line for wealth level around $W_0 = 1$. Comparing to the unconstrained strategy, Figure 10 shows that the constrained optimal strategy may further reduce weights on stocks around the reference point $W_0 = 1$. The intuition is that the constrained investor is more loss averse since a potential unlimited risk-seeking strategy in the loss region is prohibited by portfolio constraints.

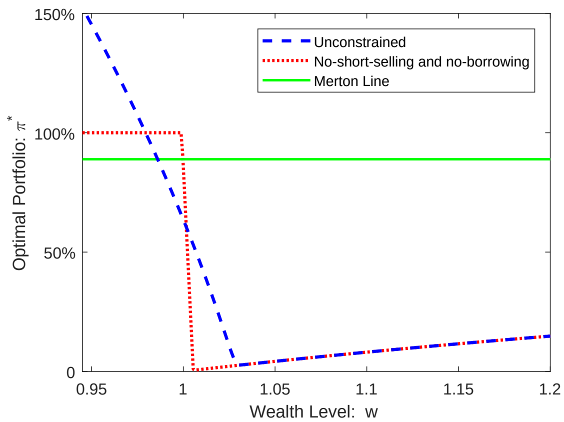

Figure 10 A comparison among the time 0 Merton's strategy (solid line), our constrained strategy (dotted line), and the unconstrained strategy (dashed line) for the non-concave portfolio optimization problem discussed in Berkelaar, Kouwenberg and Post (2004). Observe that around the wealth level $w = 1$, the unconstrained portfolio is below the Merton line, and the constrained strategy may further reduce the weights on stock. Default parameters values are $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $p = 0.5$, $\lambda = 2.25$, $W_0 = 1$, $T = 1/12$, $B = 0.5$, $\pi \in [0,1]$ and Merton line $\pi^* = (\mu - r)/((1-p)\sigma^2) = 88.9\%$.

4.2.3. The Delegated Portfolio Choice with Portfolio Constraints

Option Compensation

Now we study how portfolio constraints affect the risk incentive of a risk averse manager compensated with a call option (7). The default parameter values are set as following: $K = 1$, $\alpha = 0.2$, $C = 0.02$, $p = 0.5$, $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $W_0 = 1$, $T = 1$, $B = 0.5$, and $\pi \in [0,1]$.

In Figure 11, we plot the time 0 optimal fraction of total wealth invested in the stock π^* against wealth level for the Merton's strategy (solid line), our constrained strategy (dotted line), and the unconstrained strategy (dashed line), respectively. When the option is out of the money, the unconstrained strategy requires a large leverage ratio that exceeds the scope of the figure. Assuming no portfolio constraints, Carpenter (2000) shows that the option compensation may lead to less

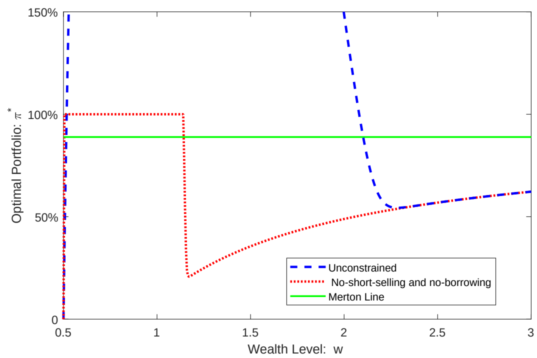

Figure 11 A comparison among the time 0 Merton's strategy (solid line), our constrained strategy (dotted line), and the unconstrained strategy (dashed line) for the delegated portfolio optimization with the option compensation scheme discussed in Carpenter (2000). Observe that the unconstrained portfolio is below the Merton line when the option is deeply in the money (e.g. $w > 2$), and the constrained portfolio is below the Merton line for a wider wealth levels (e.g. $w > 1.2$). The parameters are: $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $p = 0.5$, $K = 1$, $\alpha = 0.2$, $C = 0.02$, $W_0 = 1$, $T = 1$, $T - t = 1/12$, $B = 0.5$, $\pi \in [0, 1]$, and Merton line $\pi^* = (\mu - r)/\sigma^2/(1 - p) = 88.9\%$.

risk taking, compared with the Merton stategy. This is confirmed by Figure 11, where the portfolio weight of the unconstrained strategy is below the Merton's line when the option is deeply in the money (e.g. $w > 2$). Interestingly, Figure 11 shows that for the constrained optimal strategy, the risk reduction induced by the option compensation becomes more significant in a larger range (e.g. $w > 1.2$). This is because the fear that the option ends up out of the money leads to risk averse, while portfolio constraints further reduce risk-seeking.

Convex Flow-Performance Relationship

Now we incorporate portfolio constraints into the delegated portfolio optimization with the convex flow-performance relationship and study the cost to investors induced by fund managers' convex incentive. The default parameter values for the convex flow-performance relationship come from Basak, Pavlova and Shapiro (2007): $\eta_L = -0.08$, $\eta_H = 0.08$, $f_L = 0.8$, and $f_H = 1.5$.

We examine how portfolio constraints affect the cost of delegation, which is the utility loss to the investor due to the manager's deviating from the investor's optimal policy. Basak, Pavlova and Shapiro (2007) point out that there are two types of delegation cost: the first derives from the fact that the manager's attitude towards risk could be different from the investor's; the second derives from the incentive induced by the convex flow-performance relationship. We only consider the second type of delegation cost.

Assume that the investor is equipped with a power utility with the same risk-averse coefficient as that of the manager. Then, the investor's direct optimal policy is to follow the Merton's strategy. The cost of delegation, λ, is defined by

$$V_I(0, W_0) = \tilde{V}_I(0, (1 + \lambda)W_0), \tag{22}$$

where $V_I(0, W_0)$ is the value function of the investor with initial fund value W_0 at time 0 under the manager's optimal policy accounting for the local convex incentives, and $\tilde{V}_I(0, (1+\lambda)W_0)$ is the value function of the investor with initial fund value $(1+\lambda)W_0$ at time 0 under the Merton's policy.

Table 4 exhibits the cost of delegation with and without portfolio constraints for different sets of parameter values (e.g. time to maturity, Sharpe ratio). It can be seen that the portfolio constraints reduce the cost of delegation, and the effect is significant when the time to maturity is short. Intuitively, this is because portfolio constraints effectively prohibit an unlimited leverage that would be otherwise adopted by the fund manager as the deadline approaches.

Time to Maturity T	Cost of Delegation λ (%)	
	Unconstrained Case	Constrained Case: $\pi \in [0,1]$
Stock with Lower Sharpe Ratio		
1/12	-0.95	-0.03
1/2	-0.91	-0.47
1	-1.27	-1.11
Stock with Higher Sharpe Ratio		
1/12	-1.56	-0.04
1/2	-3.42	-1.46
1	-3.26	-3.21

Table 4 A comparison of the cost of delegation between our constrained strategy and the unconstrained strategy of Basak, Pavlova and Shapiro (2007). Observe that with portfolio constraints, the costs of misaligned incentives associated with delegated portfolio management are alleviated. The first column is the time to maturity. The second and the third columns are the cost of delegation for the cases that the manager faces no portfolio constraint and the bounded portfolio constraint ($\pi \in [0,1]$), respectively. The parameter values for the upper panel are: $\mu - r = 0.04$, $\sigma = 0.3$, $p = 0.5$, and the parameter values for the lower panel are: $\mu - r = 0.08$, $\sigma = 0.2$, $p = 0.6$. The other common parameter values are: $\eta_L = -0.08$, $\eta_H = 0.08$, $f_L = 0.8$, $f_H = 1.5$, $B = 0.5$, and $W_0 = 1$.

Convex Performance Fee Schemes

In Table 5, we show the effect of portfolio constraints on the value functions (utilities) of the managers and the investors when the traditional scheme is replaced by a first-loss scheme. The third and fourth columns list the utilities of the managers and the investors under the managers' optimal unconstrained strategies and constrained strategies, respectively. In each parentheses, the first number is the utility of the managers and the second number is the utility of the investors. The '*' in the superscript means that the utility is improved when replacing the traditional scheme with the first-loss scheme. The third and fourth columns show that, without portfolio constraints, both the managers and the investors are better off when the traditional scheme (20% performance fee) is replaced by the first-loss scheme (30% performance fee). However, if the performance fee in the first-loss scheme is 40% or above, such substitution renders investors worse off. These results

		Utilities for Unconstrained Case		Utilities for Constrained Case	
Scheme	Performance Fee	Manager	Investor	Manager	Investor
Stock with Lower Sharpe Ratio					
Traditional	$\alpha = 0.2$	0.11	0.041	0.057	0.036
First-Loss	$\alpha = 0.3$	0.12^*	0.043^*	0.046	0.060^*
First-Loss	$\alpha = 0.4$	0.13^*	0.036	0.056	0.051^*
First-Loss	$\alpha = 0.5$	0.15^*	0.027	0.065^*	0.039^*
Stock with Higher Sharpe Ratio					
Traditional	$\alpha = 0.2$	0.14	0.038	0.056	0.023
First-Loss	$\alpha = 0.3$	0.15^*	0.043^*	0.031	0.060^*
First-Loss	$\alpha = 0.4$	0.18^*	0.032	0.040	0.055^*
First-Loss	$\alpha = 0.5$	0.21^*	0.020	0.050	0.047^*
First-Loss	$\alpha = 0.6$	0.24^*	0.005	0.060^*	0.035^*

Table 5 A comparison of the utilities improvement between our constrained strategy and the unconstrained strategy of He and Kou (2018). Observe that the constrained case requires higher performance fee than the unconstrained case to make both the utilities of the manager and investor better off when the traditional scheme is replaced with the first-loss schemes. The first column is the type of compensation scheme, which would be either the traditional scheme or the first-loss scheme. The second column is the performance fee. The third and fourth columns (the fifth and sixth columns) are the utilities of the manager and his investor under the manager's optimal unconstrained strategies (constrained strategies), respectively. The '*' in the superscript means that the utility is improved when replacing the traditional scheme with the first-loss scheme. The parameter values for the upper panel are: $\mu - r = 0.04$, $\sigma = 0.3$, $p = 0.5$, **and the parameter values for the lower panel are:** $\mu - r = 0.08$, $\sigma = 0.2$, $p = 0.6$. **The other common parameter values are:** $\gamma = 0.1$, $T = 1$, $B = 0.5$, $W_0 = 1$, $\pi \in [0,1]$.

are consistent with the general findings of He and Kou (2018). However, the fifth and sixth columns show that, with the portfolio constraint $\pi \in [0,1]$, the performance fee will be as high as 50% for the upper panel (60% for the lower panel) so that both the utilities of the managers and investors are better off when the traditional scheme is replaced with the first-loss schemes. We believe that it is again because portfolio constraints prohibit unlimited risk-taking such that the managers with the first-loss scheme demand a high performance fee as compensation.

4.2.4. The Aspiration Utilities with Portfolio Constraints Now we study the implication of the optimal constrained portfolio to the demand for the skewness under the aspiration utility given in (14).

In Figure 12, we plot the skewness of the optimal terminal wealth against the wealth level for the constrained strategy (dotted line) and the unconstrained strategy (dashed line). For the unconstrained case, the skewness is negative when the wealth level is slightly smaller than the aspiration level $R = 1$, and the skewness turns positive when the wealth level moves far away from the aspiration level, consistent with the finding of Lee, Zapatero and Giga (2018) that investors' demand for skewness is endogenous in the aspiration utility model.

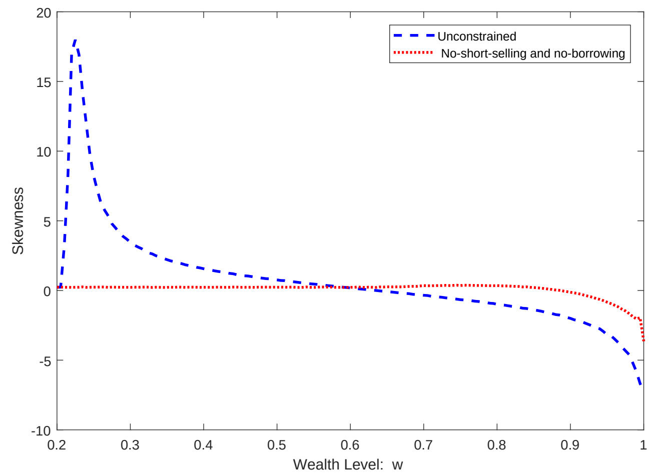

Figure 12 A comparison of the terminal wealth's skewness between our constrained strategy and the unconstrained strategy of Lee, Zapatero and Giga (2018). Observe that the demand for the positive skewness is quite weak for the constrained case. The dotted (dashed) line is the terminal wealth's skewness against the current wealth level for the constrained (unconstrained) optimal strategy. For the unconstrained case, the investor prefers negative (positive) skewness when his wealth level is close to (far away from) the aspiration level $R = 1$. The parameter values are $\mu = 0.07$, $r = 0.03$, $\sigma = 0.3$, $p = 0.5$, $c_1 = 1.2$, $c_2 = 0$, $R = 1$, $B = 0$, $W_0 = 1$, and $\pi \in [0, 1]$.

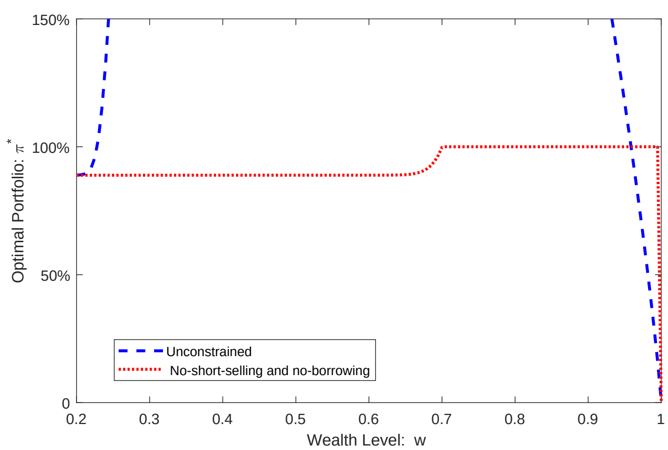

Figure 13 A comparison between our constrained strategy and the unconstrained strategy of Lee, Zapatero and Giga (2018). Observe that comparing to the unconstrained case, the weights on stock are smaller in the constrained case. The dotted line is the time 0 optimal fraction of total wealth invested in the stock π^* against the wealth level for our constrained optimal strategy, where the portfolio constraint is $\pi \in [0, 1]$. The dashed line stands for the optimal strategy of the unconstrained problem(some part that exceeds the scope of the figure is not displayed). The parameter values are: $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $p = 0.5$, $c_1 = 1.2$, $c_2 = 0$, $R = 1$, $B = 0$, $T = 1/12$ and $W_0 = 1$.

We can observe that with portfolio constraints the demand for the positive skewness is less. To investigate the reason, we plot in Figure 13 (see also the upper panel of Figure 8) the time 0 optimal fraction of total wealth invested in the stock π^* against the wealth level for the constrained case (dotted line) and the unconstrained case (dashed line), respectively. Observe that for the unconstrained case, the optimal strategy requires a high leverage as the wealth level is far away from the aspiration level $R=1$, which incurs a positive skewness. However, such a gambling strategy is prohibited by portfolio constraints so that the demand for the positive skewness is much less. Figure 12 shows that the constrained strategy may still induce a significant demand for negative skewness for the wealth level being close to the aspiration level $R=1$. This is because the investors only need a small gain to consume the status goods and thus considerably reduce the fraction of wealth invested in the stock.

5. Conclusion

We provide a general framework for non-concave portfolio optimization, where two-side portfolio constraints can be imposed and utility functions are allowed to be discontinuous. We show that portfolio constraints significantly affect investors' non-concave portfolio choice. More precisely, we find that in general (i) with two-side portfolio constraints, the concavification principle does not hold and the value function associated with non-concave portfolio optimization is generally non-concave; (2) investors are not myopic with respect to portfolio constraints, in the sense that they take actions in anticipation of portfolio constraints being potentially binding; and (3) a large short-selling or leverage ratio may induce investors to gamble in the case of underperformance.

