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Penalty Method for Portfolio Selection with Capital Gains Tax

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Many finance problems can be formulated as a singular stochastic control problem, where the associated Hamilton-Jacobi-Bellman (HJB) equation takes the form of variational inequality and its penalty approximation equation is linked to a regular control problem. The penalty method, as a finite difference scheme for the penalty equation, has been widely used to numerically solve singular control problems, and its convergence analysis in literature relies on the uniqueness of solution to the original HJB equation problem. We consider a singular stochastic control problem arising from continuous-time portfolio selection with capital gains tax, where the associated HJB equation problem admits infinitely many solutions. We show that the penalty method still works and converges to the value function which is the minimal (viscosity) solution of the HJB equation problem. Numerical results are presented to demonstrate the efficiency of the penalty method and to better understand optimal investment strategy in the presence of capital gains tax. Our approach sheds light on the robustness of the penalty method for general singular stochastic control problems.

Key words: Portfolio, taxation, optimal control

1. Introduction

There is a large body of literature on singular stochastic control problems in finance, such as portfolio selection with transaction costs (Magill and Constantinides 1976 and Davis and Norman 1990), option pricing with transaction costs (Davis, Panas and Zariphopoulou 1993), optimal dividend distribution (Guo 2002 and Choulli, Taksar and Zhou 2003), and the pricing of guarantee minimum withdrawal benefits (Dai, Kwok and Zong 2008). In general, these problems do not permit analytical solutions, so one has to resort to numerical solutions to the associated HJB equations. Since the HJB equations arising take the form of variational inequalities with gradient constraints, the penalty method, which employs a finite difference scheme for the penalty approximation to the variational inequality equations, has been widely used to numerically solve singular control problems in finance (e.g., Dai and Zhong 2010 and Huang and Forsyth 2012).

This paper is concerned with a singular control problem arising from continuous-time portfolio selection with capital gains tax (Ben Tahar, Soner and Touzi 2010, Cai, Chen and Dai 2018). Different from the singular control problems aforementioned, the resulting HJB equation problem turns out to admit infinitely many solutions. This gives rise to a solution selection puzzle: which solution of the HJB equation problem corresponds to the value function? More importantly, due to lack of analytical solutions in general, how do we numerically find the value function as well as optimal strategy?

The major contribution of this paper is to show that the penalty method still applies to the present problem, despite the associated HJB equation problem admits more than one solution. Indeed, we find that its penalty approximation problem still has a unique solution which can be solved using a finite difference scheme and converges to the value function as the penalty parameter goes to infinity (Part (ii) of Proposition 3.1 and Part (i) of Theorem 3.1). Moreover, we reveal that the value function is nothing but the minimum *constrained viscosity* solution to the original HJB equation problem (Part (ii) of Theorem 3.1).

This paper also contributes to the singular control literature by providing an explicit construction of regular controls that approximate a singular control. The construction plays a critical role in proving the convergence of the penalty method, because the penalty approximation to the original HJB equation is associated with a regular control problem. As far as we know, our paper is the first one to explicitly construct such approximation in the singular control literature. It is worthwhile emphasizing that this construction approach works for general singular control problems, shedding light on the robustness of the penalty method even if the associated HJB equation problems lack uniqueness of solution.

We conduct an extensive numerical analysis to demonstrate efficiency of the penalty method and investigate investment strategy. In particular, we numerically show that the HJB equation problem permits infinitely many *non-trivial* solutions. Interestingly, we find that the non-trivial solutions are associated with non-admissible investment strategies, which helps us better understand the optimal investment strategy in the presence of capital gains tax.

Technically, this paper complements the convergence analysis of numerical solutions to constrained viscosity solutions that involve inequality boundary conditions. As inequality boundary conditions fail to work with numerical solutions, we prescribe appropriate artificial boundary conditions in terms of financial intuition. Furthermore, we prove comparison principle in the sense of constrained viscosity solutions for the penalty approximation problem, where a key step is to show the continuity of the resulting value function. Using the comparison principle and the prescribed boundary conditions, we obtain the convergence of numerical solutions to the original value function.

Literature Review. Merton (1969, 1971) pioneers the study of continuous-time portfolio selection without market friction. Magill and Constantinides (1976) introduce transaction costs into the Merton model, which leads to a singular stochastic control problem that has been extensively studied.[1] Because of the strong path-dependency of tax basis, capital gains tax has not been incorporated into continuous-time portfolio selection until recently.[2]

Inspired by the multi-step portfolio selection model developed by Dammon, Spatt and Zhang (2001) who adopt the average tax basis as an approximation, Ben Tahar, Soner and Touzi (2007, 2010) formulate a continuous time portfolio selection model with capital gains tax. In order to examine how the asymmetric tax structure in the US market affects the behavior of investors, Dai et al. (2015) establish an investment and consumption model with long- and short-term capital gains tax rates.[3] Cai, Chen and Dai (2018) present an asymptotic analysis to characterize optimal strategy and the associated value function in the presence of capital gains taxes and regime switching. Lei, Li and Xu (2019) propose a continuous-time portfolio selection model with capital gains taxes and return predictability to examine how market returns and capital gains taxes jointly affect the optimal policy. The classical Howard algorithm and the projected SOR method are employed in the above papers to examine optimal strategies.[4] However, none of them provides a rigorous convergence analysis of the numerical methods and clarifies the fact that the value function is the minimum viscosity solution to the HJB equation problem.

In the field of financial engineering, the penalty method is first introduced to numerically price American options whose valuation model is formulated as an optimal stopping problem[5] (Forsyth and Vetzal 2002). Dai, Kwok and Zong (2008) formulate the valuation of guarantee minimum withdrawal benefits as a singular stochastic control problem and pioneer the use of the penalty method to singular stochastic control problems. Dai and Zhong (2010) conduct numerical analysis of the penalty method for portfolio selection with transaction costs, and Huang and Forsyth (2012) present a convergence proof of the penalty method for the valuation of guarantee minimum withdrawal benefits using the robust framework developed by Barles and Souganidies (1991).

The framework developed by Barles and Souganidies (1991) makes use of the notion of viscosity solutions and is able to prove the convergence of any consistent, monotone, and stable numerical scheme for general fully nonlinear partial differential equations (PDEs), provided that comparison principle holds in the sense of viscosity solutions. However, for the portfolio section problem with capital gains tax, the associated HJB equation problem has infinitely many solutions, which indicates the failure of comparison principle. As a consequence, one could not directly apply the framework of Barles and Souganidies (1991).

The remainder of the paper is organized as follows. In the next section, we present the portfolio selection model with capital gains tax proposed by Ben Tahar, Soner and Touzi (2010), and address the reason that the associated HJB equation problem admits multiple solutions. In particular, we present a class of analytical solutions which, however, are not the corresponding value function. In Section 3, we conduct theoretical analysis and show why the penalty method works. We first present a regular control problem, where the admissible investment strategies have bounded speed of trading, and the resulting HJB equation is a penalty approximation to the original HJB equation. We show that comparison principle holds for the penalty approximation problem, and any singular control can be approximated by a regular control. We then infer that the value function associated with the singular control problem is the minimum viscosity solution of the original HJB equation, which can be solved by the penalty method for the penalty approximation problem. In Section 4, we propose appropriate boundary conditions for the penalty method to ensure its convergence. Numerical results are presented to demonstrate the efficiency of the penalty method and to better understand optimal strategy. We conclude in Section 5. All technical proofs are in Appendix and E-Companion.

2. Mathematical Formulation

In this section, we first present a mathematical formulation for the continuous time investment and consumption problem with capital gains tax established by Ben Tahar, Soner and Touzi (2010), and then elaborate on the non-uniqueness of solution to the resulting HJB equation problem.

2.1. The Market

Assume that there are two assets that an investor can trade without any transaction cost. The first asset ("the bond") is a bank account growing at a continuously compounded after-tax interest rate r. The second asset ("the stock") is a risky investment and its price S_t follows

$$dS_t = \mu S_t dt + \sigma S_t d\mathcal{B}_t, \qquad S_0 = 1,$$

where μ and σ are constants with $\mu > r$, and $\mathcal{B}_t$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$ with $\mathcal{B}_0 = 0$ almost surely. Note that the initial share price is normalized to \$1.

The investor is subject to capital gains tax. We assume that (i) capital gains can be realized immediately after sale, (ii) there is no wash sale restriction, (iii) short selling is prohibited, and (iv) the tax basis used to evaluate capital gains is defined as the weighted average of past purchase prices.