Theoretically, we prove a comparison principle for discontinuous viscosity solutions associated with general non-concave portfolio optimization problems. Using the comparison principle, we show that a monotone, stable, and consistent finite difference scheme is still applicable and convergent for the general problems.

Appendix A: A New Definition of Viscosity Solution Used

Comparing with the standard definition of viscosity solution, for example Definition 7.4 of Crandall, Ishii and Lions (1992) or Definition 1.1 of Barles and Souganidis (1991), the new definition pays special attention to the asymptotic property (18).

To facilitate the presentation, define the lower semicontinuous envelope and upper semicontinuous envelope of the value function V as

$$V_*(t,w) = \liminf_{(t_1,w_1)\to(t,w)} V(t_1,w_1), \text{ and } V^*(t,w) = \limsup_{(t_1,w_1)\to(t,w)} V(t_1,w_1), \tag{23}$$

and define the Hamiltonian:

$$H(w,p,A) := \sup_{d\le\pi\le u}\{\frac{1}{2}\pi^2 w^2\sigma^2 A + \pi w\eta p\}, \quad w>0. \tag{24}$$

Then, we can rewrite the HJB equation (17) as

$$-\frac{\partial V(t,w)}{\partial t} - H\left(w, \frac{\partial V(t,w)}{\partial w}, \frac{\partial^2 V(t,w)}{\partial w^2}\right) = 0.. \tag{25}$$

Note that $H(w,p,A)$ is continuous except at $A=0$ at which it is likely lower semicontinuous for unbounded portfolio set $[d,u]$ [5]. So, we need to use its lower semicontinuousenvelope H_* and upper semicontinuous envelope H^* (cf. (23)) to define the viscosity supersolution and subsolution as following:

DEFINITION A.1. For $w \geq B$, let $K(w)=U(w)$ if the portfolio set $[d,u]$ is bounded, and $K(w)=\hat{U}(w)$ if the portfolio set $[d,u]$ is unbounded, where $\hat{U}$ is the concave envelope of U. Define $K(B-)=K(B)$ and $L=\max\{-d,u\}$. Let V be a locally bounded function.

(i). We say that V is a viscosity subsolution of the HJB equation (17) with the boundary condition (16) and the asymptotic property (18) (or (21) for unbounded portfolio set) if it satisfies the following conditions:

a) For all smooth ψ such that $V^* \leq \psi$ and $V^*(\bar{t},\bar{w})=\psi(\bar{t},\bar{w})$ for some $(\bar{t},\bar{w}) \in [0,T)\times(B,+\infty)$,

$$-\frac{\partial \psi(\bar{t},\bar{w})}{\partial t} - H^*\left(\bar{w}, \frac{\partial \psi(\bar{t},\bar{w})}{\partial w}, \frac{\partial^2 \psi(\bar{t},\bar{w})}{\partial w^2}\right) \leq 0. \tag{26}$$

b) For all $0 \leq t < T$,

$$V^*(t,B) \leq K(B). \tag{27}$$

c) For all $w \geq B$,

$$\limsup_{(t,\zeta)\to(T-,w)} V(t,\zeta) - K(w-) - 2\Phi\left(\frac{\min\{0,\log\zeta/w\}}{L\sigma\sqrt{T-t}}\right)(K(w)-K(w-)) \leq 0. \tag{28}$$

(ii). We say that V is a viscosity supersolution of the HJB equation (17) with the boundary condition (16) and the asymptotic condition (18) (or (21) for unbounded portfolio set) if it satisfies the following conditions:

a) For all smooth φ such that $V_* \geq \varphi$ and $V_*(\bar{t},\bar{w})=\varphi(\bar{t},\bar{w})$ for some $(\bar{t},\bar{w}) \in [0,T)\times(B,+\infty)$,

$$-\frac{\partial \varphi(\bar{t},\bar{w})}{\partial t} - H_*\left(\bar{w}, \frac{\partial \varphi(\bar{t},\bar{w})}{\partial w}, \frac{\partial^2 \varphi(\bar{t},\bar{w})}{\partial w^2}\right) \geq 0. \tag{29}$$

b) For all $0 \leq t < T$,

$$V_*(t,B) \geq K(B). \tag{30}$$

c) For all $w \geq B$,

$$\liminf_{(t,\zeta)\to(T-,w)} V(t,\zeta) - K(w-) - 2\Phi\left(\frac{\min\{0,\log\zeta/w\}}{L\sigma\sqrt{T-t}}\right)(K(w)-K(w-)) \geq 0. \tag{31}$$

(iii). We say that V is a viscosity solution if it is both a viscosity supersolution and subsolution.

Appendix B: The Finite Difference Scheme Used

Given a big enough upper bound $A > B$. Let $\Sigma_\Delta = \{(t_n,w_i): 0 \leq n \leq N_t, 0 \leq i \leq N_w\}$ be a discretization mesh of the domain $[0,T]\times[B,A]$ with fixed step size Δt and Δw. Let V_i^n be the solution at grid (t_n,w_i) of the following discretization version of the HJB equation (17):

$$-\frac{V_i^{n+1}-V_i^n}{\Delta t} - \sup_{d\leq\pi\leq u}\left\{\frac{\pi^2 w_i^2\sigma^2}{2}\frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{(\Delta w)^2} + \pi w_i \eta \frac{\Delta V_i^n(\pi)}{\Delta w}\right\} = 0, \tag{32}$$

[5] When the portfolio set $[d,u]$ is unbounded, the Hamiltonian H is continuous. When the portfolio set $[d,u]$ is unbounded, the Hamiltonian H is infinite for $A>0$, finite and continuous for $A<0$, and either finite or infinite at $A=0$; if it is finite at $A=0$, it is left continuous at $A=0$, and thus, it is lower-semicontinuous at $A=0$.

with the boundary and terminal conditions [6]

$$V_0^n = U(B), \quad V_{N_w}^n = U(A), \quad n = 0,1,2,\dots,N_t, \tag{33}$$

$$V_i^{N_t} = U(w_i), \quad i = 1,2\dots,N_w - 1. \tag{34}$$

The first order difference $\Delta V_i^n(\pi)$ is defined for $d \le \pi \le u$ as following:

$$\Delta V_i^n(\pi) = \begin{cases} (V_{i+1}^n - V_{i-1}^n)/2 & \text{if } \ |\pi| > \pi_i, \\ V_{i+1}^n - V_i^n & \text{if } \ |\pi| < \pi_i \text{ and } \pi\eta > 0, \\ V_i^n - V_{i-1}^n & \text{if } \ |\pi| < \pi_i \text{ and } \pi\eta < 0, \end{cases} \tag{35}$$

where $\pi_i = |\eta|\Delta w/(\sigma^2 w_i)$, and the difference method at $|\pi| = |\pi_i|$ is chosen such that the objective function of the optimization problem in (32) is upper semicontinuous. The difference form in (35) is to use maximal central differencing as that in Wang and Forsyth(2008). By Lemma EC.3.4, the scheme is monotone, consistent and stable.

(32) is a nonlinear equation which can be solved by the following iterative procedure: $V_i^{n,0} = V_i^{n+1}$, and given $V_i^{n,k}$, let

$$\pi_i^{n,k} = \arg\sup_{d\le\pi\le u} \left\{ \frac{\pi^2 w_i^2 \sigma^2}{2} \frac{V_{i+1}^{n,k} - 2V_i^{n,k} + V_{i-1}^{n,k}}{(\Delta w)^2} + \pi w_i \eta \frac{\Delta V_i^{n,k}(\pi)}{\Delta w} \right\},$$

where $\Delta V_i^{n,k}(\pi)$ is defined in (35) replacing all $\{n\}$ in superscript by $\{n,k\}$, and $V_i^{n,k+1}$ solves the following linear equations:

$$\frac{V_i^{n+1} - V_i^{n,k+1}}{\Delta t} + \frac{(\pi_i^{n,k})^2 w_i^2 \sigma^2}{2} \frac{V_{i+1}^{n,k+1} - 2V_i^{n,k+1} + V_{i-1}^{n,k+1}}{(\Delta w)^2} + \pi_i^{n,k} w_i \eta \frac{\Delta V_i^{n,k+1}(\pi_i^{n,k})}{\Delta w} = 0.$$

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[6] If the original portfolio set $[d,u]$ is unbounded, we can replace U by its concave envelope $\hat{U}$. And by Lemma EC.3.2, we can replace the unbounded portfolio constraint set $[d,u]$ by a bounded portfolio constraint set $[-C,C]\cap[d,u]$ for some large C.

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E-Companion to "Non-Concave Utility Maximization without the Concavification Principle"

Appendix EC.1: Proof of Theorem 3.1

Before proceeding to the proof of the theorem, we present the closed-form classic solutions for the constrained portfolio optimization problem under the power utility in the following lemma. The results are guaranteed by a direct verification and thus the proof is omitted.

LEMMA EC.1.1. *If the utility function $U(\cdot)$ in (4) is given by the power function:*

$$Q^{(q)}(T,w)=w^q/q, w>0, \text{ for some } \ q<1 \text{ and } \ q\neq 0, \tag{EC.1}$$

and there is no liquidation constraint, that is $B=0$, then, the optimal portfolio is given by

$$\pi_*=d\boldsymbol{1}_{\pi_0<d}+\pi_0\boldsymbol{1}_{d\le\pi_0\le u}+u\boldsymbol{1}_{\pi_0>u}, \text{ where } \ \pi_0:=\frac{\eta}{(1-q)\sigma^2}. \tag{EC.2}$$

Let $\Lambda=\sup_{d\le\pi\le u}\left\{\eta\pi-\frac{1-q}{2}\sigma^2\pi^2\right\}=\eta\pi_*-\frac{1-q}{2}\sigma^2\pi_*^2$. The value function is given by

$$Q^{(q)}(t,w)=\tilde{Q}^{(q)}(t)w^q/q, \quad w>0,\ 0\le t\le T, \tag{EC.3}$$

where

$$\tilde{Q}^{(q)}(t)=e^{q\Lambda(T-t)},\ 0\le t\le T. \tag{EC.4}$$

Proof of Theorem 3.1: We prove the theorem by contradiction. To help to derive a contradiction, set $\hat{u}=e^{\beta(t-T)}(\bar{v})^*$, $\hat{v}=e^{\beta(t-T)}(\underline{v})_*$, for some $\beta>0$, where $(\bar{v})^*$ is the upper semicontinuous envelope of $\bar{v}$ and $(\underline{v})_*$ is the lower semicontinuous envelope of $\underline{v}$ (cf. (23)). Then $\hat{u}$ ($\hat{v}$) is a subsolution (supersolution) to

$$-\frac{\partial V(t,w)}{\partial t}-H\left(w,\frac{\partial V(t,w)}{\partial w},\frac{\partial^2 V(t,w)}{\partial w^2}\right)+\beta V(t,w)=0. \tag{EC.5}$$

Assume on the contrary that

$$\hat{u}(\bar{t},\bar{w})-\hat{v}(\bar{t},\bar{w})=2\delta>0, \tag{EC.6}$$

for some $(\bar{t},\bar{w})\in(0,T)\times(B,\infty)$. We are proceeding to derive a contradiction to this assumption.

First, we show that, for each positive α, we can find an interior point of $(0,T)\times(B,\infty)\times(0,T)\times(B,\infty)$, named $(t_\alpha,w_\alpha,s_\alpha,\zeta_\alpha)$, which makes M_α of (EC.7) take its maximum. Let

$$M_\alpha(t,w,s,\zeta)=\hat{u}(t,w)-\hat{v}(s,\zeta)-\varphi(t,w,s,\zeta), \tag{EC.7}$$

where the test function φ is specified as

$$\varphi(t,w,s,\zeta) = \epsilon_3 G(t,w,s,\zeta) + \frac{\epsilon_1}{t} + \frac{\epsilon_2}{T-t} + \frac{\alpha}{2}((t-s)^2 + (w-\zeta)^2). \tag{EC.8}$$

Here, $G(t,w,s,\zeta) = Q^{(q)}(t,w) + Q^{(q)}(s,\zeta)$ and $Q^{(q)}(t,w)$ is given by (EC.3) in Lemma EC.1.1 for some q, such that $1 > q > p$, and ϵ_1, ϵ_2, and ϵ_3 are positive and sufficiently small such that (cf. (EC.6))

$$M_\alpha(\bar{t}, \bar{w}, \bar{t}, \bar{w}) > \delta. \tag{EC.9}$$

By the condition that $\hat{u}$ and $\hat{u}$ are controlled by $C_1 w^p + C_2$ for large w, they are bounded by $Q^{(q)}$ in (EC.3) for large w. Thus, M_α of (EC.7) can not take maximum as $w \to \infty$. Furthermore, it can not take take maximum at $t=0$ or $t=T$, thanks to the term $\frac{\epsilon_1}{t} + \frac{\epsilon_2}{T-t}$ in the test function φ in (EC.8). Besides, by (27) and (30) of Definition A.1, $\hat{u} \le \hat{v}$ at the liquidation boundary $w=B$. Thus, M_α takes maximum at the interior points of $(0,T)\times(B,\infty)\times(0,T)\times(B,\infty)$, and we can denote one of them as $(t_\alpha, w_\alpha, s_\alpha, \zeta_\alpha)$.

Second, we show that we can fix $\epsilon_2 > 0$ sufficiently small, s.t. for α sufficiently large we have

$$\frac{\epsilon_1}{t_\alpha^2} \ge \frac{\epsilon_2}{(T-t_\alpha)^2}. \tag{EC.10}$$

In order to prove that, for any ϵ_1, ϵ_2, ϵ_3, sending $\alpha \to \infty$, there exists a subsequence such that (see Proposition 3.7 in Crandall, Ishii and Lions (1992))

$$\lim_{\alpha\to+\infty} \alpha((t_\alpha - s_\alpha)^2 + (w_\alpha - \zeta_\alpha)^2) = 0, \tag{EC.11}$$

and both (t_α, w_α) and (s_α, ζ_α) converge to some interior point $(\hat{t}, \hat{w})$. The point depends on the choice of ϵ_1, ϵ_2, and ϵ_3. Let $(\hat{t}_0, \hat{w}_0)$ be a limit of $(\hat{t}, \hat{w})$ as $\epsilon_2 \to 0$. We assert that $\hat{t}_0 < T$. Then, (EC.10) can be done accordingly. Now, we show that $\hat{t}_0 < T$. If $\hat{t}_0 = T$, according to (28) and (31), we have

$$\begin{aligned}
&\limsup_{(\hat{t},\hat{w})\to(T-,\hat{w}_0)} (\hat{u}(\hat{t},\hat{w}) - \hat{v}(\hat{t},\hat{w})) \\
\le& \limsup_{(\hat{t},\hat{w})\to(T-,\hat{w}_0)} \hat{u}(\hat{t},\hat{w}) - K(\hat{w}_0-) - 2\Phi\left(\frac{\min\{0, \ln \hat{w}/\hat{w}_0\}}{L\sigma\sqrt{T-\hat{t}}}\right)(K(\hat{w}_0) - K(\hat{w}_0-)) \\
&- \liminf_{(\hat{t},\hat{w})\to(T-,\hat{w}_0)} \hat{v}(\hat{t},\hat{w}) - K(\hat{w}_0-) - 2\Phi\left(\frac{\min\{0, \ln \hat{w}/\hat{w}_0\}}{L\sigma\sqrt{T-\hat{t}}}\right)(K(\hat{w}_0) - K(\hat{w}_0-)) \\
\le& 0,
\end{aligned}$$

which contradicts the fact that, for each ϵ_2 and α (cf. (EC.7), (EC.8), (EC.9) and M_α takes maximum at $(t_\alpha, w_\alpha, s_\alpha, \zeta_\alpha)$),

$$\hat{u}(t_\alpha, w_\alpha) - \hat{v}(s_\alpha, \zeta_\alpha) \ge M_\alpha(t_\alpha, w_\alpha, s_\alpha, \zeta_\alpha) \ge M_\alpha(\bar{t}, \bar{w}, \bar{t}, \bar{w}) > \delta > 0. \tag{EC.12}$$

Thus, we have proved that $\hat{t}_0 < T$. Then we can choose ϵ_2 sufficiently small, s.t. $\hat{t} < \frac{T_0+\hat{t_0}}{2}$, and (EC.10) is satisfied for sufficiently large α.

Third, we apply the Ishii lemma (see Theorem 8.3 in Crandall, Ishii and Lions (1992)) to the maximum point $(t_\alpha, w_\alpha, s_\alpha, \zeta_\alpha)$ of M_α satisfying (EC.10) to derive a contradiction. For simplification, in the following, we use (t,w,s,ζ) to represent $(t_\alpha, w_\alpha, s_\alpha, \zeta_\alpha)$.

By Ishii's lemma, for any $\gamma > 0$, there exist M, N , s.t.

$$-\frac{\partial \varphi}{\partial t} - H^*\left(w, \frac{\partial \varphi}{\partial w}, M\right) + \beta \hat{u} \leq 0, \tag{EC.13}$$

$$\frac{\partial \varphi}{\partial s} - H_*\left(\zeta, -\frac{\partial \varphi}{\partial \zeta}, N\right) + \beta \hat{v} \geq 0. \tag{EC.14}$$

where φ is given in (EC.8), and

$$\begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \leq \nabla^2_{w,\zeta}\ \varphi + \gamma\left(\nabla^2_{w,\zeta}\ \varphi\right)^2 \tag{EC.15}$$

with

$$\nabla^2_{w,\zeta}\ \varphi = \begin{pmatrix} \frac{\partial^2 \varphi}{\partial w^2} & \frac{\partial^2 \varphi}{\partial w \partial \zeta} \\ \frac{\partial^2 \varphi}{\partial w \partial \zeta} & \frac{\partial^2 \varphi}{\partial \zeta^2} \end{pmatrix}.$$

By a direct calculation (cf. (EC.8)), we have

$$\frac{\partial \varphi}{\partial t} = -\frac{\epsilon_1}{t^2} + \frac{\epsilon_2}{(T-t)^2} + \alpha(t-s) + \epsilon_3 \frac{\partial G}{\partial t}, \qquad \frac{\partial \varphi}{\partial s} = -\alpha(t-s) + \epsilon_3 \frac{\partial G}{\partial s}, \tag{EC.16}$$

$$\frac{\partial \varphi}{\partial w} = \alpha(w-\zeta) + \epsilon_3 \frac{\partial G}{\partial w}, \qquad \frac{\partial \varphi}{\partial \zeta} = -\alpha(w-\zeta) + \epsilon_3 \frac{\partial G}{\partial \zeta}, \tag{EC.17}$$

and

$$\nabla^2_{w,\zeta}\ \varphi = \alpha \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \epsilon_3 \begin{pmatrix} \frac{\partial^2 G}{\partial w^2} & 0 \\ 0 & \frac{\partial^2 G}{\partial \zeta^2} \end{pmatrix}.$$

Then, by (EC.15), for any pair of x, y, we have

$$\begin{aligned} Mx^2 - Ny^2 &= (x,y) \begin{pmatrix} M, & 0 \\ 0, & -N \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &\leq (x,y) \left(\nabla^2_{w,\zeta}\ \varphi + \gamma(\nabla^2_{w,\zeta}\ \varphi)^2\right) \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \alpha(x-y)^2 + \epsilon_3 \left(\frac{\partial^2 G}{\partial w^2} x^2 + \frac{\partial^2 G}{\partial \zeta^2} y^2\right) + \gamma\,(x,y)\,(\nabla^2_{w,\zeta}\ \varphi)^2 \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

When α and ϵ_3 are fixed, (t,w,s,ζ) is always in a bounded area. Thus, we can choose γ small enough, such that

$$Mx^2 - Ny^2 = \alpha(x-y)^2 + \epsilon_3 \left(\frac{\partial^2 G}{\partial w^2} x^2 + \frac{\partial^2 G}{\partial \zeta^2} y^2\right) + o(1), \text{ for any } \ x, y. \tag{EC.18}$$

Recall (EC.13) and (EC.14). For the case that $[d,u]$ is bounded, we have $H^* = H = H_*$. For the case that $[d,u]$ is unbounded, $H_* = H$ and $H^* = H$ when $M < 0$. In (EC.18), we can take $x = y = c > 0$, that is,

$$(M-N)c^2 = \epsilon_3\left(\frac{\partial^2 G}{\partial w^2}c^2 + \frac{\partial^2 G}{\partial \zeta^2}c^2\right) + o(1) = -\epsilon_3 q(1-q)\tilde{Q}^{(q)}(t)\left(w^{q-2}c^2 + \zeta^{q-2}c^2\right) + o(1) < 0,$$

where $\tilde{Q}^{(q)}(t)$ is given by (EC.4) in Lemma EC.1.1 and is positive. Then, $M < N$. Furthermore, by (EC.14) and (24), we have $N \le 0$, thus $M < 0$. So, in (EC.13) and (EC.14), we always have $H^* = H = H_*$.

Let $x = \pi w\sigma$ and $y = \pi\zeta\sigma$. Then, by (EC.13) and (EC.14), we have

$$\begin{aligned}0 \le & \frac{\partial \varphi}{\partial t} + H\left(w, \frac{\partial \varphi}{\partial w}, M\right) - \beta\hat{u} + \frac{\partial \varphi}{\partial s} - H\left(\zeta, -\frac{\partial \varphi}{\partial \zeta}, N\right) + \beta\hat{v}\\ = & \frac{\partial \varphi}{\partial s} + \frac{\partial \varphi}{\partial t} + \sup_{d\le\pi\le u}\left\{\pi w\eta\frac{\partial \varphi}{\partial w} + \frac{1}{2}x^2 M\right\} - \sup_{d\le\pi\le u}\left\{\pi\zeta\eta(-\frac{\partial \varphi}{\partial \zeta}) + \frac{1}{2}y^2 N\right\} + \beta(\hat{v}-\hat{u}).\end{aligned}$$

Plugging in the derivatives (EC.16) and (EC.17), by (EC.10), we have

$$\begin{aligned}0 \le & -\frac{\epsilon_1}{t^2} + \frac{\epsilon_2}{(T-t)^2} + \epsilon_3\frac{\partial G}{\partial s} + \epsilon_3\frac{\partial G}{\partial t} + \sup_{d\le\pi\le u}\left\{\pi w\eta(\alpha(w-\zeta) + \epsilon_3\frac{\partial G}{\partial w})) + \frac{1}{2}Mx^2\right\}\\ & - \sup_{d\le\pi\le u}\left\{\pi\zeta\eta(\alpha(w-\zeta) - \epsilon_3\frac{\partial G}{\partial \zeta}) + \frac{1}{2}Ny^2\right\} + \beta(\hat{v}-\hat{u})\\ \le & \sup_{d\le\pi\le u}\left\{\pi\eta\alpha(w-\zeta)^2 + \epsilon_3\pi w\eta\frac{\partial G}{\partial w} + \epsilon_3\pi\zeta\eta\frac{\partial G}{\partial \zeta} + \frac{1}{2}(Mx^2 - Ny^2)\right\}\\ & + \epsilon_3\frac{\partial G}{\partial s} + \epsilon_3\frac{\partial G}{\partial t} + \beta(\hat{v}-\hat{u}).\end{aligned} \tag{EC.19}$$

Recalling $x = \pi w\sigma$ and $y = \pi\zeta\sigma$, by (EC.11), for large π, we have

$$\alpha(x-y)^2 = \pi^2\sigma^2\alpha(w-\zeta)^2 = \pi^2 o(1). \tag{EC.20}$$

Then, by (EC.11), (EC.18) and (EC.20), (EC.19) becomes

$$\begin{aligned}0 \le & \sup_{d\le\pi\le u}\left\{\epsilon_3\pi w\eta\frac{\partial G}{\partial w} + \frac{1}{2}\epsilon_3\pi^2 w^2\sigma^2\frac{\partial^2 G}{\partial w^2} + \epsilon_3\pi\zeta\eta\frac{\partial G}{\partial \zeta} + \frac{1}{2}\epsilon_3\pi^2\zeta^2\sigma^2\frac{\partial^2 G}{\partial \zeta^2} + o(1)(1+\pi+\pi^2)\right\}\\ & + \epsilon_3\frac{\partial G}{\partial s} + \epsilon_3\frac{\partial G}{\partial t} + \beta(\hat{v}-\hat{u}).\end{aligned}$$

Note that $\frac{\partial^2 G}{\partial w^2}$ and $\frac{\partial^2 G}{\partial \zeta^2}$ are strictly negative, thus $o(1)(1+\pi+\pi^2)$ in brace is negligible even when $[d,u]$ is unbounded. Thus, we have

$$0 \le \epsilon_3\left(\frac{\partial G}{\partial t} + H\left(w, \frac{\partial G}{\partial w}, \frac{\partial^2 G}{\partial w^2}\right)\right) + \epsilon_3\left(\frac{\partial G}{\partial s} + H\left(\zeta, \frac{\partial G}{\partial \zeta}, \frac{\partial^2 G}{\partial \zeta^2}\right)\right) + \beta(\hat{v}-\hat{u}) + o(1). \tag{EC.21}$$

Recall that $G(t,w,s,\zeta) = Q^{(q)}(t,w) + Q^{(q)}(s,\zeta)$ and $Q^{(q)}(t,w)$ given by (EC.3) in Lemma EC.1.1 satisfies the HJB equation (25), thus, (EC.21) can be simplified as,

$$0 \le \beta(\hat{v}-\hat{u}) + o(1).$$

Using (EC.12), we have $\beta\delta < 0$. Note that $\beta > 0$, then $\delta < 0$, which contradicts to (EC.6). And the theorem is proved. $\square$

Appendix EC.2: Proof of Theorem 3.2

We will take three steps to verify that the value function satisfies the conditions a), b), and c) in Definition A.1. Then, the value function of (4) is a viscosity solution of the HJB equation (17) with the boundary condition (16) and the asymptotic condition (18) (or (21) for unbounded portfolio set). The uniqueness of the viscosity solution and the continuity of the value function before the maturity are guaranteed by the comparison principle in Theorem 3.1. The property of the value function at the maturity is fully characterized by the the asymptotic condition (18) (or (21) for unbounded portfolio set). And thus Theorem 3.2 is proved.

Step 1: We verify Condition a) in Definition A.1.

In the standard viscosity solution approach (see. e.g. Chapter V of Fleming and Soner, 2006), a dynamic programming principle based on the continuity of value functions is used to verify Condition a). In our model, the utility function U is upper semicontinuous. Similar to Bouchard and Touzi (2011), we first give a weak version of dynamic programming for the value function V.

Proposition EC.2.1 **(Weak Dynamic Programming)**. *Denote $W_s^{t,w,\pi}$ as the wealth process W_s starting from $W_t = w$ under the portfolio π. For any stopping time τ taking values within $[t,T]$, and $(t,w) \in [0,T) \times (B,+\infty)$, we have*

$$V(t,w) \le \sup_{d\le\pi\le u} \mathbb{E}[V^*(\tau, W_\tau^{t,w,\pi})] \tag{EC.22}$$

and

$$V(t,w) \ge \sup_{d\le\pi\le u} \mathbb{E}[V_*(\tau, W_\tau^{t,w,\pi})]. \tag{EC.23}$$

Proof of Proposition EC.2.1: First, the inequality (EC.22) follows from the inequality (3.1) of Bouchard and Touzi (2011), which is a direct consequence of the law of iterated expectations.

Next, we give the proof for the inequality (EC.23). For each portfolio π, denote

$$J(t,w;\pi) := \mathbb{E}[U(W_T^{t,w,\pi})]. \tag{EC.24}$$

Bouchard and Touzi (2011) prove the inequality (EC.23) in their Corollary 3.6 under the assumption that $J(t,w;\pi)$ in (EC.24) is lower semicontinuous in (t,w), for any $\pi \in [d,u]$. In our problem, by Assumption 2.1, the utility function U is upper semicontinuous and bounded by a power function w^p when w is sufficiently large for some $0 < p < 1$. Then, $J(t,w;\pi)$ is bounded by the value function $Q^{(p)}(t,w)$ given in (EC.3). By the dominated convergence theorem and the continuity of the controlled process $W_s^{t,w,\pi}, t \le s < T$, the function $J(\cdot,\cdot;\pi)$ is upper semicontinuous. However, by Lemma 3.5 of Reny(1999), $J(\cdot,\cdot;\pi)$ can be approximated from above by a sequence of continuous functions (or lower semicontinuous functions in a wider domain of functions) on each compact domain. So, we can mimic the proof of Bouchard and Touzi (2011) by firstly replacing $J(\cdot,\cdot;\pi)$ by its continuous approximations, and then take a limit to recover $J(\cdot,\cdot;\pi)$. We give below the detail of the proof.

By definition, for fixed $\epsilon>0$ and any (s,ζ), there exists a portfolio $\pi^{(s,\zeta)}\in[d,u]$, such that

$$J(s,\zeta;\pi^{(s,\zeta)})\geq V(s,\zeta)-\epsilon.$$

Consider any lower semicontinuous function $\psi^{(s,\zeta)}(\cdot)\geq J(\cdot;\pi^{(s,\zeta)})$ and any upper semicontinuous function $\varphi\leq V$ (later, we will send $\psi^{(s,\zeta)}(\cdot)\to J(\cdot;\pi^{(s,\zeta)})$ and $\varphi\to V_*$). By the definition of semi-continuity, there exists $r_{(s,\zeta)}>0$, such that

$$\begin{aligned}\psi^{(s,\zeta)}(s_1,\zeta_1)\geq\psi^{(s,\zeta)}(s,\zeta)-\epsilon,\varphi(s_1,\zeta_1)\leq\varphi(s,\zeta)+\epsilon,\\ \text{for any}\ \ (s_1,\zeta_1)\in B(s,\zeta,r_{(s,\zeta)}),\end{aligned}$$

where $B(s,\zeta,r_{(s,\zeta)}):=\{(s_1,\zeta_1):|s_1-s|<r_{(s,\zeta)},|\zeta_1-\zeta|<r_{(s,\zeta)}\}$. Then

$$\begin{aligned}\psi^{(s,\zeta)}(s_1,\zeta_1)&\geq\psi^{(s,\zeta)}(s,\zeta)-\epsilon\geq J(s,\zeta;\pi^{(s,\zeta)})-\epsilon\geq V(s,\zeta)-2\epsilon\geq\varphi(s,\zeta)-2\epsilon\\ &\geq\varphi(s_1,\zeta_1)-3\epsilon,\ \text{for any}\ \ (s_1,\zeta_1)\in B(s,\zeta,r_{(s,\zeta)}).\end{aligned}\tag{EC.25}$$

$\{B(s,\zeta,r_{(s,\zeta)})\}_{(s,\zeta)}$ forms an open covering of $[0,T]\times[B,\infty)$. Then by the Lindelöf covering theorem, there exists a countable sequence $(s_i,\zeta_i)_{i\geq1}$, such that the sequence of open sets $\{B(s_i,\zeta_i,r_{(s_i,\zeta_i)})\}_{(s_i,\zeta_i),i\geq1}$ forms an open covering of $[0,T]\times[B,\infty)$. Then we can define a disjoint partition $\{A_n\}_{n\geq1}$:

$$A_1:=B(s_1,\zeta_1,r_{(s_1,\zeta_1)}),A_n:=B(s_n,\zeta_n,r_{(s_n,\zeta_n)})\setminus(\cup_{1\leq i\leq n-1}A_i),\ \text{for}\ \ n>1.$$