Let x_t, y_t, and k_t be the amount in the bank account, the current dollar value of, and the purchase price of stock holdings, respectively. We introduce two right-continuous (with left limits), nonnegative, and nondecreasing $\{\mathscr{F}_t\}_{t\geqslant 0}$-adapted processes L_t and M_t with $L_0 = M_0 = 0$, where dL_t represents the *dollar amount* transferred from the bank to the stock account at time t (corresponding to a purchase of stock), while $dM_t \leqslant 1$ represents the *proportion of shares* transferred from the stock account to the bank at time t (corresponding to a sale of stock). Bear in mind that the average tax basis is used to evaluate capital gains. Hence, when one sells stock at time t, the purchase price k_t declines by the same proportion dM_t as the dollar value of stock holdings y_t does. As such, the evolution processes of x_t, y_t, and k_t are

$$\begin{cases} dx_t = (rx_{t-} - c_t)dt - dL_t + [y_{t-} - \alpha(y_{t-} - k_{t-})]dM_t, \\ dy_t = \mu y_{t-}dt + \sigma y_{t-}d\mathcal{B}_t + dL_t - y_{t-}dM_t, \\ dk_t = dL_t - k_{t-}dM_t, \end{cases} \tag{2.1}$$

where α and $c_t \geq 0$ are the tax rate and the consumption rate, respectively.

Remark 2.1. If transaction costs are incurred, the evolution equation for x_t becomes

$$dx_t = (rx_{t-} - c_t)dt - (1+b)dL_t + (1-\hat{b})[y_{t-} - \alpha(y_{t-} - k_{t-})]dM_t,$$

where $b \in [0,\infty)$ and $\hat{b} \in [0,1)$ are transaction cost rates. From mathematical analysis point of view, including transaction cost usually makes problems more regular. In this paper, we exclude transaction costs, though the analysis presented here can be carried over to the transaction costs case.

Remark 2.2. When $y_{t-} - k_{t-} < 0$, the investor faces a capital loss and the term $-\alpha(y_{t-} - k_{t-})dM_t$ means a tax rebate.

Remark 2.3. The investor may perform a simultaneous sell and buy transaction. For example, as shown in Ben Tahar, Soner and Touzi (2010), when $y_{t-} < k_{t-}$, making a wash-sale is optimal, i.e., selling all stock holding to realize tax rebate and immediately buying a certain amount of stock. Note that the action of first buying dL_t amount of stock and then selling dM_t portion of stock holding can be realized by the action of first selling dM_t portion of stocking holding and then buying $(1-dM_t)dL_t$ amount of stock. Thus, any successive series of simultaneous sell and buy actions can be simplified by a single sell-buy action. Without loss of generality, we always assume that in the case of the simultaneous sell and buy, sell precedes buy, which is precisely described by (2.1).

2.2. The investor's Problem

Due to capital gains tax, we denote by $z_t := x_t + (1-\alpha)y_t + \alpha k_t$ the investor's after-tax wealth at time t. It is helpful to notice that although we can trade discontinuously, the after-tax wealth process z_t evolves continuously. Since the after-tax wealth is required to be non-negative, we define the solvency region to be the closure $\bar{D}$ of D, where

$$D = \{(x,y,k) \mid y > 0, k > 0, z := x + (1-\alpha)y + \alpha k > 0\}.$$

Assume that the investor is given an initial position in $\bar{D}$. An investment and consumption strategy $\{(c_t, L_t, M_t)\}_{t\geq 0}$ is admissible for (x,y,k) starting from time 0, if (x_t, y_t, k_t) given by (2.1) with $(x_0, y_0, k_0) = (x,y,k)$ is in $\bar{D}$ for all $t \geq 0$. The investor's problem is to choose an admissible strategy to maximize the intertemporal consumption, i.e.,

$$\mathbb{E}\left[\int_0^\infty \mathbf{U}(c_t,t)dt\right],$$

where $\mathbf{U}(c_t,t)=e^{-\beta t}U(c_t)$ with utility function $U(\cdot)$. We will focus on the following CRRA utility with relative risk aversion level $p\in(0,1)$, i.e.,

$$U(c)=\frac{c^p}{p}. \tag{2.2}$$

We always assume

$$\beta>\beta_p:=p\Big[r+\frac{(\mu-r)^2}{2(1-p)\sigma^2}\Big]. \tag{2.3}$$

2.3. Value function and HJB equation

Note that to guarantee solvency, the portfolio has to be liquidated once $z_t=0$ for some $t\geq 0$. This motivates us to introduce a stopping time for any $\mathbf{x}:=(x,y,k)\in\bar{D}$,

$$\tau_{\mathbf{x}}:=\sup\{t>0\mid z_s>0\ \forall\, s\in[0,t)\}; \tag{2.4}$$

here $\tau_{\mathbf{x}}=0$ if $z_{0-}=0$. Hence we can instead consider the investor's problem by stopping time τ by assuming the portfolio is automatically liquidated at time $\tau_{\mathbf{x}}$. In this way, every strategy in the set

$$\mathcal{A}=\{(C,L,M):=\{(c_t,L_t,M_t)\}_{t\geq 0}\mid c_t\geqslant 0,\quad dL_t\geqslant 0,\quad 0\leqslant dM_t\leqslant 1,\ t\geq 0\}$$

becomes admissible. We then define the value function:

$$V_*(\mathbf{x})=\sup_{(C,L,M)\in\mathcal{A}}\mathbb{E}\left[\int_0^{\tau_{\mathbf{x}}}\mathbf{U}(c_t,t)dt\right]\quad\forall\,\mathbf{x}\in\bar{D}; \tag{2.5}$$

For later use, we divide the boundary ∂D of $\bar{D}$ into three parts:

$$\begin{aligned}\Gamma_1&=\{(x,y,k)\in\partial D\mid z=0\},\\ \Gamma_2&=\{(x,y,k)\in\partial D\mid y=0\},\\ \Gamma_3&=\{(x,y,k)\in\partial D\mid k=0\}.\end{aligned}$$

Since $\tau_{\mathbf{x}}=0$ when $z=0$, we have

$$V_*(\mathbf{x})=0\quad\forall\,\mathbf{x}\in\Gamma_1.$$

Ben Tahar, Soner and Touzi (2007, 2010) prove the following proposition, which indicates that the value function defined above is a viscosity solution to the corresponding HJB equation.

Proposition 2.1. *Let V_* be as defined in* (2.5)*. Then V_* is a viscosity solution of*

$$\mathscr{F}_*[u]=0 \quad in \quad D, \quad -\mathscr{F}_*[u]\leqslant 0 \ on \ \Gamma_2\cup\Gamma_3, \quad u=0 \ on \ \Gamma_1, \tag{2.6}$$

where[6]

$$\begin{aligned}\mathscr{F}_*[u] &= \max\{\mathscr{L}u+U^*(u_x),\ \mathscr{B}u,\ \mathscr{S}u\},\\ \mathscr{L}u &= \tfrac{1}{2}\sigma^2y^2u_{yy}+\mu y u_y+rxu_x-\beta u, \quad U^*(s)=\sup_{c\geqslant 0}\Big\{U(c)-cs\Big\} \quad \forall s\geqslant 0,\\ \mathscr{B}u &= -u_x+u_y+u_k, \quad \mathscr{S}u=[(1-\alpha)y+\alpha k]u_x-yu_y-ku_k.\end{aligned} \tag{2.7}$$

With the boundary condition $-\mathscr{F}_*[u]\leqslant 0$ on $\Gamma_2\cup\Gamma_3$, the viscosity solution of $\mathscr{F}_*[u]=0$ in D is often referred to as the *state-constraint* or *constrained* viscosity solution.

2.4. Non-uniqueness of Viscosity Solution to the HJB equation

It has been shown in Ben Tahar, Soner and Touzi (2010) that the value function V_* is bounded from above by the Merton's solution (i.e., the value function in the absence of capital gains tax) and from below by a function associated with a suboptimal strategy, namely

$$A_Lz^p\le V_*(\mathbf{x})\le A_Mz^p, \tag{2.8}$$

where $A_M=\frac{1}{p}[\frac{1}{1-p}(\beta-p(r+\frac{(\mu-r)^2}{2\sigma^2(1-p)}))]^{p-1}$, and $A_L=\frac{1}{p}[\frac{1}{1-p}(\beta-p(r+\frac{(\mu-\frac{r}{1-\alpha})^2}{2\sigma^2(1-p)}))]^{p-1}$. The upper bound indicates that an investor cannot take advantage of tax rebate to be better off.

It is not hard to verify that the function

$$\hat{V}(\mathbf{x}):=Az^p, \text{ for any constant } A\ge A_M \tag{2.9}$$

is a classical solution of (2.6). One can further verify that $\hat{V}$ as given above is indeed a state-constraint viscosity solution of (2.6).[7] This indicates that (2.6) admits infinitely many viscosity solutions.