Recalling (EC.25), for $(\tau,W_\tau^{t,w,\pi})\in A_i$, by setting $\pi(s)=\pi^{(s_i,\zeta_i)}(s)$ for $\tau\leq s\leq T$, we have

$$\psi^{(s_i,\zeta_i)}(\tau,W_\tau^{t,w,\pi})1_{(\tau,W_\tau^{t,w,\pi})\in A_i}\geq\varphi(\tau,W_\tau^{t,w,\pi})1_{(\tau,W_\tau^{t,w,\pi})\in A_i}-3\epsilon.$$

Now, we are ready to prove (EC.23). For any portfolio $d\leq\pi\leq u$, define the portfolio

$$\pi_n(s):=\pi(s)1_{t\leq s\leq\tau}+1_{\tau\leq s\leq T}\left(\sum_{i=1}^n\pi^{(s_i,\zeta_i)}1_{(\tau,W_\tau^{t,w,\pi})\in A_i}+\pi(s)1_{(\tau,W_\tau^{t,w,\pi})\notin\cup_{1\leq i\leq n}A_i}\right).$$

Then

$$\sum_{i=1}^n\psi^{(s_i,\zeta_i)}(\tau,W_\tau^{t,w,\pi})1_{(\tau,W_\tau^{t,w,\pi})\in A_i}\geq\sum_{i=1}^n\varphi(\tau,W_\tau^{t,w,\pi})1_{(\tau,W_\tau^{t,w,\pi})\in A_i}-3\epsilon,$$

and

$$\begin{aligned}&\mathbb{E}[\sum_{i=1}^{n}\psi^{(s_i,\zeta_i)}(\tau,W_\tau^{t,w,\pi})1_{(\tau,W_\tau^{t,w,\pi})\in A_i}+J((\tau,W_\tau^{t,w,\pi});\pi)1_{(\tau,W_\tau^{t,w,\pi})\notin\cup_{1\le i\le n}A_i}]\\ \ge&\mathbb{E}[\varphi(\tau,W_\tau^{t,w,\pi})1_{(\tau,W_\tau^{t,w,\pi})\in\cup_{1\le i\le n}A_i}+J(\tau,W_\tau^{t,w,\pi};\pi)1_{(\tau,W_\tau^{t,w,\pi})\notin\cup_{1\le i\le n}A_i}]-3\epsilon.\end{aligned}\quad\text{(EC.26)}$$

As aforementioned, for each (s_i,ζ_i), $J(s,\zeta;\pi^{(s_i,\zeta_i)})$ is upper semicontinuous. By Lemma 3.5 of Reny(1999), there exists a sequence of continuous functions $\{\tilde{\psi}_j^{(s_i,\zeta_i)}:j\ge 1\}$ such that

$$\tilde{\psi}_j^{(s_i,\zeta_i)}(s,\zeta)\ge J(s,\zeta;\pi^{(s_i,\zeta_i)})$$

and $\lim_{j\to\infty}\tilde{\psi}_j^{(s_i,\zeta_i)}(s,\zeta)=J(s,\zeta;\pi^{(s_i,\zeta_i)})$, for all $(s,\zeta)\in A_i$. Let $\psi_j^{(s_i,\zeta_i)}:=\max_{j\ge j'}\tilde{\psi}_{j'}^{(s_i,\zeta_i)}$, then $\psi_j^{(s_i,\zeta_i)}$ is nonincreasing in j and converges to $J(s,\zeta;\pi^{(s_i,\zeta_i)})$ as j tends to infinity. Similarly, we can find a nondecreasing sequence of continuous functions $\{\varphi_j:j\ge 1\}$ such that $\varphi_j(s,\zeta)\le V_*(s,\zeta)$ and $\lim_{j\to\infty}\varphi_j(s,\zeta)=V_*(s,\zeta)$ on $\cup_{1\le i\le n}A_i$. Replacing $\psi^{(s_i,\zeta_i)}$ by $\psi_j^{(s_i,\zeta_i)}$ on the left hand side of (EC.26), and replacing φ by φ_j on the right hand side of (EC.26), sending $j\to\infty$, by the monotone convergence theorem, we have

$$\begin{aligned}&\mathbb{E}[\sum_{i=1}^{n}J(\tau,W_\tau^{t,w,\pi};\pi^{(s_i,\zeta_i)})1_{(\tau,W_\tau^{t,w,\pi})\in A_i}+J(\tau,W_\tau^{t,w,\pi};\pi)1_{(\tau,W_\tau^{t,w,\pi})\notin\cup_{1\le i\le n}A_i}]\\ \ge&\mathbb{E}[V_*(\tau,W_\tau^{t,w,\pi})1_{(\tau,W_\tau^{t,w,\pi})\in\cup_{1\le i\le n}A_i}+J(\tau,W_\tau^{t,w,\pi};\pi)1_{(\tau,W_\tau^{t,w,\pi})\notin\cup_{1\le i\le n}A_i}]-3\epsilon.\end{aligned}\quad\text{(EC.27)}$$

Note that the left hand side of (EC.27) is $\mathbb{E}[J(\tau,W_\tau^{t,w,\pi};\pi_n)]$. Now, sending $n\to\infty$ on the right side of (EC.27), by the dominated convergence theorem, we have

$$\mathbb{E}[J(\tau,W_\tau^{t,w,\pi};\pi_n)]\ge\mathbb{E}[V_*(\tau,W_\tau^{t,w,\pi})]-3\epsilon.$$

By the law of iterated expectations

$$\begin{aligned}V(t,w)\ge&\mathbb{E}[J(t,w;\pi_n)]=\mathbb{E}[\mathbb{E}[U(W_T^{t,w,\pi_n})|\mathcal{F}(\tau)]]\\ =&\mathbb{E}[J(\tau,W_\tau^{t,w,\pi};\pi_n)]\ge\mathbb{E}[V_*(\tau,W_\tau^{t,w,\pi})]-3\epsilon.\end{aligned}\quad\text{(EC.28)}$$

By the arbitrariness of π and ϵ, (EC.23) follows. □

Then, Condition a) is verified by Corollary 5.6 of Bouchard and Touzi (2011).

Step 2: We verify Condition b) in Definition A.1 by the following proposition.

Proposition EC.2.2. *For all $0\le t,\bar{t}<T$ and $w>B\ge 0$, we have,*

$$\lim_{(t,w)\to(\bar{t},B)}V(t,w)=U(B).\quad\text{(EC.29)}$$

Proof of Proposition EC.2.2: For any portfolio π, define the first passage time $\tau^b_{t,w,\pi}=\inf\{s\geq t: W_s=b|W_t=w\}$. First, we show that, for any $\zeta>0$,

$$\limsup_{(t,w)\to(\bar{t},B)} \mathbb{P}\left[\tau^{B+\zeta}_{t,w,\pi}\leq \min\{\tau^B_{t,w,\pi},T\}\right]=0. \tag{EC.30}$$

Define $\tau_{t,w,\pi}=\min\{\tau^B_{t,w,\pi},\tau^{B+\zeta}_{t,w,\pi}\}$, and let $f(t,w)$ be the maximum probability that the wealth process W hits $B+\zeta$ before hitting B by maturity without portfolio constraints, provided that the wealth process W starts at $W_t=w$, that is,

$$\begin{aligned} f(t,w) &= \sup_{\pi}\mathbb{P}\left[\tau^{B+\zeta}_{t,w,\pi}\leq \min\{\tau^B_{t,w,\pi},T\}\right] \\ &= \sup_{\pi}\mathbb{E}[\mathbf{1}_{\{W^{t,w,\pi}_{\tau_{t,w,\pi}\wedge T}\geq B+\zeta\}}]=\sup_{\pi}\mathbb{E}[\mathbf{1}_{\{W^{t,w,\pi}_T\geq B+\zeta\}}], \end{aligned} \tag{EC.31}$$

where $W^{t,w,\pi}_T$ is the wealth at time T under portfolio π with initial value $W_t=w$. Without portfolio constraints, the market is complete. A standard martingale approach gives that the optimal wealth never hits the upper boundary $B+\zeta$ before maturity, and at maturity, it equals either $B+\zeta$ or B, depending on whether an event $\{\rho^T_t<c\}$ happens for some constant c to be determined by the budget constraint:

$$w=\mathbb{E}\left[\rho^T_t W^{t,w,\pi}_T\right]=\mathbb{E}\left[\rho^T_t\left(B*\mathbf{1}_{\{\rho^T_t>c\}}+(B+\zeta)*\mathbf{1}_{\{\rho^T_t<c\}}\right)\right]=B+\zeta\mathbb{E}\left[\rho^T_t*\mathbf{1}_{\{\rho^T_t<c\}}\right], \tag{EC.32}$$

where ρ^T_t is the pricing kernel, $\rho^T_t=\exp\{-\theta(\mathcal{B}_T-\mathcal{B}_t)-\frac{1}{2}\theta^2(T-t)\}$, and $\theta=\eta/\sigma$ is the market price of risk. Let $w\to B$ in (EC.32), we have $c\to 0$. Then, by (EC.31), we have,

$$\limsup_{(t,w)\to(\bar{t},B)} \mathbb{P}\left[\tau^{B+\zeta}_{t,w,\pi}\leq \min\{\tau^B_{t,w,\pi},T\}\right]\leq \limsup_{(t,w)\to(\bar{t},B)} f(t,w)=\limsup_{(t,w)\to(\bar{t},B)} \mathbb{P}[\rho^T_t<c]=0. \tag{EC.33}$$

That is, (EC.30) is proved.

Second, we prove (EC.29). For any $\zeta>0$,

$$V(t,w)\leq \sup_{d\leq\pi\leq u} E[U(W^{t,w,\pi}_T)\mathbf{1}_{\tau^{B+\zeta}_{t,w,\pi}\leq\min\{\tau^B_{t,w,\pi},T\}}]+\sup_{d\leq\pi\leq u} E[U(W^{t,w,\pi}_T)\mathbf{1}_{\tau^{B+\zeta}_{t,w,\pi}>\min\{\tau^B_{t,w,\pi},T\}}]. \tag{EC.34}$$

The second term on the right hand side of (EC.34) is bounded by $U(B+\zeta)$. If the first term goes to zero as $(t,w)\to(\bar{t},B)$, then

$$\limsup_{(t,w)\to(\bar{t},B)} V(t,w)\leq U(B+\zeta).$$

Letting ζ decrease to zero, since $U(\cdot)$ is nondecreasing and right-continuous, we get

$$\limsup_{(t,w)\to(\bar{t},B)} V(t,w)\leq U(B).$$

On the other side, it is trivial that

$$\liminf_{(t,w)\to(\bar{t},B)} V(t,w) \geq U(B).$$

Thus, (EC.29) holds and the proposition is proved.

Last, we verify that the first term on the right hand side of (EC.34) goes to zero as $(t,w)\to(\bar{t},B)$. If U is bounded by a finite constant M, by (EC.33)

$$\begin{aligned}&\limsup_{(t,w)\to(\bar{t},B)} \sup_{d\leq\pi\leq u} E[U(W_T^{t,w,\pi})\mathbf{1}_{\tau_{t,w,\pi}^{B+\zeta}\leq\min\{\tau_{t,w,\pi}^{B},T\}}]\\ \leq& M \limsup_{(t,w)\to(\bar{t},B)} \sup_{d\leq\pi\leq u} E[\mathbf{1}_{\tau_{t,w,\pi}^{B+\zeta}\leq\min\{\tau_{t,w,\pi}^{B},T\}}]=0.\end{aligned}$$

If U is unbounded,

$$\begin{aligned}&\sup_{d\leq\pi\leq u} E[U(W_T^{t,w,\pi})\mathbf{1}_{\tau_{t,w,\pi}^{B+\zeta}\leq\min\{\tau_{t,w,\pi}^{B},T\}}]\\ =&\sup_{d\leq\pi\leq u} E[U(W_T^{t,w,\pi})\mathbf{1}_{\tau_{t,w,\pi}^{B+\zeta}\leq\min\{\tau_{t,w,\pi}^{B},T\},U(W_T^{t,w,\pi})>U(A)}]\\ &+\sup_{d\leq\pi\leq u} E[U(W_T^{t,w,\pi})\mathbf{1}_{\tau_{t,w,\pi}^{B+\zeta}\leq\min\{\tau_{t,w,\pi}^{B},T\},U(W_T^{t,w,\pi})\leq U(A)}]\\ \leq& U(A)\sup_{d\leq\pi\leq u} E[U(W_T^{t,w,\pi})/U(A)\mathbf{1}_{U(W_T^{t,w,\pi})>U(A)}]+U(A)\mathbf{1}_{\tau_{t,w,\pi}^{B+\zeta}\leq\min\{\tau_{t,w,\pi}^{B},T\}}\\ \leq& U(A)\sup_{d\leq\pi\leq u} E[(U(W_T^{t,w,\pi})/U(A))^q\mathbf{1}_{U(W_T^{t,w,\pi})>U(A)}]+U(A)f(t,w)\\ \leq&\sup_{d\leq\pi\leq u} E[(W_T^{t,w,\pi})^{pq}]/(U(A))^{q-1}+U(A)f(t,w)\\ \leq& Q^{(pq)}(t,w)/(U(A))^{q-1}+U(A)f(t,w),\ \text{for any}\ \ A>B,\end{aligned} \tag{EC.35}$$

where $q>1$ is chosen such that $pq<1$ and $Q^{(pq)}(t,w)$ is given by (EC.3). The third inequality holds since $(U(w))^q$ is bounded from above by w^{pq} for large w (cf. Assumption 2.1). When $(t,w)\to(\bar{t},B)$, by (EC.33), $f(t,w)\to 0$. We can choose $A\to\infty$, such that $U(A)\to 0$ and $U(A)f(t,w)\to 0$. Thus, by (EC.35), we have

$$\lim_{(t,w)\to(\bar{t},B)} \sup_{d\leq\pi\leq u} E[U(W_T^{t,w,\pi})\mathbf{1}_{\tau_{t,w,\pi}^{B+\zeta}\leq\min\{\tau_{t,w,\pi}^{B},T\}}]=0.$$

□

Step 3: We verify Condition c) in Definition A.1 by the following proposition.

Proposition EC.2.3. *Denote $\hat{U}$ as the concave envelope of U.*
(i). When the portfolio set $[d,u]$ is bounded, denoting $L=\max\{u,-d\}$ and $U(B-)=U(B)$, for any $w\geq B$, we have

$$\lim_{(t,\zeta)\to(T-,w)} V(t,\zeta)-U(w-)-2\Phi\left(\frac{\min\{0,\log\zeta/w\}}{L\sigma\sqrt{T-t}}\right)(U(w)-U(w-))=0.$$

Here $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variable, and $U(w-)$ is the left limit of U at w.

(ii). When the portfolio set $[d,u]$ is unbounded, for any $w \geq B$, we have

$$\lim_{(t,\zeta)\to(T-,w)} V(t,\zeta) = \hat{U}(w).$$

To prove Proposition EC.2.3(i), we need Proposition EC.2.4 below which focuses on the simplest discontinuous case: the goal-reaching problem. We present Proposition EC.2.4 first, while its proof will be proposed at the end of the proof for Proposition EC.2.3(i).

Proposition EC.2.4. *Assume that d,u are finite. For $t < T$ and $0 \leq w \leq 1$, let $V(t,w) := \sup_{d\leq\pi\leq u} \mathbb{E}[\mathbf{1}_{W_T\geq 1}|W_t = w]$ be the value function of the goal-reaching problem, then*

$$\limsup_{t\to T^-} \sup_{0\leq w\leq 1} \left| V(t,w) - f(\frac{\log w}{\sqrt{T-t}}) \right| = 0, \tag{EC.36}$$

where $f(z) := 2\Phi(\frac{\min\{0,z\}}{L\sigma})$, $L := \max\{u, -d\} < +\infty$, and $\Phi(\cdot)$ is the cumulative probability of the standard normal distribution. More specifically,

$$f\left(\frac{\log w}{\sqrt{T-t}} - a\sqrt{T-t}\right) \leq V(t,w) \leq f\left(\frac{\log w}{\sqrt{T-t}} + a\sqrt{T-t}\right), \tag{EC.37}$$

where $a := \max_{d\leq\pi\leq u} \{|\eta\pi - \frac{1}{2}\sigma^2\pi^2|\} < +\infty$.

Proof of Proposition EC.2.3(i): Here, the portfolio set $[d,u]$ is bounded. Let a and $f(\cdot)$ be given in Proposition EC.2.4. Define $\tilde{V}(t,y) := V(t,w)$, where $y = \log w$, then $\tilde{V}$ is the value function associated with the utility $\tilde{U}(y) := U(e^y) \leq C_1 + C_2 e^{py}$ and the log-wealth process $Y_s^{t,y,\pi} = \log W_s$, satisfying

$$dY_s^{t,y,\pi} = (\eta\pi_s - \frac{1}{2}\sigma^2\pi_s^2)ds + \sigma\pi_s d\mathcal{B}_s, \qquad Y_t^{t,y,\pi} = y. \tag{EC.38}$$

We aim to prove that, for any y_0,

$$\lim_{(t,y)\to(T-,y_0)} \tilde{V}(t,y) - \tilde{U}(y_0-) - f\left(\frac{y-y_0}{\sqrt{T-t}}\right)(\tilde{U}(y_0) - \tilde{U}(y_0-)) = 0. \tag{EC.39}$$

First, we show that under bounded portfolio constraints, the controlled process Y of (EC.38) will not move too far away from its initial point in short time. To be more specific, for a fixed y_0 and any $\epsilon > 0$, since $\tilde{U}$ is nondecreasing and right-continuous, there exists a $\delta > 0$, s.t.

$$\tilde{U}(y_0-) - \epsilon \leq \tilde{U}(y) \leq \tilde{U}(y_0-), \text{ when } \; y_0 - \delta \leq y < y_0, \tag{EC.40}$$

$$\tilde{U}(y_0) \leq \tilde{U}(y) \leq \tilde{U}(y_0) + \epsilon, \text{ when } \; y_0 \leq y \leq y_0 + \delta. \tag{EC.41}$$

Then, for any y such that $|y-y_0|\le\delta/2$, by the second inequality of (EC.37), for any $\pi\in[d,u]$, we have

$$\mathbb{P}(Y_T^{t,y,\pi}\ge y_0+\delta)=\mathbb{P}(Y_T^{t,y-y_0-\delta,\pi}\ge 0)\le f\left(\frac{y-y_0-\delta}{\sqrt{T-t}}+a\sqrt{T-t}\right). \tag{EC.42}$$

Note that for $|y-y_0|\le\delta/2$, $y-y_0-\delta<-\delta/2$, thus the right hand of (EC.42) goes to zero as $t\to T$. Similarly, considering the process $-Y$ instead of Y, by the second inequality of (EC.37), for any $\pi\in[d,u]$, we have

$$\mathbb{P}(Y_T^{t,y,\pi}\le y_0-\delta)=\mathbb{P}(-Y_T^{t,y-y_0+\delta,\pi}\ge 0)\le f\left(\frac{y_0-y-\delta}{\sqrt{T-t}}+a\sqrt{T-t}\right). \tag{EC.43}$$

The right hand of (EC.43) also goes to zero as $t\to T$, since $y_0-y-\delta<-\delta/2$ for $|y-y_0|\le\delta/2$. That is, as the control $\pi\in[d,u]$ is bounded, the terminal log-wealth $Y_T^{t,y,\pi}$ concentrates on the domain $(y_0-\delta,y_0+\delta)$ when $T-t$ is small.

Second, we prove (EC.39). Let $\tilde U_1(y):=\tilde U(y_0-)+\mathbf{1}_{y\ge y_0}(\tilde U(y_0)-\tilde U(y_0-))$, which is a linear transformation of the goal-reaching utility. Let $\tilde V_1(t,y)$ be the value function for the optimization problem with the utility $\tilde U_1(y)$ and the log-wealth process Y of (EC.38). Then, it is a linear transformation of the value function in Proposition EC.2.4. By (EC.36), we have

$$\lim_{(t,y)\to(T-,y_0)}\tilde V_1(t,y)-\tilde U(y_0-)-f\left(\frac{y-y_0}{\sqrt{T-t}}\right)(\tilde U(y_0)-\tilde U(y_0-))=0. \tag{EC.44}$$

By (EC.40), (EC.41) and the definition of $\tilde U_1$, we have

$$\mathbb{E}[|\tilde U(y)-\tilde U_1(y)|]\le\epsilon,\ \text{when}\ \ |y-y_0|\le\delta.$$

Then, for any strategy π,

$$\begin{aligned}&\mathbb{E}[|\tilde U(Y_T^{t,y,\pi})-\tilde U_1(Y_T^{t,y,\pi})|]\\ \le&\epsilon\ \mathbb{E}[1_{\{|Y_T^{t,y,\pi}-y_0|\le\delta\}}]+\mathbb{E}[|\tilde U(Y_T^{t,y,\pi})-\tilde U(y_0)|1_{\{Y_T^{t,y,\pi}>y_0+\delta\}}]+\mathbb{E}[|\tilde U(Y_T^{t,y,\pi})-\tilde U(y_0-)|1_{\{Y_T^{t,y,\pi}<y_0-\delta\}}]\\ \le&\epsilon+\mathbb{E}[(C_1+C_2e^{pY_T^{t,y,\pi}}+|\tilde U(y_0)|)1_{\{Y_T^{t,y,\pi}>y_0+\delta\}}]+\mathbb{E}[(C_1+C_2e^{-pY_T^{t,y,\pi}}+|\tilde U(y_0-)|)1_{\{Y_T^{t,y,\pi}<y_0-\delta\}}].\end{aligned}$$

By integration by part, we have

$$\begin{aligned}&\mathbb{E}[|\tilde U(Y_T^{t,y,\pi})-\tilde U_1(Y_T^{t,y,\pi})|]\\ \le&\epsilon+(C_1+|\tilde U(y_0)|+C_2e^{p(y_0+\delta)})\mathbb{P}[Y_T^{t,y,\pi}>y_0+\delta]+(C_1+|\tilde U(y_0-)|+C_2e^{-p(y_0-\delta)})\mathbb{P}[Y_T^{t,y,\pi}<y_0-\delta]\\ &+C_2\int_\delta^\infty pe^{p(y_0+v)}\mathbb{P}(Y_T^{t,y,\pi}\ge y_0+v)dv+C_2\int_\delta^\infty pe^{-p(y_0-v)}\mathbb{P}(Y_T^{t,y,\pi}\le y_0-v)dv.\end{aligned}$$

By (EC.42) and (EC.43), we have

$$
\begin{aligned}
&\mathbb{E}[|\tilde{U}(Y_T^{t,y,\pi}) - \tilde{U}_1(Y_T^{t,y,\pi})|]\\
\le&\epsilon + (C_1 + |\tilde{U}(y_0)| + C_2 e^{p(y_0+\delta)}) f(\frac{y - y_0 - \delta + a(T-t)}{\sqrt{T-t}})\\
&+ (C_1 + |\tilde{U}(y_0-)| + C_2 e^{-p(y_0-\delta)}) f(\frac{y_0 - y - \delta + a(T-t)}{\sqrt{T-t}})\\
&+ C_2 \int_\delta^\infty \left(p e^{p(y_0+v)} f(\frac{y - y_0 - v + a(T-t)}{\sqrt{T-t}}) + p e^{-p(y_0-v)} f(\frac{y_0 - y - v + a(T-t)}{\sqrt{T-t}}) \right) dv.
\end{aligned}
$$

The bound is independent with π. Then taking supremum for $\pi \in [d,u]$ and sending $t \to T$, since $y - y_0 - v < -\delta/2$ and $y_0 - y - v < -\delta/2$ for $|y - y_0| \le \delta/2$ and $v \ge \delta$, we have

$$\limsup_{t\to T} \sup_{d \le \pi \le u} \mathbb{E}[|\tilde{U}(Y_T^{t,y,\pi}) - \tilde{U}_1(Y_T^{t,y,\pi})|] \le \epsilon.$$

Then

$$\limsup_{(t,y)\to(T-,y_0)} |\tilde{V}(t,y) - \tilde{V}_1(t,y)| \le \limsup_{(t,y)\to(T,y_0)} \sup_{d\le\pi\le u} \mathbb{E}[|\tilde{U}(Y_T^{t,y,\pi}) - \tilde{U}_1(Y_T^{t,y,\pi})|] \le \epsilon.$$

Since ϵ is arbitrary, we have $\lim_{(t,y)\to(T-,y_0)} |\tilde{V}(t,y) - \tilde{V}_1(t,y)| = 0$. Then (EC.39) follows by (EC.44). Hence, Proposition EC.2.3(i) is proved. □

We now proceed to prove Proposition EC.2.4, which relies on Lemma EC.2.1 below. After the logarithmic transformation $Y_s^{t,y,\pi} = \log W_s, s \ge t$ and $Y_t^{t,y,\pi} = y = \log w$, the optimization problem in Proposition EC.2.4 can be reformulated as

$$\tilde{V}(t,y) := \sup_{d\le\pi\le u} \mathbb{E}[\mathbf{1}_{Y_T^{t,y,\pi} \ge 0}], \tag{EC.45}$$

where the log-wealth process Y is given in (EC.38). Then, $\tilde{V}(t,y) = V(t,w)$ with $y = \log w$.

As t approaches T, it is anticipated that the volatility term (of order $\sqrt{T-t}$) rather than the drift term (of order $T-t$) plays a dominant role. Thus, we first study a goal-reaching problem discarding the drift term, which turns out to have a closed-form solution.

LEMMA EC.2.1. *For $0 \le t \le T, y \le 0$, let $G(t,y)$ be the value function of the following problem:*

$$
\begin{aligned}
G(t,y) := &\sup_{d\le\pi\le u} \mathbb{E}[\mathbf{1}_{\bar{Y}_T^{t,y,\pi} \ge 0}]\\
&s.t. \quad d\bar{Y}_s^{t,y,\pi} = \sigma \pi_s d\mathcal{B}_s, \ t < s \le T, \ and \ \bar{Y}_t^{t,y,\pi} = y.
\end{aligned} \tag{EC.46}
$$

Then $G(t,y) = f(\frac{y}{\sqrt{T-t}})$, where f is as defined in Proposition EC.2.4.

Note that $f(\frac{y}{\sqrt{T-t}}) = \mathbb{P}(\max_{t\le s\le T} \bar{Y}_s^{t,y,\pi} \ge 0)$, where $\pi \equiv l := u * 1_{\{u \ge -d\}} + d * 1_{\{u < -d\}}$. Thus, the lemma says that it is optimal to apply the maximum leverage or short-sale $\pi_s = l$ and switch to $\pi_s = 0$ once the goal is reached.

Proof of Lemma EC.2.1: First, we show that $G(t,y) = g(\frac{y}{\sqrt{T-t}})$, for some function $g(z)$. Consider any two points (y_1,t_1) and (y_2,t_2) such that $t_t, t_2 < T$, $y_1, y_2 < 0$ and $\frac{y_1}{\sqrt{T-t_1}} = \frac{y_2}{\sqrt{T-t_2}}$. For any adapted strategy π_s, $t_1 \le s \le T$, and a time-change $C(s) := t_1 + \frac{T-t_1}{T-t_2}(s-t_2)$ for $s \in [t_2, T]$, we can define an adapted strategy $\hat{\pi}_s := \pi_{C(s)}$ and a new Brownion motion $\hat{\mathcal{B}}_s := \sqrt{\frac{T-t_2}{T-t_1}}\left(\mathcal{B}_{C(s)} - \mathcal{B}_{t_1}\right)$ for $s \in [t_2, T]$. By (EC.46),

$$\begin{aligned}\bar{Y}_T^{t_1,y_1,\pi} - y_1 &= \int_{t_1}^T \sigma\pi_s d\mathcal{B}_s = \int_{C(t_2)}^{C(T)} \sigma\pi_s d\mathcal{B}_s = \int_{t_2}^T \sigma\pi_{C(s)} d\mathcal{B}_{C(s)} = \sqrt{\frac{T-t_1}{T-t_2}}\int_{t_2}^T \sigma\hat{\pi}_s d\hat{\mathcal{B}}_s \\ &\overset{d}{=} \sqrt{\frac{T-t_1}{T-t_2}}\left(\bar{Y}_T^{t_2,y_2,\hat{\pi}} - y_2\right), \end{aligned} \tag{EC.47}$$

where $\overset{d}{=}$ represents "equal in distribution". Recalling that $\frac{y_1}{\sqrt{T-t_1}} = \frac{y_2}{\sqrt{T-t_2}}$, we have,

$$\begin{aligned}\mathbb{P}(\bar{Y}_T^{t_1,y_1,\pi} \ge 0) =& \mathbb{P}(\bar{Y}_T^{t_1,y_1,\pi} - y_1 \ge -y_1) = \mathbb{P}(\sqrt{\frac{T-t_1}{T-t_2}}\left(\bar{Y}_T^{t_2,y_2,\hat{\pi}} - y_2\right) \ge -\frac{\sqrt{T-t_1}}{\sqrt{T-t_2}} y_2) \\ =& \mathbb{P}(\bar{Y}_T^{t_2,y_2,\hat{\pi}} - y_2 \ge -y_2) = \mathbb{P}(\bar{Y}_T^{t_2,y_2,\hat{\pi}} \ge 0).\end{aligned}$$

This means $G(t_1,y_1) \le G(t_2,y_2)$. For the same reason, $G(t_2,y_2) \le G(t_1,y_1)$.

Second, we show that $g(z)$ is the unique viscosity solution to the ODE

$$\begin{cases} -zg'(z) - \sup_{d\le\pi\le u}\{\sigma^2\pi^2 g''(z)\} = 0,\ z<0, \\ g(0) = 1,\ \ \lim\limits_{z\to-\infty} g(z) = 0. \end{cases} \tag{EC.48}$$

The uniqueness holds by a standard approach to proving the comparison principle of this ODE, which we omit here. We only verify that $g(z)$ is a viscosity solution to the above ODE. By Proposition EC.2.1 (Weak Dynamic Programming) and Corollary 5.6 of Bouchard and Touzi (2011), if $h \in C^{1,2}([0,T)\times(-\infty,0))$ and $h - G^*$ attains local minimum 0 at $(\bar{t}, y_0)$, then

$$-\frac{\partial}{\partial t}h(\bar{t},y_0) - \sup_{d\le\pi_t\le u}\left\{\frac{1}{2}\sigma^2\pi_t^2\frac{\partial^2}{\partial y^2}h(\bar{t},y_0)\right\} \le 0.$$

Now consider a function $\phi \in C^2((-\infty,0])$, such that $\phi(z) - g^*(z)$ attains local minimum 0 at interior point $-\infty < z_0 < 0$, then $\phi(\frac{y}{\sqrt{T-t}}) \in C^{1,2}([0,T)\times(-\infty,0))$, and $\phi(\frac{y}{\sqrt{T-t}}) - G^*(t,y)$ attains local minimum 0 at point $(t, \sqrt{T-t}\ z_0)$ for any $0 \le t < T$. Then, for any $0 \le t < T$, we have

$$-\frac{\partial}{\partial t}\phi(\frac{y}{\sqrt{T-t}})\big|_{\{y=\sqrt{T-t}\ z_0\}} - \sup_{d\le\pi_t\le u}\left\{\frac{1}{2}\sigma^2\pi_t^2\frac{\partial^2}{\partial y^2}\phi(\frac{y}{\sqrt{T-t}})\big|_{\{y=\sqrt{T-t}\ z_0\}}\right\} \le 0.$$

By a direct calculation, we have

$$-z_0\phi'(z_0) - \sup_{d\le\pi\le u}\left\{\sigma^2\pi^2\phi''(z_0)\right\} \le 0.$$

Since $g^*(0) = 1$ and $\lim\limits_{z\to-\infty} g^*(w) = 0$, we infer that g^* is a viscosity subsolution of (EC.48). A similar argument shows that g_* is a viscosity supersolution. Therefore, g is a viscosity solution of (EC.48).