The reason for such non-uniqueness is that for any function u, one has $\mathscr{S}u=0$ on $\Gamma_2\cap\Gamma_3=\{y=k=0\}$ and as such,

$$-\mathscr{F}_*[u]=-\max\{\mathscr{L}u+U^*(u_x),\ \mathscr{B}u,\ \mathscr{S}u\}\le-\mathscr{S}u=0.$$

Hence, the boundary condition $-\mathscr{F}_*[u] \le 0$ at $y = k = 0$, as given in (2.6), does not provide any information there, which makes solution non-unique.

Since the constant A as given in (2.9) is bigger than A_M, one may question whether the value function would be the unique non-trivial solution to the HJB equation problem (2.6) by imposing the Mertons' solution as an upper bounded. We will numerically show in Section 4.2 that it is not true, because the HJB equation problem (2.6) has infinitely many non-trivial solutions, bounded from above by the Mertons' solution.

As a consequence, it is necessary to find a criterion to identify the right solution of the HJB equation problem that corresponds to the value function. Since analytical solutions are generally unavailable, it is also desirable to design an efficient numerical method to solve for the value function. Later we can see that the two targets can be achieved in a unified framework.

3. Penalty Approximation

Problem (2.5) is known as a singular control problem, because the state processes (x_t, y_t, k_t) are likely discontinuous due to control effort. One objective of this paper is to show that a singular control problem can be approximated by a regular control problem whose HJB equation is a penalty approximation to the original HJB equation.

3.1. A Regular Control Problem

Define a subset of admissible strategies

$$\mathcal{A}_\lambda = \{(C, L, M) \in \mathcal{A} \mid \; dL_t = l_t z_t dt, \quad dM_t = m_t dt, \quad 0 \le l_t, m_t \le \lambda\}, \tag{3.1}$$

where $\lambda > 0$ is a constant measuring the maximum rate of trading. We consider the following control problem restricted to $\mathcal{A}_\lambda$:

$$V(\mathbf{x}; \lambda) = \sup_{(C,L,M) \in \mathcal{A}_\lambda} \mathbb{E}\left[\int_0^{\tau_{\mathbf{x}}} \mathbf{U}(c_t, t) dt\right] \quad \forall \mathbf{x} \in \bar{D}. \tag{3.2}$$

Note that the above control problem is a regular control problem, which, formally, is associated to a penalty approximation to the original HJB equation problem (2.6), that is

$$\mathscr{F}[u, \lambda] = 0 \;\text{ in }\; D, \quad -\mathscr{F}[u, \lambda] \leqslant 0 \text{ on } \;\Gamma_2 \cup \Gamma_3, \quad u = 0 \text{ on } \;\Gamma_1, \tag{3.3}$$

with $\mathscr{F}[u, \lambda] =: \mathscr{L} u + U^*(u_x) + \lambda z (\mathscr{B} u)^+ + \lambda (\mathscr{S} u)^+$.

Different from the HJB equation problem (2.6), the penalty approximation problem (3.3) turns out to possess a unique (state-constraint) viscosity solution, and the value function V is nothing but the unique solution. Indeed, we are able to show that the comparison principle holds in the sense of viscosity solution for the penalty approximation problem (3.3), which yields the uniqueness of solution. Intuitively, in contrast to the singularity of problem (2.6) that provides no information at $y = k = 0$, the boundary condition of the penalty problem (3.3) at $y = k = 0$ is reduced to

$$-[\mathscr{L}u + U^*(u_x) + \lambda z(\mathscr{B}u)^+] \leq 0,$$

which implies that one should either buy stock or take no action when all money is in bank account. This explains why the uniqueness of solution holds for the penalty problem (3.3) but not for the original problem (2.6).

We now summarize the above results as a proposition.

Proposition 3.1. *Consider the penalty approximation problem* (3.3)*, where* λ *is a positive constant.*

(i) (Comparison Principle) Let $\mathcal{C}$ *be defined as*

$$\mathcal{C} := \left\{ u \,\middle|\, \sup_{(x,y,k)\in D} \frac{|u(x,y,k)|}{|x + (1-\alpha)y + \alpha k|^p} < \infty \right\}. \tag{3.4}$$

If $u \in \mathcal{C}$ *is a state-constraint viscosity subsolution,* $v \in \mathcal{C}$ *is a state-constraint viscosity supersolution, and* v *is continuous, then* $u \leqslant v$ *on* $\bar{D}$*. Consequently,* (3.3) *admits at most one state-constraint viscosity solution in* $\mathcal{C}$*.*

(ii) The value function $V(\cdot;\lambda)$ *as defined in* (3.2) *is the unique (state-constraint) viscosity solution of* (3.3)*.*

The proof is placed in E-Companion EC.1. A critical step is to prove the continuity of the value function $V(\cdot;\lambda)$, because we use the definition of continuous viscosity sub-/super-solutions. It is worthwhile pointing out that if the upper (lower) semicontinuous envelopes are employed to define viscosity subsolution (supersolution), we would face a technical obstacle when proving the corresponding comparison principle.

3.2. Approximating Singular Control via Regular Control

Proposition 3.1 indicates that one can find the value function $V(\cdot;\lambda)$ by numerically solving the penalty problem (3.3). A natural question is whether $V(\cdot;\lambda)$ converges to the original value function $V_*(\cdot)$. The following theorem gives a positive answer, and moreover, $V_*(\cdot)$ proves to be the minimum state-constraint viscosity solution to the original HJB equation problem (2.6).

THEOREM 3.1. *Let $V(\mathbf{x};\lambda)$ and $V_*(\mathbf{x})$ be the value functions as defined in* (3.2) *and* (2.5)*, respectively. Then*

(i) $\lim_{\lambda\to\infty} V(\mathbf{x};\lambda) = V_*(\mathbf{x}) \quad \forall \mathbf{x}\in\bar{D}$.

(ii) $V_*(\mathbf{x})$ *is the minimum state-constraint viscosity solution to (2.6).*

Proof of part (ii): Note that any viscosity solution $v(\mathbf{x})$ to (2.6) must be a supersolution to (3.3). By the comparison principle for problem (3.3) (i.e., part (i) in Proposition 3.1), we have

$$v(\mathbf{x}) \geq V(\mathbf{x};\lambda) \quad \forall \lambda > 0. \tag{3.5}$$

Sending $\lambda\to\infty$ in (3.5) and combining with part (i), we have

$$v(\mathbf{x}) \geq \lim_{\lambda\to+\infty} V(\mathbf{x};\lambda) = V_*(\mathbf{x}),$$

which yields the desired result as we have shown in Proposition 2.1 that $V_*(\cdot)$ is a viscosity solution to (2.6).

The proof of part (i) is quite technical and is placed in E-Companion EC.2. For completeness, we present a sketched proof in Appendix A. The key point is to explicitly construct a sequence of regular controls to approximate any singular control. We emphasize that our construction approach is general and can be extended to other singular control problems. Hence, we essentially show that any singular control problem can be approximated by a series of regular control problems.

In summary, despite that the value function of the singular control problem cannot be determined by the associated HJB equation problem for which the comparison principle fails, we are able to instead solve its penalty approximation for which the comparison principle holds, and the solution does converges to the value function as the penalty parameter goes to infinity. Moreover, our theoretical analysis reveals that the value function is the minimum viscosity solution to the original HJB equation problem.

4. Numerical Analysis

In this section, we propose a penalty method, namely, a finite difference scheme for the penalty approximation problem (3.3). Numerical results are presented to demonstrate the efficiency of this method and examine the optimal strategy.