Third, a direct calculation shows that f is a classical solution to

$$\begin{cases} -zf'(z) - L^2\sigma^2 f''(z) = 0, \; z < 0, \\ f(0) = 1, \;\; \lim\limits_{z\to-\infty} f(z) = 0. \end{cases}$$

By the convexity of f, f is also a classical solution to

$$\begin{cases} -zf'(z) - \sup_{d\le\pi\le u}\{\sigma^2\pi^2 f''(z)\} = 0, \; z < 0, \\ f(0) = 1, \;\; \lim\limits_{z\to-\infty} f(z) = 0. \end{cases}$$

By the uniqueness of viscosity solution, we prove the lemma. □

Proof of Proposition EC.2.4: For any $(t,y) \in [0,T) \times (-\infty, 0]$, let $Y_s^{t,y,\pi}$ be given in (EC.38) with starting point $Y_t^{t,y,\pi} = y$ and portfolio $\pi_s, t \le s \le T$, and let $\bar{Y}_s^{t,y+a(T-t),\pi}$ be given in (EC.46) with starting point $\bar{Y}_s^{t,y+a(T-t),\pi} = y + a(T-t)$ and the same portfolio $\pi_s, t \le s \le T$. Recalling that $a := \max\limits_{d\le\pi\le u}\{|\eta\pi - \frac{1}{2}\sigma^2\pi^2|\} < +\infty$, by comparison, we have $Y_T^{t,y,\pi} \le \bar{Y}_T^{t,y+a(T-t),\pi}$, a.s. Similarly $Y_T^{t,y,\pi} \ge \bar{Y}_T^{t,y-a(T-t),\pi}$, a.s. Then,

$$G(t, y - a(T-t)) \le \tilde{V}(t,y) \le G(t, y + a(T-t)),$$

where $G(t,y) = f(\frac{y}{\sqrt{T-t}})$ is given in Lemma EC.2.1. So,

$$f(\frac{y}{\sqrt{T-t}} - a\sqrt{T-t}) \le \tilde{V}(t,y) \le f(\frac{y}{\sqrt{T-t}} + a\sqrt{T-t}).$$

By the uniform continuity of f, the proposition is proved. □

Bian, Chen and Xu (2019) give a proof for Proposition EC.2.3(ii). Here, we provide an alternative proof. First, we introduce two lemmas.

Lemma EC.2.2. *In the case $u = +\infty$ or $d = -\infty$, we have that*

$$V(t, \lambda w_1 + (1-\lambda)w_2) \ge \lambda U(w_1) + (1-\lambda)U(w_2), \quad \textit{for all} \;\; 0 < \lambda < 1, t < T. \tag{EC.49}$$

Proof of Lemma EC.2.2: First, consider the case that $u = +\infty$. For $t < T$, given w_1 and w_2, denote $w = \lambda w_1 + (1-\lambda)w_2$, and let $W_s^{t,w,n}, t \le s \le T$ be the wealth process with $W_t^{t,w,n} = w$ and a constant proportion strategy $\pi_s \equiv n$. Define the stopping times

$$\begin{cases} \tau_1^{(n)} := \inf\{s \ge t | W_s^{t,w,n} = w_1\}, \\ \tau_2^{(n)} := \inf\{s \ge t | W_s^{t,w,n} = w_2\}. \end{cases}$$

Next, we show that

$$\lim_{n\to+\infty} \mathbb{P}(\tau_1^{(n)} < \min\{\tau_2^{(n)}, T\}) = \lambda, \tag{EC.50}$$

$$\lim_{n\to+\infty} \mathbb{P}(\tau_2^{(n)} < \min\{\tau_1^{(n)}, T\}) = 1 - \lambda. \tag{EC.51}$$

Note that

$$dW_s^{t,w,n}/W_s^{t,w,n} = \eta n ds + \sigma n d\mathcal{B}_s, \quad t \le s \le T.$$

By rescaling the process, define $\tilde{W}_s^{t,w,n} := W_{t+(s-t)/n^2}^{t,w,n}, t \le s \le t + n^2(T-t)$ and $\tilde{W}_t^{t,w,n} = w$, then

$$d\tilde{W}_s^{t,y,n}/\tilde{W}_s^{t,y,n} = \frac{\eta}{n} ds + \sigma d\mathcal{B}_s^{(n)}, \quad t \le s \le t + n^2(T-t), \tag{EC.52}$$

where $\mathcal{B}_s^{(n)}$ is a standard Brownian motion. Define the stopping times

$$\begin{cases} \tilde{\tau}_1^{(n)} := \inf\{s \ge t | \tilde{W}_s^{t,w,n} = w_1\}, \\ \tilde{\tau}_2^{(n)} := \inf\{s \ge t | \tilde{W}_s^{t,w,n} = w_2\}, \end{cases}$$

Then, by definition

$$\mathbb{P}(\tau_1^{(n)} < \min\{\tau_2^{(n)}, T\}) = \mathbb{P}(\tilde{\tau}_1^{(n)} < \min\{\tilde{\tau}_2^{(n)}, t + n^2(T-t)\}), \tag{EC.53}$$

$$\mathbb{P}(\tau_2^{(n)} < \min\{\tau_1^{(n)}, T\}) = \mathbb{P}(\tilde{\tau}_2^{(n)} < \min\{\tilde{\tau}_1^{(n)}, t + n^2(T-t)\}). \tag{EC.54}$$

As $n \to \infty$, $\mathbf{1}_{\tilde{\tau}_1^{(n)} < \min\{\tilde{\tau}_2^{(n)}, n^2(T-t)+t\}} \to \mathbf{1}_{\tilde{\tau}_1^{(\infty)} < \tilde{\tau}_2^{(\infty)}}$ in distribution, where

$$d\tilde{W}_s^{t,y,\infty}/\tilde{W}_s^{t,y,\infty} = \sigma d\mathcal{B}_s, \quad s \ge t, \tag{EC.55}$$

then

$$\lim_{n\to+\infty} \mathbb{P}(\tilde{\tau}_1^{(n)} < \min\{\tilde{\tau}_2^{(n)}, n^2(T-t)+t\}) = \lambda,$$
$$\lim_{n\to+\infty} \mathbb{P}(\tilde{\tau}_2^{(n)} < \min\{\tilde{\tau}_1^{(n)}, n^2(T-t)+t\}) = 1-\lambda.$$

By (EC.53) and (EC.54), we prove (EC.50) and (EC.51).

Now, take the following strategy

$$\pi_s^{(n)} = \begin{cases} n, \ t \le s \le \tau_1^{(n)} \wedge \tau_2^{(n)} \wedge T, \ \text{if } u = +\infty, \\ 0, \ \text{otherwise}, \end{cases}$$

then

$$V(t, \lambda w_1 + (1-\lambda) w_2) \ge \limsup_{n\to\infty} \mathbb{E}[W_T^{t,w,\pi^{(n)}}] \ge \lambda U(w_1) + (1-\lambda) U(w_2).$$

That is, we prove (EC.49) for the case $u = +\infty$. For the case $d = -\infty$, replacing n by $-n$ yields the desired result. $\square$

Lemma EC.2.3. *Consider a concave function $f(w)$ on $B \le w < +\infty$, which satisfies $\lim_{w\to+\infty} \frac{f(w)}{w^p} = 0$ for some $0 < p < 1$. Then for a fixed point $w_0 \ge B$ and a fixed number $\epsilon > 0$, there exists a function of type*

$$f_1(w) = C_1 w^q + C_2, \quad p \le q < 1.$$

s.t. $f_1(w) \ge f(w)$, $\forall w \ge B$, and $f_1(w_0) \le f(w_0) + \epsilon$.

Proof of Lemma EC.2.3: First, since f is concave, there exists a tangent line $h(w)$ of this function, s.t. $h(w)$ has a finite positive slope, $h(w) \geq f(w)$, and $h(w_0) \leq f(w_0) + \frac{\epsilon}{2}$. What's more, assume $f(w) \leq \frac{h(n)-h(B)}{n^p} w^p + h(B) \leq h(w)$ when $w \geq n$.

Now choose $C_2 = h(B)$ and $C_1 = \frac{h(n)-h(B)}{n^q}$, and let $f_1(w) = C_1 w^q + C_2$. For any $p \leq q < 1$, we have $f_1(B) = h(B) \geq f(B)$ and $f_1(n) = h(n) \geq f(n)$. By the concavity of f_1, we have $f_1 \geq h \geq f$ in $[0,n]$, and $f_1(w) \geq \frac{h(n)-h(B)}{n^p} w^p + h(B) \geq f(w)$ when $w \geq n$. That is, $f_1(w) \geq f(w)$, $\forall w \geq B$.

Sending $q \to 1-$ gives $f_1(w_0) \to h(w_0)$. Thus, there exists $q \in [p,1)$, such that $f_1(w_0) \leq f(w_0) + \epsilon$. □

Proof of Proposition EC.2.3(ii): First, we show that $V(t,w) \geq \hat{U}(w)$ for all $t < T, w \geq B$. If not, by Lemma EC.2.2, there exists an $\epsilon > 0$, s.t.

$$\hat{U}(w) \geq V(t,w) + \epsilon \geq \lambda U(w_1) + (1-\lambda) U(w_2) + \epsilon, \quad \forall\ \lambda w_1 + (1-\lambda) w_2 = w,\ \lambda \in (0,1).$$

While the region between $\hat{U}$ and $f \equiv U(B)$ is the smallest convex hull of

$$\{(F,w) | U(B) \leq F \leq U(w), B \leq w < +\infty\}.$$

By the separation theorem of convex sets, $\hat{U}(w)$ has a positive distance from this convex hull, which leads to a contradiction. Then we have

$$\liminf_{(t,\zeta)\to(T-,w)} V(t,\zeta) \geq \hat{U}(w).$$

For the other side inequality, consider the case with utility $f_1(w) = C_1 w^q + C_2$ given by Lemma EC.2.3. By Lemma EC.1.1, it has a classical solution $V_1(t,\zeta)$ which is continuous at the terminal time T. Then for any $\epsilon > 0$,

$$\limsup_{(t,\zeta)\to(T-,w)} V(t,\zeta) \leq \limsup_{(t,\zeta)\to(T-,w)} V_1(t,\zeta) = f_1(w) \leq \hat{U}(w) + \epsilon.$$

Thus, part (ii) of Proposition EC.2.3 is proved by sending $\epsilon \to 0$. □

The verification of Condition c) is now finished.

Appendix EC.3: Proof of Theorem 3.3

The HJB equation of our non-concave utility maximization problem may involve unbounded wealth level, unbounded portfolio constraints, and discontinuous utility function. In this section, we first present three lemmas to show that the value function of the original problem can be approximated by the value function under the case that the wealth processes are controlled in a bounded domain, the portfolio set is bounded, and the utility function is continuous. Then, we only need to prove the convergence of the numerical algorithm for the case that the wealth processes are controlled in a bounded domain, the portfolio set is bounded, and the utility function is continuous.

The first lemma shows that the portfolio optimization problem with unbounded wealth processes allowed can be approximated by the problem with bounded wealth processes restriction. To prepare the lemma, for any $A>0$, define $U_A(w)=U(w)$ for $w\le A$ and $U_A(w)=U(A)$ for $w>A$, and $\hat{U}_A(w)=\hat{U}(w)$ for $w\le A$ and $\hat{U}_A(w)=\hat{U}(A)$ for $w>A$, where $\hat{U}$ is the concave envelope of U.

LEMMA EC.3.1. *For any $A>0$, denote V_A as the value function under the utility U_A ($\hat{U}_A$) when the portfolio set $[d,u]$ is bounded (unbounded). Then, we have*

$$\lim_{A\to\infty} V_A(t,w)=V(t,w). \tag{EC.56}$$

Proof of Lemma EC.3.1: Denote by $W_s^{t,w,\pi}$ the wealth process W_s starting from $W_t=x$ under the portfolio π. When the portfolio set $[d,u]$ is bounded,

$$\begin{aligned}|V_A(t,w)-V(t,w)| &= \left|\sup_{d\le\pi\le u} E[U_A(W_T^{t,w,\pi})]-\sup_{d\le\pi\le u} E[U(W_T^{t,w,\pi})]\right| \\ &\le \sup_{d\le\pi\le u} E[\left|U_A(W_T^{t,w,\pi})-U(W_T^{t,w,\pi})\right|] \\ &= \sup_{d\le\pi\le u} E[\left|U(A)-U(W_T^{t,w,\pi})\right| 1_{\{W_T^{t,w,\pi}>A\}}].\end{aligned}$$

First, we show that (EC.56) holds when U is bounded by a finite constant M. In this case, by Chebyshev inequality,

$$\begin{aligned}|V_A(t,w)-V(t,w)| &\le 2M\sup_{d\le\pi\le u} P[W_T^{t,w,\pi}>A]=2M\sup_{d\le\pi\le u} P[(W_T^{t,w,\pi})^q>A^q] \\ &\le 2M\sup_{d\le\pi\le u} E[(W_T^{t,w,\pi})^q]/A^q\le 2MQ^{(q)}(t,w)/A^q\to 0 \text{ as } A\to\infty,\end{aligned}$$

where $0<q<1$, and $Q^{(q)}(t,w)$ is given by (EC.3) in Lemma EC.1.1 which is finite.

Second, we show that (EC.56) holds when U is unbounded. In this case, since U is increasing, we have

$$\begin{aligned}|V_A(t,w)-V(t,w)| &\le 2\sup_{d\le\pi\le u} E[|U(W_T^{t,w,\pi})|1_{\{W_T^{t,w,\pi}>A\}}] \\ &=2U(A)\sup_{d\le\pi\le u} E[U(W_T^{t,w,\pi})/U(A)1_{\{U(W_T^{t,w,\pi})>U(A)\}}] \\ &\le 2U(A)\sup_{d\le\pi\le u} E[(U(W_T^{t,w,\pi})/U(A))^q 1_{\{U(W_T^{t,w,\pi})>U(A)\}}] \\ &\le 2\sup_{d\le\pi\le u} E[\max\{0,U(W_T^{t,w,\pi})^q\}]/(U(A))^{q-1} \\ &\le 2\sup_{d\le\pi\le u} Q^{(pq)}(t,w)/(U(A))^{q-1}\to 0 \text{ as } A\to\infty,\end{aligned}$$

where $q>1$ is chosen such that $pq<1$ and $Q^{(pq)}(t,w)$ is given by (EC.3). The last inequality holds since $(U(w))^q$ is bounded from above by w^{pq} for large w. Thus, $\lim_{A\to\infty} V_A(t,w)=V(t,w)$. The proof for the case that $[d,u]$ is unbounded is similar. □

The second lemma shows that the unbounded portfolio constraint problem can be approximated by a bounded constraint problem under the concave envelope of the utility function.

LEMMA EC.3.2. *If the portfolio set $[d,u]$ is unbounded, for any $C>0$, let $V^C(t,w)$ denote the value function of the optimization problem with portfolio constraint $\pi\in[-C,C]\cap[d,u]$ under the utility $\hat{U}$, the concave envelope of U. Then, we have*

$$\lim_{C\to+\infty} V^C(t,w)=V(t,w).$$

Proof of Lemma EC.3.2: It is obvious that

$$\limsup_{C\to+\infty} V^C(t,w)\le V(t,w).$$

On the other side, given any strategy π, note that in order to well define the wealth process, we have (see. e.g., Definition 1.2.1 of Karatzas and Shreve, 1998),

$$\int_t^T |\pi_s\eta|ds<+\infty,\ \text{and}\quad \int_t^T (\pi_s\sigma)^2 ds<+\infty,\quad \text{almost surely.}$$

For each $C>0$, define $\pi_s^{(C)}=\pi_s 1_{\{-C\le\pi_s\le C\}}$, $t\le s\le T$. Then, there exists a subsequence C_n, such that $C_n\to\infty$ as $n\to\infty$, and (see. e.g., Problem 3.2.27 and its proof in Karatzas and Shreve, 1991),

$$\int_t^T \pi_s^{(C_n)}\eta ds\to\int_t^T \pi_s\eta ds\quad\text{as}\quad n\to\infty,\quad\text{almost surely,}$$
$$\int_t^T \pi_s^{(C_n)}\sigma d\mathcal{B}_s\to\int_t^T \pi_s\sigma d\mathcal{B}_s\quad\text{as}\quad n\to\infty,\quad\text{almost surely.}$$

Let $W_s^{t,w,\pi}$ denote the wealth W_s starting from $W_t=x$ under the portfolio π. Recalling (2), we have $W_T^{t,w,\pi^{(C_n)}}\to W_T^{t,w,\pi}$ as $n\to\infty$ almost surely. By the dominated convergence theorem (the expectation is finite when the problem is well defined), we have

$$\limsup_{n\to+\infty}\mathbb{E}[\hat{U}(W_T^{t,w,\pi^{(C_n)}})]=\mathbb{E}[\limsup_{n\to+\infty}\hat{U}(W_T^{t,w,\pi^{(C_n)}})]=\mathbb{E}[\hat{U}(W_T^{t,w,\pi})]. \tag{EC.57}$$

For any $\pi\in[d,u]$,

$$\limsup_{C\to+\infty} V^C(t,w)\ge\limsup_{n\to+\infty}\sup_{d\le\pi\le u}\mathbb{E}[\hat{U}(W_T^{t,w,\pi^{(C_n)}})]=\limsup_{n\to+\infty}\mathbb{E}[\hat{U}(W_T^{t,w,\pi^{(C_n)}})].$$

Thus, by (EC.57),

$$\limsup_{C\to+\infty} V^C(t,w)\ge\sup_{d\le\pi\le u}\limsup_{n\to+\infty}\mathbb{E}[\hat{U}(W_T^{t,w,\pi^{(C_n)}})]=\sup_{d\le\pi\le u}\mathbb{E}[\hat{U}(W_T^{t,w,\pi})]=V(t,w).$$

Noting that V^C is increasing in C, the lemma is proved. □

The third lemma shows that when the portfolio set $[d,u]$ is bounded, the value function associated with a discontinuous utility can be approximated by a value function associated with some continuous utility that approximates the discontinuous utility.

LEMMA EC.3.3. *There exist an increasing sequence of continuous function $U_{d,k}$ and a decreasing sequence of continuous function $U_{u,k}$, such that for any $k>0$, $U(w-\frac{1}{k})\le U_{d,k}(w)\le U(w)$ and $U(w)\le U_{u,k}(w)\le U(w+\frac{1}{k})$ for all $w\ge B$. When the portfolio set $[d,u]$ is bounded, let $V_{u,k}$ and $V_{d,k}$ be the value function associated with the utility $U_{u,k}$ and $U_{d,k}$, respectively. Then, we have*

$$V(t,w)=\lim_{k\to\infty}V_{d,k}(t,w)=\lim_{k\to\infty}V_{u,k}(t,w),\qquad t<T.$$

Proof of Lemma EC.3.3: For each $k>0$, we can use the classical convolution method to construct continuous utility functions $U_{u,k}$ and $U_{d,k}$. Consider a function ϕ, s.t. ϕ is nonnegative, supported on $[-1,1]$, $\phi\in C^{\infty}_{\mathbb{R}}$, and $\int_{-1}^{1}\phi(x)dx=1$. Then

$$U_{u,k}(w):=\int_{-\infty}^{\infty}2k\phi(2k\zeta)*U(w+\frac{1}{2k}-\zeta)d\zeta,$$

is what we need. Replacing $U(w+\frac{1}{2k}-\zeta)$ by $U(w-\frac{1}{2k}-\zeta)$, we get $U_{d,k}$.

Firstly, it is obvious that $V_{d,k}$ and $V_{u,k}$ are monotonic, and $V_{d,k}\le V\le V_{u,k}$ for any k.

Next we show that for any $\delta>0$ and $t<T-\delta$,

$$\lim_{k\to\infty}V_{d,k}(t,w)\ge V(t+\delta,w). \tag{EC.58}$$

To see this, let $W_s^{t,w,\pi}$ denote the wealth W_s starting from $W_t=x$ under the portfolio π, and note that

$$V_{d,k}(t,w)=\sup_{d\le\pi\le u}\mathbb{E}[U_{d,k}(W_T^{t,w,\tilde{\pi}})]=\sup_{d\le\pi\le u}\mathbb{E}[U_{d,k}(W_{T+\delta}^{t+\delta,w,\tilde{\pi}})]. \tag{EC.59}$$

For any admissible strategy π_s, $t+\delta\le s\le T$, let $\tilde{\pi}_t=\pi_s$ for $t+\delta\le s\le T$, and $\tilde{\pi}_t=l$ for $s>T$,where $l=u$ if $u>-d$ and $l=d$ if $u\le -d$. Define $\tau^k:=\inf\{s\ge T|W_s^{t+\delta,w,\tilde{\pi}}\ge W_T^{t+\delta,w,\tilde{\pi}}+\frac{1}{k}\}$ and a strategy $\tilde{\pi}_t^k$ as following:

$$\tilde{\pi}_s^k=\begin{cases}\pi_s,\ t+\delta\le s\le T,\\ l,\quad T<s\le\tau^k,\\ 0,\quad \tau^k<s\le T+\delta.\end{cases}$$

Due to the local oscillation of Brownian Motion and strict positiveness of X_T, we have $\lim\limits_{k\to+\infty}\tau^k\to T$, a.s., which means $\liminf\limits_{k\to+\infty}U_{d,k}(W_{T+\delta}^{t+\delta,w,\tilde{\pi}^k})\ge U(W_T^{t+\delta,w,\pi})$, a.s. So by Fatou's lemma,

$$\liminf_{k\to+\infty}\mathbb{E}[U_{d,k}(W_{T+\delta}^{t+\delta,w,\tilde{\pi}^k})]\ge\mathbb{E}[U(W_T^{t+\delta,w,\pi})],$$

then, by(EC.59),

$$\begin{aligned}V_{d,k}(t,w)=\sup_{d\le\pi\le u}\mathbb{E}[U_{d,k}(W_{T+\delta}^{t+\delta,w,\tilde{\pi}})]&\ge\sup_{d\le\pi\le u}\mathbb{E}[U_{d,k}(W_{T+\delta}^{t+\delta,w,\tilde{\pi}^k})]\\ &\ge\sup_{d\le\pi\le u}\liminf_{k\to+\infty}\mathbb{E}[U_{d,k}(W_{T+\delta}^{t+\delta,w,\tilde{\pi}^k})]\ge\sup_{d\le\pi\le u}\mathbb{E}[U(W_T^{t+\delta,w,\pi})]=V(t+\delta,w),\end{aligned}$$

that is, (EC.58) holds.

Recall that V is continuous when $t<T$ by Theorem 3.2. Sending $\delta \to 0$, we have $\lim_{k\to\infty} V_{d,k}(t,w) \geq V(t,w)$, for $t<T$. The converse inequality holds since $U_{d,k} \leq U$. Thus, we have proved the part for $V_{d,k}$. And the part for $V_{u,k}$ follows by similar arguments. □

With the help of Lemmas EC.3.1, EC.3.2 and EC.3.3, we now prove Theorem 3.3.

Proof of Theorem 3.3: By Lemmas EC.3.1, EC.3.2 and EC.3.3, we only need to consider the case that the portfolio set $[d,u]$ is bounded, the diffusions are controlled in a bounded domain $[B,A]$ for a big enough upper bounded A, and the utility function U is continuous.

Let $\Sigma_\Delta = \{(t_n, w_i) : 0 \leq n \leq N_t, 0 \leq i \leq N_w\}$ be a discretization mesh of the domain $[0,T]\times[B,A]$ with fixed time and spatial step sizes Δt and Δw. Let V_i^n be the solution at grid (t_n, w_i) of a monotone, stable, and consistent finite difference scheme for the HJB equation (17) with terminal and boundary conditions (34) and (33). If the discretization solution V_i^n assumes the terminal and boundary conditions uniformly, that is,

$$\lim_{\substack{(t_n,w_i)\to(T-,w)\\ \Delta t\to 0,\Delta w\to 0}} V_i^n = U(w), \tag{EC.60}$$

and

$$\lim_{\substack{(t_n,w_i)\to(t,B)\\ \Delta t\to 0,\Delta w\to 0}} V_i^n = U(B), \text{ and } \lim_{\substack{(t_n,w_i)\to(t,A)\\ \Delta t\to 0,\Delta w\to 0}} V_i^n = U(A), \tag{EC.61}$$

then with the help of the Comparison Principle, by Theorem 2.1 of Barles and Souganidis (1991), the solution of the finite difference scheme converges to the unique viscosity solution, that is, the value function of our problem.

To finish the proof, we show that the discretization solution V_i^n assumes the terminal and boundary conditions (EC.60) and (EC.61). The convergence property (EC.60) is proved by Lemma IX.5.3 of Fleming and Soner (2006). Thus, we only need to show that the discretization solution V_i^n assumes the boundary conditions (EC.61).

First, we show that

$$\limsup_{\substack{(t_n,w_i)\to(t,B)\\ \Delta t\to 0,\Delta w\to 0}} V_i^n \leq U(B), \text{ and } \limsup_{\substack{(t_n,w_i)\to(t,A)\\ \Delta t\to 0,\Delta w\to 0}} V_i^n \leq U(A). \tag{EC.62}$$

Let $\overline{V}$ be the value function of the unconstrained portfolio optimization problem with the utility $\hat{U}$, the concave envelope of U in the bounded domain $[B,A]$, and the boundary conditions $\overline{V}(t,B) = U(B)$ and $\overline{V}(t,A) = U(A)$. By Theorem 4.2 of Bian, Chen and Xu (2019), $\overline{V}$ is the unique classical solution to the associated HJB equation, continuous at the boundaries and the terminal time. Then,

$\overline{V}$ is a classical supersulution of the original constraint problem. By Lemma IX.5.2 of Fleming and Soner (2006), we have that, for every $a>0$, there exists $h_0>0$, such that, for $\Delta w<h_0$,

$$V_i^n \le \overline{V}(t_n,w_i)+a, \text{ for all } \ n,i,$$

By the continuity of $\overline{V}$, we have

$$\limsup_{\substack{(t_n,w_i)\to(t,B)\\ \Delta t\to 0,\Delta w\to 0}} V_i^n \le \limsup_{\substack{(t_n,w_i)\to(t,B)\\ \Delta t\to 0,\Delta w\to 0}} \overline{V}(t_n,w_i)+a=\overline{V}(t,B)+a=U(B)+a,$$

and

$$\limsup_{\substack{(t_n,w_i)\to(t,A)\\ \Delta t\to 0,\Delta w\to 0}} V_i^n \le \limsup_{\substack{(t_n,w_i)\to(t,A)\\ \Delta t\to 0,\Delta w\to 0}} \overline{V}(t_n,w_i)+a=\overline{V}(t,A)+a=U(A)+a.$$

By the arbitrariness of a, we get (EC.62).

Second, we show that

$$\liminf_{\substack{(t_n,w_i)\to(t,B)\\ \Delta t\to 0,\Delta w\to 0}} V_i^n \ge U(B), \text{ and } \liminf_{\substack{(t_n,w_i)\to(t,A)\\ \Delta t\to 0,\Delta w\to 0}} V_i^n \ge U(A). \tag{EC.63}$$

The constant $U(B)$ is a discrete subsolution, by Lemma IX.4.2 of Fleming and Soner (2006) or Theorem 5.2 of Forsyth and Labahn (2007), we have $V_i^n \ge U(B)$ for all n,i. So, we get the first inequality of (EC.63). For the second inequality of (EC.63), by the discretization version of the HJB equation (17), we have $V_i^n \ge V_i^{n+1}$ for all n,i [7]. Then by recursion, $V_i^n \ge V_i^{N_t}=U(w_i)$ for all n,i. By the continuity of U, we have

$$\liminf_{\substack{(t_n,w_i)\to(t,A)\\ \Delta t\to 0,\Delta w\to 0}} V_i^n \ge \liminf_{\substack{(t_n,w_i)\to(t,A)\\ \Delta t\to 0,\Delta w\to 0}} U(w_i)=U(A).$$

So, we get the secnd inequality of (EC.63). □

At the end of this section, we show that the the finite difference scheme given in Appendix B is monotone, stable, and consistent. Then, by Theorem 3.3, the discretization solution of the scheme will converge to the value function of our problem. By Lemmas EC.3.1 and EC.3.2, we only need to consider the case that the portfolio set $[d,u]$ is bounded and the state variable are in a bounded domain, that is, $w\in[B,A]$ for a bounded $A>0$.

[7] For example, by the discretization equation (32), we have

$$V_i^n = V_i^{n+1} + \Delta t * \sup_{d\le\pi\le u}\left\{\frac{\pi^2 w_i^2\sigma^2}{2}\frac{V_{i+1}^n-2V_i^n+V_{i-1}^n}{(\Delta w)^2}+\pi w_i\eta\frac{\Delta V_i^n(\pi)}{\Delta w}\right\}\ge V_i^{n+1}.$$

LEMMA EC.3.4. *Assume that the portfolio set $[d,u]$ is bounded. B is the lower liquidation boundary, and let $A>B$ be a finite cutoff, such that the wealth level is restricted in the bounded domain $[B,A]$. The finite difference scheme (32-35) given in Appendix B is monotone, stable, and consistent (cf. Barles and Souganidis(1991) for the definition of monotonicity, stability and consistence.)*

Proof of Lemma EC.3.4: Denote the left hand side of the discretization HJB equation (32) as

$$S\left((\Delta t,\Delta w),(t_n,w_i),V_i^n,\{V_{i-1}^n,V_{i+1}^n,V_i^{n+1}\}\right). \tag{EC.64}$$

First, S can be rewritten as (cf. (32) and (35))

$$-\frac{V_i^{n+1}-V_i^n}{\Delta t}-\sup_{d\le\pi\le u}\left\{\alpha_i(\pi)V_{i-1}^n-(\alpha_i(\pi)+\beta_i(\pi))V_i^n+\beta_i(\pi)V_{i+1}^n\right\}, \tag{EC.65}$$

where $\alpha_i(\pi)$ and $\beta_i(\pi)$ are defined for $d\le\pi\le u$ as following:

$$\begin{cases}\alpha_i(\pi)=\frac{\pi^2w_i^2\sigma^2}{2(\Delta w)^2}-\frac{\pi w_i\eta}{2\Delta w},\ \beta_i(\pi)=\frac{\pi^2w_i^2\sigma^2}{2(\Delta w)^2}+\frac{\pi w_i\eta}{2\Delta w}, & \text{if } |\pi|\ge\pi_i;\\ \alpha_i(\pi)=\frac{\pi^2w_i^2\sigma^2}{2(\Delta w)^2}, \qquad \beta_i(\pi)=\frac{\pi^2w_i^2\sigma^2}{2(\Delta w)^2}+\frac{\pi w_i\eta}{\Delta w}, & \text{if } |\pi|<\pi_i \text{ and } \pi\eta>0;\\ \alpha_i(\pi)=\frac{\pi^2w_i^2\sigma^2}{2(\Delta w)^2}-\frac{\pi w_i\eta}{\Delta w},\ \beta_i(\pi)=\frac{\pi^2w_i^2\sigma^2}{2(\Delta w)^2}, & \text{if } |\pi|<\pi_i \text{ and } \pi\eta<0,\end{cases} \tag{EC.66}$$

and $\pi_i=|\eta|\Delta w/(\sigma^2 w_i)$.

First, we prove the monotonicity. Note that $w_i>0$ and $\Delta w>0$. By definition, we have

$$\alpha_i(\pi)\ge 0,\ \text{and}\ \beta_i(\pi)\ge 0,\ \text{for all}\ \ d\le\pi\le u. \tag{EC.67}$$

Then, S is non-increasing with respective to V_{i-1}^n, V_{i+1}^n and V_i^{n+1}, so the scheme S is monotone.