4.1. Numerical Scheme

Theorem 3.1 allows us to instead solve for the value functions $V(\mathbf{x};\lambda)$. Since both $V(\mathbf{x};\lambda)$ and $V_*(\mathbf{x})$ are homogeneous of degree p in $\mathbf{x}$, we make the following similarity reduction

$$V(\mathbf{x};\lambda) = z^p W(\eta,\xi;\lambda), \quad V_*(\mathbf{x}) = z^p W_*(\eta,\xi), \quad \eta = \frac{\alpha k}{z}, \quad \xi = \frac{(1-\alpha)y}{z}. \tag{4.1}$$

It is easy to verify that $W(\eta,\xi;\lambda)$ is the unique constrained viscosity solution to the two-dimensional penalized PDE

$$\mathscr{F}_1[W,\lambda] := \mathscr{L}_1 W + (\frac{1}{p}-1)\,(pW - \eta W_\eta - \xi W_\xi)^{\frac{p}{p-1}} + \lambda(\mathscr{B}_1 W)^+ + \lambda(\mathscr{S}_1 W)^+ = 0 \tag{4.2}$$

in $\eta > 0,\ \xi > 0$, with boundary conditions

$$-\mathscr{F}_1[W,\lambda] \le 0 \tag{4.3}$$

on $\eta = 0$ or $\xi = 0$, where

$$\mathscr{B}_1 W := (1-\alpha)W_\xi + \alpha W_\eta, \quad \mathscr{S}_1 W := -\xi W_\xi - \eta W_\eta, \tag{4.4}$$

and

$$\begin{aligned}\mathscr{L}_1 W :=& \frac{1}{2}\sigma^2\xi^2\left(\eta^2 W_{\eta\eta} - 2\eta(1-\xi)W_{\xi\eta} + (1-\xi)^2 W_{\xi\xi}\right)\\ &+ \left(-r(1-\xi-\eta) + \mu(1-\xi) - (1-p)\sigma^2\xi(1-\xi)\right)\xi W_\xi\\ &+ (\sigma^2\gamma\xi^2 - r(1-\xi-\eta) - \mu\xi)\eta W_\eta + \left(-\beta + rp(1-\xi-\eta) + p\mu\xi - \frac{1}{2}p\gamma\sigma^2\xi^2\right)W.\end{aligned}$$

Since a finite difference scheme only works with a bounded region, we restrict attention to a truncated, triangle domain $\widetilde{D}_{M_0} = \{(\eta,\xi) | 0 < \eta < \frac{\alpha}{1-\alpha}M_0\xi,\ 0 < \xi < M_0\}$, where $M_0 > 1$ is a constant. We then apply the finite difference discretization to the penalized PDE (4.2) in $\widetilde{D}_{M_0}$ with the following artificial boundary conditions

$$\begin{cases} -\mathscr{B}_1 W = 0 \quad \text{on } \eta = 0, & (4.5)\\ -\mathscr{B}_1 W = 0 \quad \text{on } \eta = \frac{\alpha}{1-\alpha}M_0\xi, & (4.6)\\ -\mathscr{S}_1 W = 0 \quad \text{on } \xi = M_0, & (4.7)\end{cases}$$

where (4.5)-(4.7) imply that one should sell stock on $\xi = M_0$ and buy stock on $\eta = 0$ and on $\eta = \frac{\alpha}{1-\alpha} M_0 \xi$. The detailed scheme is presented in Appendix B.

It is worthwhile pointing out that the comparison principle holds for the PDE (4.2) in $\widetilde{D}_{M_0}$ with boundary conditions (4.5)-(4.7) in the sense of viscosity solution (cf. Dupuis and Ishii 1990). Hence, by standard arguments in Barles and Souganidies (1991), Huang and Forsyth (2012), and Dai and Zhong (2010), we are able to show that for fixed M_0 and λ, the finite difference scheme is convergent to the corresponding viscosity solution, denoted by $\widetilde{W}(\eta, \xi; \lambda, M_0)$. We can further show that $\widetilde{W}(\eta, \xi; \lambda, M_0)$ converges to $W_*(\eta, \xi)$ as M_0 and λ go to infinity (See Appendix C).

To characterize optimal strategy, we define the buy region **BR**, sell region **SR**, and the no-trading region **NTR** as follows:

$$\mathbf{BR} = \{(\eta, \xi) \in \widetilde{D}_{M_0} : \mathscr{B}_1 W = 0\},$$
$$\mathbf{SR} = \{(\eta, \xi) \in \widetilde{D}_{M_0} : \mathscr{S}_1 W = 0\},$$
$$\mathbf{NTR} = \{(\eta, \xi) \in \widetilde{D}_{M_0} : \mathscr{B}_1 W < 0,\ \mathscr{S}_1 \hat{W} < 0\},$$

which imply that the optimal strategy is to sell stock in **SR**, buy stock in **BR**, and take no action in **NTR**. Note that the intersection of **SR** and **BR** is the wash-sale region in which one should first liquidate all stock holdings and then buy stock.

4.2. Numerical Results

Theoretically M_0 should be big enough to ensure convergence. In computation, we only need to choose $M_0 > 1$ such that $\xi = M_0$ is contained in the sell region. Let us first examine the impact of λ on numerical solutions for a fixed M_0.

4.2.1. Impact of λ The default parameter values are $r = 0.01$, $\alpha = 0.35$, $\sigma = 0.3$, $\mu = 0.05$, and $\beta = 0.2$. We fix $M_0 = 2$, and take the numerical solution with $\lambda = 10^9$ as the benchmark. We then compute relative errors between numerical solutions and the benchmark solution, measured in $\mathbb{L}_2$ norm. Table 1 reports the relative errors against λ for the case $p = 0.1$ (the upper panel) and the case $p = 0.5$ (the lower panel). It can be seen that the errors apparently tend to zero as λ increases, which implies the convergence of numerical results with penalty parameter.

Note that along $y = k$ or equivalently $\eta = \frac{\alpha}{1-\alpha}\xi$, trading does not incur capital gains tax. As a consequence, the reduced value function $W_*(\eta, \xi)$ must be constant on $\eta = \frac{\alpha}{1-\alpha}\xi$.

λ	10^1	10^2	10^3	10^4	10^5	10^6	10^7
				$p=0.1$			
Relative error	2.0×10^{-4}	2.4×10^{-5}	3.1×10^{-6}	3.4×10^{-7}	3.4×10^{-8}	3.4×10^{-9}	3.4×10^{-10}
				$p=0.5$			
Relative error	7.4×10^{-4}	1.0×10^{-4}	1.5×10^{-5}	1.6×10^{-6}	1.7×10^{-7}	1.7×10^{-8}	1.7×10^{-9}

Table 1 Relative errors against the penalty parameter λ for risk aversion level $p=0.1$ (upper panel) and $p=0.5$ (lower panel), respectively. The default parameter values are $r=0.01$, $\alpha=0.35$, $\sigma=0.3$, $\mu=0.05$, $\beta=0.2$, and $M_0=2$. The relative error, measured in $\mathbb{L}_2$ norm, is defined as $\frac{1}{\sqrt{\frac{1}{(n+1)^2}\sum_{0\le i,j\le n}\hat{W}(Z_{i,j};\Lambda,M_0)^2}}\sqrt{\frac{1}{(n+1)^2}\sum_{0\le i,j\le n}[\hat{W}(Z_{i,j};\lambda,M_0)-\hat{W}(Z_{i,j};\Lambda,M_0)]^2}$, where $\hat{W}(Z_{i,j};\lambda,M_0)$ refers to the numerical solution at the mesh point $Z_{i,j}$, $0\le i,j\le n$ for given λ and M_0, and the case $\lambda=\Lambda\equiv10^9$ is used as the benchmark.

However, for a fixed λ, numerical solutions are unlikely constant on $\eta=\frac{\alpha}{1-\alpha}\xi$. For different λ, we report in Table 2 the mean value and standard deviation of numerical solutions on the line $\eta=\frac{\alpha}{1-\alpha}\xi$ for two batches of parameter values, and as a comparison, we also report A_M and A_L which, according to (2.8) and (4.1), are the theoretical upper bound and lower bound of numerical solutions, respectively for sufficiently large λ. It can be seen that as we expect, the standard deviation shrinks and the mean value converges when λ increases. Moreover, the mean values fall within the interval $[A_L,A_M]$ when λ is large.

4.2.2. Optimal Strategy Figure 1 presents the optimal strategy in the $\xi-\eta$ plane (left) and in the $\xi-b$ plane (right), respectively, where $b=k/y$ represents the basis-price ratio, and the default parameter values are $r=0.01$, $\alpha=0.35$, $\sigma=0.3$, $p=0.1$, $\mu=0.05$, $\beta=0.2$, $\lambda=10^6$, and $M_0=1.2$. Without loss of generality, we always elaborate on optimal strategy in the $\xi-b$ plane, for the ease of comparison with those numerical results presented in Dai et al. (2015) and Cai, Chen and Dai (2018).

It can be seen from the right part of Figure 1 that $\{b>1\}$ always belongs to the wash sale region, which indicates that investors should wash-sell stock to receive tax rebates whenever there are capital gains losses. When there are capital gains profits, i.e. $b<1$, investors have incentive to defer realization of capital gains due to interest consideration, and we can observe that for a given $b<1$, there exist two boundaries $\xi_+(b)$ and $\xi_-(b)$ such that the sell region (**SR**), the buy region (**BR**), and the no-trading region (**NTR**) in $b<1$ can be described as follows using $\xi^+(b)$ and $\xi_-(b)$,

$$\mathbf{SR}\cap\{b<1\}=\{(\xi,b):\ \xi\ge\xi_+(b),\quad b<1\},$$

λ	10^{-2}	10^{0}	10^{2}	10^{4}	10^{6}	A_M	A_L
			$p=0.1$				
Mean	38.4069	38.9745	39.0306	39.0315	39.0315	39.0657	39.0217
Sd	(3.1×10^{-1})	(6.4×10^{-2})	(4.9×10^{-4})	(4.9×10^{-6})	(4.9×10^{-8})		
			$p=0.5$				
Mean	3.2037	3.2524	3.2624	3.2626	3.2627	3.2782	3.2587
Sd	(1.9×10^{-2})	(9.2×10^{-3})	(1.5×10^{-4})	(1.5×10^{-6})	(1.5×10^{-8})		

Table 2 Mean values and standard deviations of the numerical solution on $\eta=\frac{\alpha}{1-\alpha}\xi$ against λ for risk aversion level $p=0.1$ (upper panel) and $p=0.5$ (lower panel), respectively. The mean value and standard deviation are defined as $\mathrm{M}(\hat{W})\equiv\frac{1}{\#\{Z_{i,j}|Z_{i,j}\in\ \eta=\frac{\alpha}{1-\alpha}\xi\}}\sum_{Z_{i,j}\in\ \eta=\frac{\alpha}{1-\alpha}\xi}\hat{W}(Z_{i,j};\lambda,M_0)$ and $\sqrt{\frac{1}{\#\{Z_{i,j}|Z_{i,j}\in\ \eta=\frac{\alpha}{1-\alpha}\xi\}}\sum_{Z_{i,j}\in\ \eta=\frac{\alpha}{1-\alpha}\xi}[\hat{W}(Z_{i,j};\lambda,M_0)-\mathrm{M}(\hat{W})]^2}$, respectively, where $\hat{W}(Z_{i,j};\lambda,M_0)$ refers to the numerical solution at the mesh point $Z_{i,j}$, $0\le i,j\le n$ for given λ and M_0, and $\#\{Z_{i,j}|Z_{i,j}\in\ \eta=\frac{\alpha}{1-\alpha}\xi\}$ represents the number of grid points on $\eta=\frac{\alpha}{1-\alpha}\xi$. The default parameter values are $r=0.01$, $\alpha=0.35$, $\sigma=0.3$, $\mu=0.05$, $\beta=0.2$, and $M_0=2$.

$$\mathbf{BR}\cap\{b<1\}=\{(\xi,b):\ \xi\le\xi_-(b),\quad b<1\},$$
$$\mathbf{NTR}\cap\{b<1\}=\{(\xi,b):\ \xi_-(b)<\xi<\xi_+(b),\quad b<1\}.$$

Here $\xi^+(b)$ and $\xi_-(b)$ are called the optimal sell boundary and buy boundary, respectively. The shape of these regions reflects the trade-off between the benefit of tax deferral and the cost of sub-optimal risk exposure.

For a given $b<1$, when the fraction of wealth in stock is higher (lower) than the sell (buy) boundary, investors will immediately sell (buy) the minimum amount to reach the sell (buy) boundary. The trading direction in the sell region **SR** is vertically downward in the $\xi-b$ plane (e.g. A to B) because the basis-price b does not change upon sale; the trading direction in the buy region **BR** is upward but is not vertical (e.g. C to D) because the basis-price ratio b increases upon purchase provided there is a capital gain profit.

It is interesting to note that the buy and sell boundaries intersect with $b=1$ at a unique point, denoted by $(1,\xi^*)$, that is,

$$\lim_{b\to1-}\xi_+(b)=\lim_{b\to1-}\xi_-(b)=:\xi^*,$$

where ξ^* represents the initial tax-adjusted optimal fraction of wealth in stock if the initial endowment is all in the bank account. The fraction is also related to the wash sale strategy:

in the wash sale region $b > 1$, investors will first liquidate all stock holdings, then rebalance to the tax-adjusted fraction ξ^* (e.g. E vertically to the line $\xi = 0$, then to $(1, \xi^*)$).

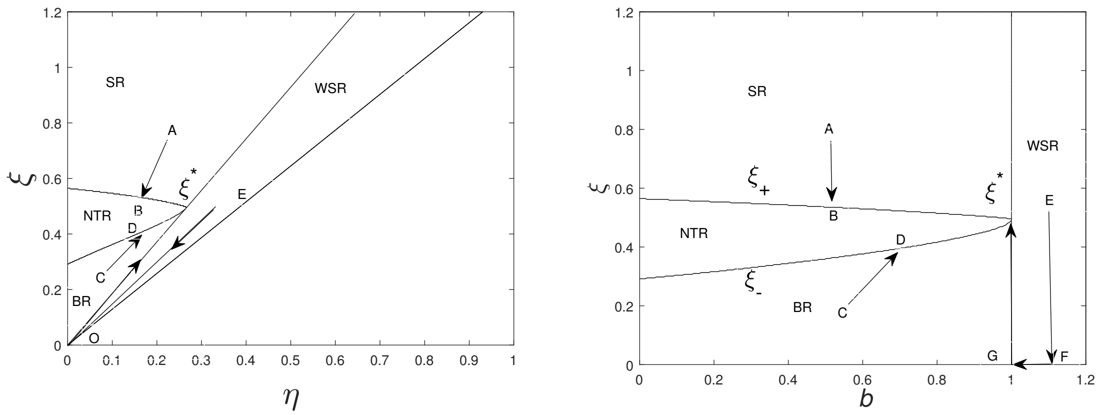

Figure 1 Sell region, buy region, and no-trading region in the $\eta - \xi$ plane (left) and in the $b - \xi$ plane (right). The arrows indicate the direction of optimal portfolio adjustment. In particular, in wash sell region, the investor first liquidates all stock, then buys stock along the dashed line. The parameter values are $r = 0.01$, $\alpha = 0.35$, $\sigma = 0.3$, $p = 0.1$, $\mu = 0.05$, $\beta = 0.2$, $\lambda = 10^6$, and $M_0 = 1.2$.

For a given λ, the resulting optimal strategy is associated with the portfolio choice problem with bounded transaction speed in the presence of capital gains tax. In Figure 2, we plot the sell boundaries and buy boundaries for different λ. It is found that as λ increases, the buy boundary and sell boundary move upwards, which implies that investors are inclined to invest more in stock when the transaction speed constraint is relaxed.

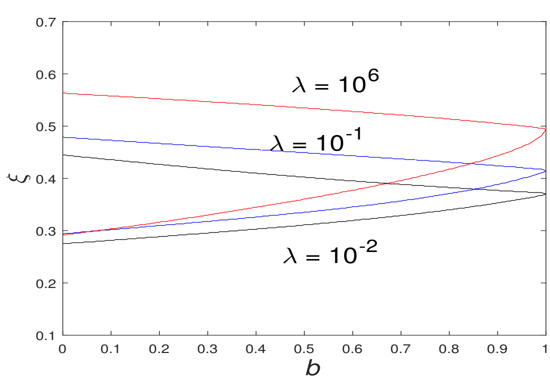

Figure 2 Optimal sell and buy boundaries for different λ. The default parameter values are $r = 0.01$, $\alpha = 0.35$, $\sigma = 0.3$, $\mu = 0.05$, $\beta = 0.2$, $p = 0.1$, $M_0 = 2$. This figure indicates that investors are inclined to invest more in stock when the transaction speed constraint is relaxed.

4.2.3. Sub-optimal Strategy, Non-admissible Strategy, and Non-trivial Solution We have presented some trivial solutions (2.9) to the HJB equation problem (2.6). Now we would like to numerically verify the existence of non-trivial solutions to the problem, which can help us better understand optimal strategy.

The numerical verification is based on two observations aforementioned: (i) the reduced value function $W_*(\eta,\xi)$ is constant on $\eta=\frac{\alpha}{1-\alpha}\xi$ or equivalently on $b=1$; and (ii) the optimal buy boundary and sell boundary intersect at a unique point on $b=1$. The observations motivate us to restrict attention to the region $\{\xi\geq 0,\eta\leq\frac{\alpha}{1-\alpha}\xi\}$ and impose a *constant* Dirichlet condition on $\eta=\frac{\alpha}{1-\alpha}\xi$, namely,

$$W(\eta,\xi)=A_0 \text{ on } \ \eta=\frac{\alpha}{1-\alpha}\xi, \tag{4.8}$$

where $A_0\in[A_L,A_M)$ is a given constant. We choose $M_0=1$, replace the boundary condition (4.6) by (4.8), and other default parameter values are $r=0.01$, $\alpha=0.35$, $\sigma=0.3$, $\mu=0.05$, $\beta=0.2$, $p=0.1$, and $\lambda=10^8$. Once numerical solutions are obtained, we can similarly define **BR**, **SR**, and **NTR** as before.