Second, we prove the consistence. For a smooth function ϕ, by Taylor expansion (cf. the left hand side of (32)),

$$\begin{aligned}&S\left((\Delta t,\Delta w),(t,w),\phi(t,w)+\xi,\{\phi(t,w-\Delta w)+\xi,\phi(t,w+\Delta w)+\xi,\phi(t+\Delta t,w)+\xi\}\right)\\ =&S\left((\Delta t,\Delta w),(t,w),\phi(t,w),\{\phi(t,w-\Delta w),\phi(t,w+\Delta w),\phi(t+\Delta t,w)\}\right)\\ =&-\left(\frac{\partial\phi(t,w)}{\partial t}+o(\Delta t)\right)-\sup_{d\le\pi\le u}\left\{w\eta\pi\left(\frac{\partial\phi(t,w)}{\partial w}+o(\Delta w)\right)+\frac{w^2\sigma^2\pi^2}{2}\left(\frac{\partial^2\phi(t,w)}{\partial w^2}+o((\Delta w)^2)\right)\right\}.\\ =&-\left(\frac{\partial\phi(t,w)}{\partial t}+o(\Delta t)\right)-H\left(w,\frac{\partial\phi(t,w)}{\partial w}+o(\Delta w),\frac{\partial^2\phi(t,w)}{\partial w^2}+o((\Delta w)^2)\right),\end{aligned}$$

where H is the Hamiltonian defined in (24). H is continuous since the portfolio set $[d,u]$ is bounded. Then

$$\begin{aligned}&\lim_{\Delta t\to0,\Delta w\to0}S\left((\Delta t,\Delta w),(t,w),\phi(t,w)+\xi,\{\phi(t,w-\Delta w)+\xi,\phi(t,w+\Delta w)+\xi,\phi(t+\Delta t,w)+\xi\}\right)\\ =&-\frac{\partial\phi(t,w)}{\partial t}-H\left(w,\frac{\partial\phi(t,w)}{\partial w},\frac{\partial^2\phi(t,w)}{\partial w^2}\right).\end{aligned}$$

So, the scheme S is consistent.

Third, we prove the stability. Let π_i^n be the optimal of (EC.65). Recalling (EC.67), for $i = 2, \dots, N_w - 1$, by (EC.65), we have

$$\begin{aligned}|V_i^n|(1+\Delta t(\alpha_i(\pi_i^n)+\beta_i(\pi_i^n))) =& |V_i^{n+1}+\Delta t(\alpha_i(\pi_i^n)V_{i-1}^n+\beta_i(\pi_i^n)V_{i+1}^n))| \\ \leq & ||V^{n+1}||_\infty + \Delta t(\alpha_i(\pi_i^n)+\beta_i(\pi_i^n))\max\{||V^n||_\infty, C\},\end{aligned}$$

where $||V^n||_\infty = \max\{|V_2^n|, \dots, |V_{N_w-1}^n|\}$ and $C = \max\{|U(B)|, |U(A)|\}$. Let $i \in \{2, \dots, N_w - 1\}$ be choose such that $|V_i^n| = ||V^n||_\infty$. Then

$$||V^n||_\infty(1+\Delta t(\alpha_i(\pi_i^n)+\beta_i(\pi_i^n))) \leq ||V^{n+1}||_\infty + \Delta t(\alpha_i(\pi_i^n)+\beta_i(\pi_i^n))\max\{||V^n||_\infty, C\}.$$

It follows that $||V^n||_\infty \leq ||V^{n+1}||_\infty$ if $||V^n||_\infty \geq C$, otherwise, $||V^n||_\infty$ is bounded by the convex combination of $||V^{n+1}||_\infty$ and C. So, for $n = 0, 1, \dots, N_t - 1$, we have,

$$||V^n||_\infty \leq \max\{||V^{n+1}||_\infty, C\}.$$

Then,

$$||V^n||_\infty \leq \max\{\max\{||V^{n+2}||_\infty, C\}, C\} = \max\{||V^{n+2}||_\infty, C\} \leq \cdots \leq \max\{||V^{N_t}||_\infty, C\} = C,$$

where the last equality holds by the terminal condition (34). So, the scheme S is stable. □

Appendix EC.4: Additional Numerical Results

In Figure EC.1, we plot the time 0 value functions against wealth level for the constrained case (no-short-selling and no-borrowing: $\pi \in [0,1]$) and unconstrained case, respectively, for the models of Berkelaar, Kouwenberg and Post (2004), Carpenter (2000), Basak, Pavlova, and Shapiro (2007), and He and Kou (2018). The dashed line stands for the value function without portfolio constraints, which is globally concave. The dotted line is the value function with bounded portfolio constraints, which turns out to be locally convex and locally concave.

In Figures EC.2-EC.5, we plot the time 0 optimal fraction of total wealth invested in the stock π^* against wealth level for the constrained case and the unconstrained strategy, respectively, for the models of Berkelaar, Kouwenberg and Post (2004), Carpenter (2000), Basak, Pavlova, and Shapiro (2007), and He and Kou (2018). The dotted (dashed) line represents the optimal strategy for the constrained (unconstrained) case. In each figure, the portfolio constraints for the constrained case are $\pi \in [0,1]$ for the upper panel and $\pi \in [-2,1]$ for the lower panel, respectively. These figures show that the non-myopic and the gambling properties of the optimal strategy are common for non-concave optimization problems with bounded portfolio constraints.

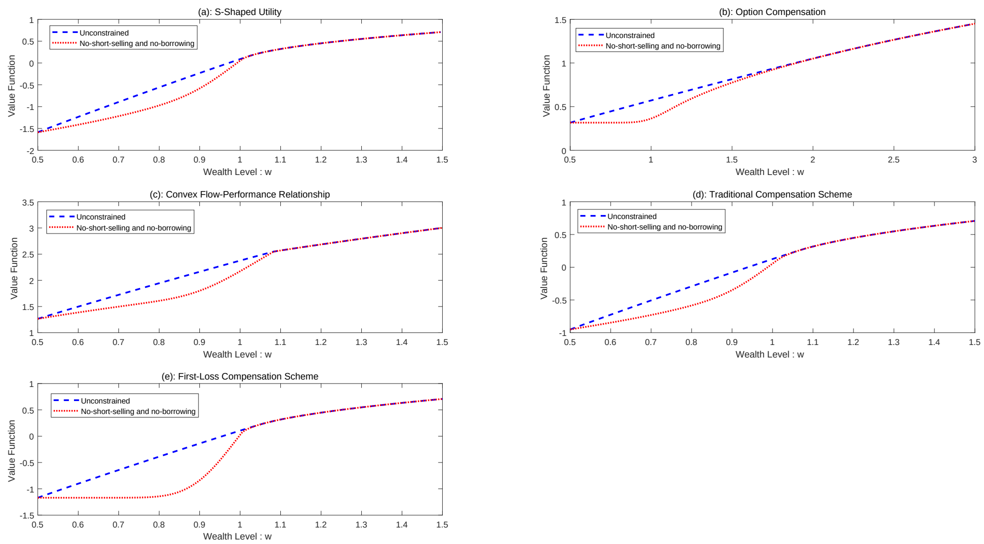

Figure EC.1 A comparison between the constrained value function and the unconstrained value function. In all models, the value function is globally concave in the unconstrained case but is not concave in the constrained case. The dotted (dashed) line is the time 0 value function against wealth level with (without) portfolio constraints. The portfolio constraints in the constrained case are no-borrowing and no-short-sale, i.e., $\pi \in [0,1]$. The five panels (a)-(e) correspond to the models of Berkelaar, Kouwenberg and Post (2004), Carpenter (2000), Basak, Pavlova, and Shapiro (2007), He and Kou (2018) (the traditional scheme) and He and Kou (2018) (the first-loss scheme). A compulsory liquidation at $w = 0.5$ is imposed.

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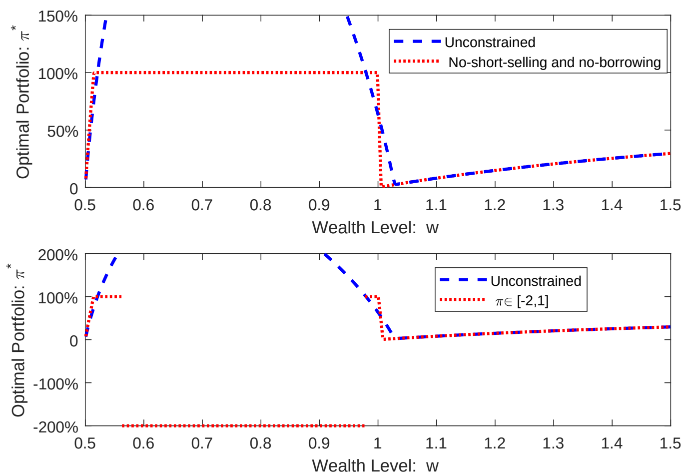

Figure EC.2 A comparison among Merton's strategy, our constrained optimal strategy and the unconstrained optimal strategy for the non-concave portfolio optimization problem discussed in Berkelaar, Kouwenberg and Post (2004). The upper panel indicates that the constrained investors are non-myopic with respect to portfolio constraints such that an early action is made before portfolio constraints being binding. The lower panel indicates that given a relatively large loss, short-selling is likely optimal even with a positive risk premium, provided that a large short-selling ratio ($d=-2$) is permitted. The dotted line is the time 0 optimal fraction of total wealth invested in te stock π^* against wealth level for our constrained optimal strategy, where the portfolio constraint is $\pi\in[0,1]$ (upper panel) and $\pi\in[-2,1]$ (lower panel), respectively. The dashed line stands for the unconstrained optimal strategy (some part that exceeds the scope of the figure is not displayed). Default parameters values are $r=0.03$, $\mu=0.07$, $\sigma=0.3$, $p=0.5$, $\lambda=2.25$, $W_0=1$, $T=1/12$, $B=0.5$.

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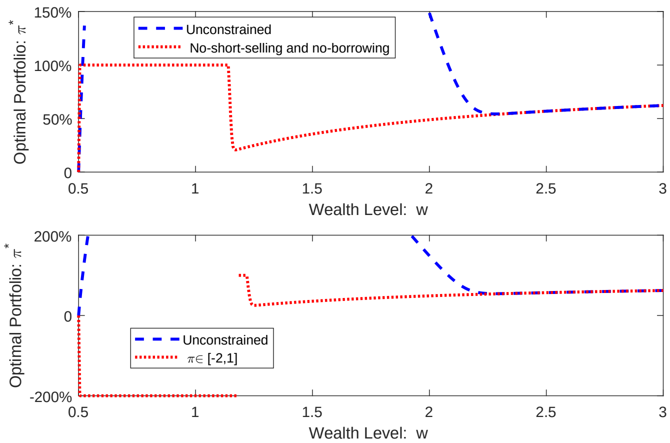

Figure EC.3 A comparison between the constrained strategy and the unconstrained strategy for the non-concave utility optimization problem discussed in Carpenter (2000). The upper panel indicates that the constrained investors are non-myopic with respect to portfolio constraints such that an early action is made before portfolio constraints being binding. The lower panel indicates that given a relatively large loss, short-selling is likely optimal even with a positive risk premium, provided that a large short-selling ratio ($d=-2$) is permitted. The dotted (dashed) line is the time 0 optimal fraction of total wealth invested in the stock π^* against wealth level for the constrained (unconstrained) case. The portfolio constraints in the constrained case are $\pi\in[0,1]$ (upper panel) and $\pi\in[-2,1]$ (lower panel), respectively. The parameter values are: $r=0.03$, $\mu=0.07$, $\sigma=0.3$, $p=0.5$, $K=1$, $\alpha=0.2$, $C=0.02$, $W_0=1$, $T=1$, $T-t=1/12$, $B=0.5$.

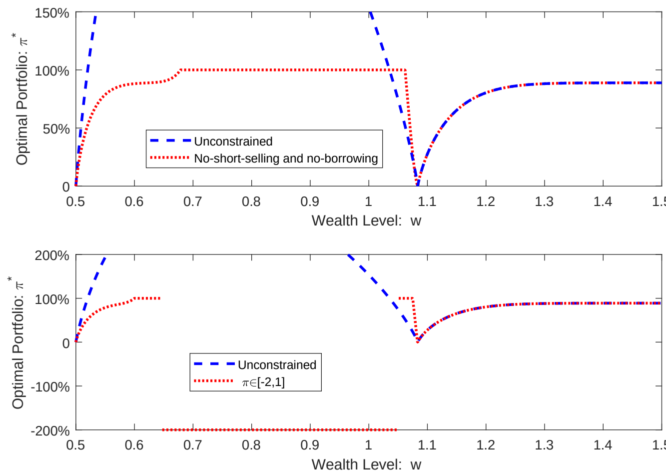

Figure EC.4 A comparison between the constrained strategy and the unconstrained strategy for the non-concave utility optimization problem discussed in Basak, Pavlova, and Shapiro (2007). The upper panel indicates that the constrained investors are non-myopic with respect to portfolio constraints such that an early action is made before portfolio constraints being binding. The lower panel indicates that given a relatively large loss, short-selling is likely optimal even with a positive risk premium, provided that a large short-selling ratio ($d=-2$) is permitted. The dotted (dashed) line is the time 0 optimal fraction of total wealth invested in the stock π^* against wealth level for the constrained (unconstrained) case. The portfolio constraints in the constrained case are $\pi \in [0,1]$ (upper panel) and $\pi \in [-2,1]$ (lower panel), respectively. The parameter values are: $r=0.03$, $\mu=0.07$, $\sigma=0.3$, $p=0.5$, $\eta_L=-0.08$, $\eta_H=0.08$, $f_L=0.8$, $f_H=1.5$, $W_0=1$, $T=1/12$, $B=0.5$.

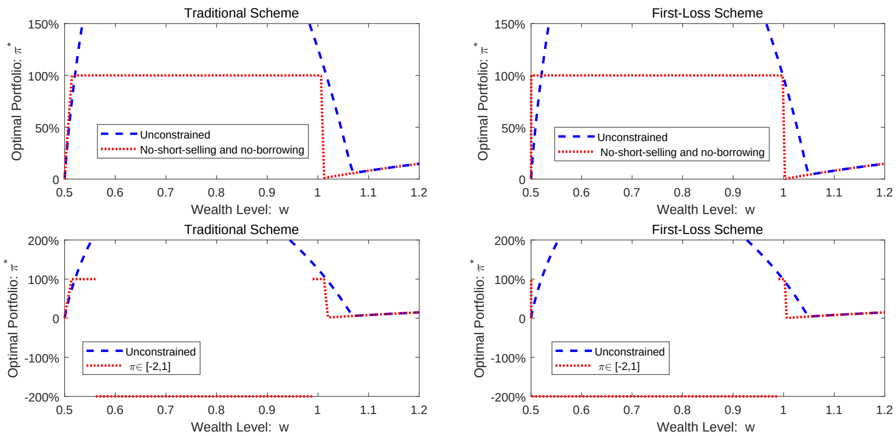

Figure EC.5 A comparison between the constrained strategy and the unconstrained strategy for the non-concave utility optimization problem discussed in He and Kou (2018) for the traditional scheme (the left column: $\gamma = 0.1, \alpha = 0.2$) and the first-loss scheme (the right column: $\gamma = 0.1, \alpha = 0.3$). The upper panel indicates that the constrained investors are non-myopic with respect to portfolio constraints such that an early action is made before portfolio constraints being binding. The lower panel indicates that given a relatively large loss, short-selling is likely optimal even with a positive risk premium, provided that a large short-selling ratio ($d = -2$) is permitted. The dotted (dashed) line is the time 0 optimal fraction of total wealth invested in the stock π^* against wealth level for the constrained (unconstrained) case. The portfolio constraints in the constrained case are $\pi \in [0,1]$ (upper panel) and $\pi \in [-2,1]$ (lower panel), respectively. The other parameter values are: $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $p = 0.5$, $\lambda = 2.25$, $W_0 = 1$, $T = 1/12$, $B = 0.5$.