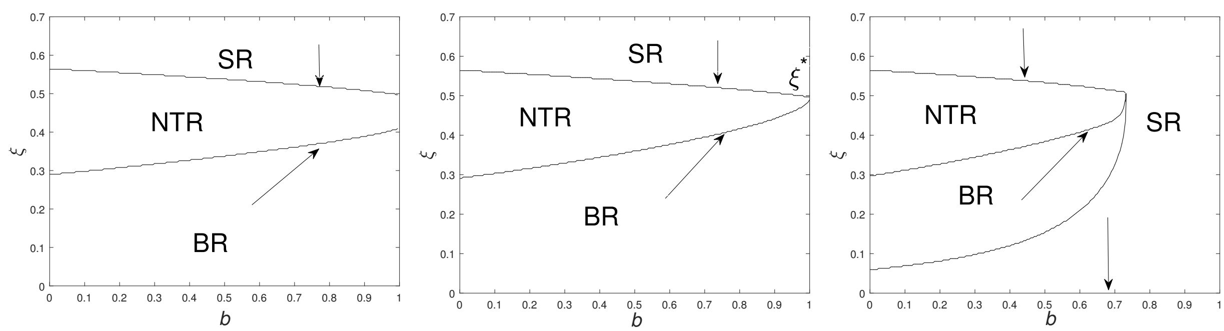

Figure 3 Optimal strategy for different A_0 (the left: $A_0=39.0216$, the middle: $A_0=39.0315$ and the right:$A_0=39.0373$). The default parameter values are $r=0.01$, $\alpha=0.35$, $\sigma=0.3$, $\mu=0.05$, $\beta=0.2$, $p=0.1$, $M_0=1$, and $\lambda=10^8$.

Figure 3 presents the **BR**, **SR**, and **NTR** in the $b-\xi$ plane for three different values of A_0, where the middle sub-figure stands for the case in which $A_0=39.0315$ is the proper value associated with the singular control problem on $b=1$, while the left (right) sub-figure stands for the case in which A_0 is lower (larger) than the proper value.

The left sub-figure suggests a strategy: the buy and sell boundaries intersect with $b=1$ at two points. The strategy is admissible because given an initial endowment, one can immediately rebalance to the crossing point of the buy boundary and $b=1$, and there will be a chance of entering the no-trading region. Thus, the strategy is a sub-optimal strategy. It is not surprising that the solution associated with the sub-optimal strategy is lower than the true value function not only on $b=1$ but also in $b<1$.[8] Hence, to obtain an optimal strategy, we may raise the value of A_0 such that the buy and sell boundaries intersect with $b=1$ at a unique point (i.e., as shown in the middle sub-figure).

In the right sub-figure, given A_0 bigger than the proper value, the buy and sell boundaries do not intersect with $b=1$. Unfortunately, this strategy is not admissible. Indeed, as we can see, (b,ξ) is always in the sell region provided b is sufficiently close to 1, which indicates that given an initial endowment, one must always stay on $b=1$ and would have no chance to enter the no-trading region. Hence, the solution does not correspond to any admissible strategy. This abnormal strategy comes from the higher value imposed on $b=1$, implying that an admissible strategy never leads to a higher utility than the optimal strategy does.

The solution associated with the right sub-figure can be trivially extended to the region $b>0$ by setting $W=A_0$ for $b>1$. Since the extended solution is sufficiently smooth for $b\geq 1$, it must be a non-trivial solution to the HJB equation problem (2.6). The non-trivial solution is always bigger than the value function in the whole solution region.

4.2.4. An Alternative Numerical Scheme Figure 3 and the above numerical analysis suggest the following alternative numerical scheme restricted to the region $\eta\leq\frac{\alpha}{1-\alpha}\xi$ (i.e., the region $b\leq 1$).

Step 1. Set $A_0=A_L$.

Step 2. Solve the PDE problem (4.2) in the region $\eta\leq\frac{\alpha}{1-\alpha}\xi$ with (4.5), (4.7), and the imposed Dirichlet boundary condition (4.8). Here either the projected SOR approach or the penalty method may be used.

Step 3. Stop if the resulting buy and sell boundaries intersect with $\eta=\frac{\alpha}{1-\alpha}\xi$ at one point.

Step 4. If the resulting buy and sell boundaries intersect with $\eta=\frac{\alpha}{1-\alpha}\xi$ at two points, we raise the value of A_0; If the resulting buy and sell boundaries do not intersect with $\eta=\frac{\alpha}{1-\alpha}\xi$, we reduce the value of A_0; Go to Step 2.

Our numerical experiments show that the above numerical scheme is convergent and yields the same numerical results as presented above.

5. Summary

The penalty method has been widely used to numerically solve the HJB equation problem associated with singular control problems. However, existing literature requires comparison principle to guarantee the convergence of this approach. In this paper, we study a singular stochastic control problem arising from portfolio choice with capital gains tax, where the associated HJB equation problem has infinitely many solutions and comparison principle fails to hold. Interestingly, we find that the corresponding penalty approximation problem has a unique solution and the penalty method still works. The key step of theoretical analysis is to prove that any admissible singular control can be approximated by a sequence of regular controls related to the penalized equation problem. Moreover, we show that the value function of the singular control problem is the minimal viscosity solution to the original HJB equation problem.

Our approach sheds light on the robustness of the penalty method for general singular stochastic control problems. Numerical results are presented to demonstrate the efficiency of the penalty method and to better understand optimal investment strategy in the presence of capital gains tax.

Appendix A: A Sketched Proof for Theorem 3.1 (i)

The detailed proof is given in E-Companion EC.2. To illustrate the key idea, we give a sketched proof here.

1. Fixing $\mathbf{x}$, for any $\epsilon > 0$, we can find an admissible strategy $\pi_t := (c_t, L_t, M_t) \in \mathcal{A}$ under which

$$\mathbb{E}[\int_0^{+\infty} e^{-\beta t} U(c_t) dt] \geq V_*(\mathbf{x}) - \epsilon.$$

By integrability, there exists a $T > 0$ such that

$$\mathbb{E}[\int_0^{T} e^{-\beta t} U(c_t) dt] \geq \mathbb{E}[\int_0^{+\infty} e^{-\beta t} U(c_t) dt] - \epsilon \geq V_*(\mathbf{x}) - 2\epsilon. \tag{A.1}$$

2. For any $\delta > 0$, there is an $N > 0$, s.t. $\mathbb{P}(\Omega_N) \geq 1 - \delta$, where $\Omega_N := \{\omega \in \Omega | L_T \vee M_T \leq N\}$. Especially, choose N sufficiently large, s.t. $\mathbb{P}(\Omega_N)$ is sufficiently large. By integrability, we have

$$\mathbb{E}[\int_0^{T} e^{-\beta t} U(c_t) \mathbf{1}_{\Omega_N} dt] \geq \mathbb{E}[\int_0^{T} e^{-\beta t} U(c_t) dt] - \epsilon. \tag{A.2}$$

3. Given n, we define $t_i = \frac{i}{n}$, $i = 0, 1, 2....$ To make the strategy adapted, on the time interval $[t_i, t_{i+1}]$, choose $l_t^{(n)}$ ($m_t^{(n)}$) to smooth the strategy L_t (M_t) in previous time step $[t_{i-1}, t_i]$.

$$l_t = \begin{cases} 0 & t \in (t_i, t_i + \frac{2}{5}(t_{i+1} - t_i)] \cup (t_i + \frac{3}{5}(t_{i+1} - t_i), t_{i+1}] \\ \dfrac{L_{t_i} - L_{t_{i-1}}}{\frac{1}{5}h} & t \in (t_i + \frac{2}{5}(t_{i+1} - t_i), t_i + \frac{3}{5}(t_{i+1} - t_i)], \end{cases}$$

$$m_t = \begin{cases} 0 & t \in (t_i + \frac{1}{3}(t_{i+1} - t_i), t_i + \frac{2}{3}(t_{i+1} - t_i)] \\ \dfrac{M_{t_i} - M_{t_{i-1}}}{\frac{2}{3}h} & t \in (t_i, t_i + \frac{1}{3}(t_{i+1} - t_i)] \cup (t_i + \frac{2}{3}(t_{i+1} - t_i), t_{i+1}]. \end{cases}$$

Noticing that L_T and M_T are bounded on Ω_N, the processes $l_t^{(n)}$ and $m_t^{(n)}$ can be uniformly Lipschitz on $\Omega_N \times [0, T]$. By Fatou lemma,

$$\liminf_{n\to+\infty} \mathbb{E}[\int_0^{T\wedge\tau^{(n)}} e^{-\beta t} U(c_t) \mathbf{1}_{\Omega_N} dt] \geq \mathbb{E}[\int_0^T e^{-\beta t} U(c_t) \mathbf{1}_{\Omega_N} dt], \tag{A.3}$$

where $\tau^{(n)}$ is the first stopping time of wealth z_t being less than $\frac{1}{n}$ under strategy $(c_t, l_t^{(n)}, m_t^{(n)})$. On Ω_N, before time $\tau^{(n)} \wedge T$, the strategy $(c_t, l_t^{(n)}, m_t^{(n)}) \in \mathcal{A}_{\lambda_{n,N}}$, where $\lambda_{n,N}$ is a constant depending on n, N.

A combination of (A.1)-(A.3) yields the desired result.

Appendix B: Finite Difference Scheme and Its Convergence for Fixed M_0 and λ

B.1. Finite Difference Scheme

Let us present the detailed finite difference scheme. Since the truncated region $\widetilde{D}_{M_0}$ is a triangle, we instead consider a rectangle region $\hat{D} := \{(b, \xi) | 0 \leq \xi \leq M_0,\ 0 \leq b = k/y \leq M_0\}$ in the $\xi - b$ plane, where the grids used are $\hat{Z}_{i,j} = (\frac{i}{n} M_0, \frac{j}{n} M_0)$, $0 \leq i \leq n$, $0 \leq j \leq n$. Denote by δ_b and δ_ξ the step sizes in b and ξ, respectively. In the $\xi - b$ plane, (4.2) becomes

$$\mathscr{L}_2 W + (\frac{1}{p} - 1)\,(pW - \xi W_\xi)^{\frac{p}{p-1}} + \lambda(\mathscr{B}_2 W)^+ + \lambda(\mathscr{S}_2 W)^+ = 0,$$

where $\mathscr{B}_2 W = \frac{1-\alpha}{\xi}(\xi W_\xi + (1-b) W_b)$, $\mathscr{S}_2 W = -\xi W_\xi$, and

$$\mathscr{L}_2 W := a_{0,0} W + a_{1,0} W_b + a_{0,1} W_\xi + a_{2,0} W_{bb} + a_{1,1} W_{\xi b} + a_{0,2} W_{\xi\xi},$$

with $a_{0,0} := -\beta + p[r(1 - \xi + \frac{\alpha}{1-\alpha} b\xi) + \mu\xi - \frac{1}{2}\sigma^2(1-p)\xi^2] < 0,\ a_{1,0} := (-\mu + (1-p)\sigma^2)\, b,\ a_{0,1} := \Big(-r(1 - \xi + \frac{\alpha}{1-\alpha} b\xi) + \mu(1-\xi) + \sigma^2(1-p)\xi(\xi - 1)\Big)\, \xi,\ a_{2,0} := \frac{1}{2}\sigma^2 b^2 \geq 0,\ a_{1,1} := \sigma^2 b\xi(1-\xi),\ a_{0,2} := \frac{1}{2}\sigma^2 \xi^2 (1-\xi)^2 \geq 0.$

Denote by $W_{i,j}^l$ the value at $\hat{Z}_{i,j}$ in l-th iteration. Given l-th guess $W_{i,j}^l$, $0 \leq i, j \leq n$, we deduce $W_{i,j}^{l+1}$, $0 \leq i, j \leq n$ by the following procedure. We apply the upwind scheme for first order term.

$$a_{1,0} W_b^{l+1}|_{\hat{Z}_{i,j}} \sim \begin{cases} a_{1,0} \dfrac{1}{\delta_b}(W_{i+1,j}^{l+1} - W_{i,j}^{l+1}), & a_{1,0} > 0, \\ a_{1,0} \dfrac{1}{\delta_b}(W_{i,j}^{l+1} - W_{i-1,j}^{l+1}), & a_{1,0} < 0. \end{cases}$$

$$a_{0,1} W_b^{l+1}|_{\hat{Z}_{i,j}} \sim \begin{cases} a_{0,1} \dfrac{1}{\delta_b}(W_{i,j+1}^{l+1} - W_{i,j}^{l+1}), & a_{0,1} > 0, \\ a_{0,1} \dfrac{1}{\delta_b}(W_{i,j}^{l+1} - W_{i,j-1}^{l+1}), & a_{0,1} < 0. \end{cases}$$

For second order terms, we use the central difference scheme for W_{bb}^{l+1} and $W_{\xi\xi}^{l+1}$, while for cross derivative term,

$$a_{1,1}W_{\xi b}^{l+1}|_{\hat{Z}_{i,j}} \sim \begin{cases} a_{1,1}\dfrac{1}{2\delta_b\delta_\xi}[W_{i+1,j+1}^{l+1}+W_{i-1,j-1}^{l+1}+2W_{i,j}^{l+1}-W_{i+1,j}^{l+1}-W_{i,j+1}^{l+1}-W_{i-1,j}^{l+1}-W_{i,j-1}^{l+1}], & a_{1,1}>0, \\ -a_{1,1}\dfrac{1}{2\delta_b\delta_\xi}[W_{i-1,j+1}^{l+1}+W_{i+1,j-1}^{l+1}+2W_{i,j}^{l+1}-W_{i+1,j}^{l+1}-W_{i,j-1}^{l+1}-W_{i-1,j}^{l+1}-W_{i,j+1}^{l+1}], & a_{1,1}<0. \end{cases}$$

For nonlinear terms including the consumption term and penalty term, we apply the non-smooth Newton iteration as given in Dai and Zhong (2010), that is, given the l-th estimate W^l, we linearize $(\mathscr{S}_2W)^+$ as

$$\begin{cases} \mathscr{S}_2W^{l+1}, & \text{if } \mathscr{S}_2W^l \geq 0, \\ 0, & \text{if } \mathscr{S}_2W^l < 0, \end{cases}$$

while in the discretization for $\mathscr{S}_2W$, the upwind scheme should be adopted. For $(\mathscr{B}_2W)^+$, similar method is applied. For consumption term $(pW-\xi W_\xi)^{\frac{p}{p-1}}$, we linearize it as

$$(pW^l-\xi W_\xi^l)^{\frac{p}{p-1}}+\frac{p}{p-1}(pW^l-\xi W_\xi^l)^{\frac{1}{p-1}}\left[p(W^{l+1}-W^l)-\xi(W_\xi^{l+1}-W_\xi^l)\right].$$

Noticing $\frac{p}{p-1}<0$, the upwind scheme here is $W_\xi^{l+1}|_{\hat{Z}_{i,j}}=\frac{1}{\delta_\xi}(W_{i,j+1}^{l+1}-W_{i,j}^{l+1})$.

On the boundaries $\xi=M_0$, $b=0$, and $b=M_0$, we discretize the Neumann boundary conditions by the upwind scheme:

$$\begin{cases} \mathscr{S}_2W_{i,n}^{l+1}=0, & 0\leq i\leq n, \\ \mathscr{B}_2W_{0,j}^{l+1}=0, & 1\leq j\leq n-1, \\ \mathscr{B}_2W_{n,j}^{l+1}=0, & 1\leq j\leq n-1. \end{cases}$$

On the boundary $\xi=0$,

$$\begin{cases} \dfrac{1}{\delta_b}(W_{i+1,0}^{l+1}-W_{i,0}^{l+1})=0, & i\delta_b<1, \\ \dfrac{1}{\delta_b}(W_{i-1,0}^{l+1}-W_{i,0}^{l+1})=0, & i\delta_b>1, \\ \dfrac{1}{\delta_\xi}(W_{i,1}^{l+1}-W_{i,0}^{l+1})=0, & i\delta_b=1. \end{cases}$$

Then we iteratively solve this linear system until W^l converges.

We point out that the above scheme is equivalent to the finite difference scheme in the triangle region $\widetilde{D}_{M_0}$ with grid points $Z_{i,j}=(\frac{\alpha}{1-\alpha}\frac{i}{n}\frac{j}{n}M_0, \frac{j}{n}M_0)$, $0\leq i,j\leq n$.

B.2. Convergence Analysis for Fixed M_0 and λ

The finite difference scheme is apparently consistent with the penalty equation problem. Assume the convergence of the Newton iteration and the monotonicity of the numerical scheme. In order to apply the convergence result of Barles and Souganidies (1991), we only need to verify the stability.

Denote by $\hat{W}$ the numerical solution. We assert that A_M is an upper bound of $\hat{W}$. Let us prove it by contradiction. Suppose not, and assume the maximum is attained at some point $\hat{Z}_{i,j}$. According to the Neumann boundary condition, we can assume $\hat{Z}_{i,j}$ to be an interior point. Due to the upwind treatment, in the discrete sense, $a_{1,0}\hat{W}_b$, $a_{0,1}\hat{W}_\xi \leq 0$ at $\hat{Z}_{i,j}$. Similarly, $(\mathscr{B}_2\hat{W})^+$, $(\mathscr{S}_2\hat{W})^+ \leq 0$ at $\hat{Z}_{i,j}$. By our numerical scheme and $a_{1,1}^2 = 4a_{2,0}a_{0,2}$, we have the non positiveness of sum of second order terms. Thus, it remains to consider the zero order term and the consumption term.

For the consumption term, according to our scheme, $\xi\hat{W}_\xi \leq 0$ in discrete sense, hence $(\frac{1}{p} - 1)\left(p\hat{W} - \xi\hat{W}_\xi\right)^{\frac{p}{p-1}} < (\frac{1}{p}-1)A_M^{\frac{p}{p-1}}$. For the zero order term, $a_{0,0} := -\beta + p[r(1-\xi+\frac{\alpha}{1-\alpha}b\xi) + \mu\xi - \frac{1}{2}\sigma^2(1-p)\xi^2] < -\beta + p[r + \frac{(\mu-r)^2}{2(1-p)\sigma^2}]$. Summing up all the terms, we infer that the value of $\mathscr{L}_2\hat{W}$ at $\hat{Z}_{i,j}$ is strictly lower than $(-\beta + p[r+\frac{(\mu-r)^2}{2(1-p)\sigma^2}])A_M + (\frac{1}{p}-1)(pA_M)^{\frac{p}{p-1}} = 0$, which is in contradiction with our numerical scheme.

Similarly, we can prove $\hat{W} \geq 0$. Thus the scheme is stable. Thanks to Barles and Souganidies (1991), the convergence of the numerical scheme to $\widetilde{W}(\eta,\xi;\lambda,M_0)$ then follows.

Appendix C: Convergence of $\widetilde{W}(\eta,\xi;\lambda,M_0)$ to the value function as M_0, $\lambda \to +\infty$

It suffices to prove the following two assertions:

(i) For any $(\eta_0,\xi_0) \in \widetilde{D}_{M_0}$,

$$\begin{aligned}&\limsup_{M_0\to+\infty}\ \sup\left\{\widetilde{W}(\eta,\xi;\lambda,M_0)|(\eta,\xi)\in\widetilde{D}_{M_0}\cap B\left((\eta_0,\xi_0);1/\sqrt{M_0}\right)\right\}\\ =&\liminf_{M_0\to+\infty}\ \inf\left\{\widetilde{W}(\eta,\xi;\lambda,M_0)|(\eta,\xi)\in\widetilde{D}_{M_0}\cap B\left((\eta_0,\xi_0);1/\sqrt{M_0}\right)\right\},\end{aligned} \tag{C.1}$$

where $B\left((\eta_0,\xi_0);1/\sqrt{M_0}\right)$ is the ball centered at (η_0,ξ_0) with radius $1/\sqrt{M_0}$. (C.1) implies the existence of the limit, denoted by $\widetilde{W}(\eta_0,\xi_0;\lambda)$.

(ii) For any $\mathbf{x}$,

$$\lim_{\lambda\to+\infty} z^p\widetilde{W}(\eta,\xi;\lambda) = V_*(\mathbf{x}). \tag{C.2}$$

C.1. Proof of (C.1)

According to Lemma 6.1 in Crandall, Ishii and Lions (1992), the function

$$\underline{\widetilde{W}}(\xi_0,\eta_0;\lambda) := \liminf_{M_0\to+\infty}\ \inf\{\widetilde{W}(\eta,\xi;\lambda,M_0)|(\eta,\xi)\in\widetilde{D}_{M_0}, \text{and } |(\eta,\xi)-(\xi_0,\eta_0)|\leq 1/\sqrt{M_0}\}$$

is a supersolution to

$$\mathscr{F}_1[W,\lambda] = 0 \tag{C.3}$$

in $\{(\eta,\xi)|\xi\geq 0, \eta\geq 0\}$ with boundary conditions

$$\begin{cases} -\mathscr{B}_1 W = 0, \quad \text{on } \eta = 0, & \text{(C.4)}\\ -\mathscr{B}_1 W = 0, \quad \text{on } \xi = 0, & \text{(C.5)}\end{cases}$$

where the differential operators $\mathscr{F}_1[\cdot,\lambda]$ and $\mathscr{B}_1$ are as given in (4.2) and (4.4), respectively.

Similarly, we can define $\widetilde{\widetilde{W}}(\xi_0,\eta_0;\lambda)$ by replacing lim sup and sup with lim inf and inf, respectively, and $\widetilde{\widetilde{W}}(\xi_0,\eta_0;\lambda)$ is a subsolution to problem (C.3)-(C.5). To obtain the desired result, we need to show that comparison principle holds for problem (C.3)-(C.5), that is, if u, v are respectively a subsolution and a supersolution to problem (C.3) to (C.5) and are bounded above by constant A_M, then $u \le v$.

Note that $z^p u$ and $z^p v$ are respectively a subsolution and a supersolution to

$$\mathscr{F}[V,\lambda]=0 \tag{C.6}$$

in D with boundary conditions

$$\begin{cases} -\mathscr{B}V=0, & \text{on } \Gamma_2\cup\Gamma_3, \qquad \text{(C.7)}\\ V=0, & \text{on } \Gamma_1, \qquad \text{(C.8)}\end{cases}$$

where the differential operators $\mathscr{F}[\cdot,\lambda]$ and $\mathscr{B}$ are as given in (3.3) and (2.7), respectively. Thus, it is sufficient to show that comparison principle holds for problem (C.6) to (C.8). This can be achieved by using a similar (but simpler) argument as in the proof of part (i) of Proposition 3.1, where $\Phi(\eta,\xi)=|i(\eta,\xi)+\epsilon\mathbf{n}|^2$ is replaced by $|i(\eta,\xi)|^2$, and there is no need of proving the continuity of the value function.

C.2. Proof of (C.2)

Define $\widetilde{V}(\mathbf{x};\lambda) := z^p\widetilde{W}(\eta,\xi;\lambda)$. We assert that V_* is a supersolution to the problem (C.6)-(C.8). Indeed, it is apparently true in D and on Γ_1. If it fails at $\mathbf{x}_0\in\Gamma_2\cup\Gamma_3$, then there exists some smooth function φ such that $\varphi(\mathbf{x}_0)=V_*(\mathbf{x}_0)$, $\varphi-V_*$ attains a local maximum at $\mathbf{x}_0$, and

$$\max\{-\mathscr{F}[\varphi,\lambda],-\mathscr{B}\varphi\}|_{\mathbf{x}_0}<0.$$

It follows $\mathscr{B}\varphi|_{\mathbf{x}_0}=-\varphi_x+\varphi_y+\varphi_k|_{\mathbf{x}_0}>0$. Noticing $\varphi-V_*$ attains local maximum 0 at $\mathbf{x}_0$, we have for sufficiently small $\delta>0$, $V_*(\mathbf{x}_0+\delta(-1,1,1))\ge\varphi(\mathbf{x}_0+\delta(-1,1,1))>\varphi(\mathbf{x}_0)=V_*(\mathbf{x}_0)$. On the other hand, by definition of V_*, $V_*(\mathbf{x}_0+\delta(-1,1,1))\le V_*(\mathbf{x}_0)$ since for any admissible strategy (C,L,M) with starting point $\mathbf{x}_0+\delta(-1,1,1)$, the investor can achieve the same utility with starting point $\mathbf{x}_0$ by first buying δ dollar of stock then following (C,L,M). Thus, V_* is also a supersolution on $\Gamma_2\cap\Gamma_3$.

Applying comparison principle of problem (C.6)-(C.8) gives $V_*(\mathbf{x})\ge\widetilde{V}(\mathbf{x};\lambda)$. Since $\widetilde{V}(\mathbf{x};\lambda)$ is a continuous supersolution to (3.3), by the comparison principle established in Proposition 3.1, $\widetilde{V}(\mathbf{x};\lambda)\ge V(\mathbf{x};\lambda)$. Combining with Theorem 3.1, we then infer $\lim\limits_{\lambda\to+\infty}\widetilde{V}(\mathbf{x};\lambda)=V_*(\mathbf{x})$, which yields the desired result.

Endnotes

1. See, e.g., Constantinides (1986), Davis and Norman (1990), Shreve and Soner (1994), Liu and Loewenstein (2002), Liu (2004), Dai and Yi (2009), Kallsen and Muhle-Karbe (2010), Bichuch and Shreve (2013), Chen and Dai (2013), Gerhold et al. (2014), Hobson, Tse and Zhu (2019), and references therein.
2. There does exist a literature on the discrete time model with capital gains tax, e.g., Constantinides (1983, 1984), Dybvig and Koo (1996), Dammon, Spatt and Zhang (2001, 2004), DeMiguel and Uppal (2005), Gallmeyer, Kaniel and Tompaidis (2006), Marekwica (2012), and Ehling et al. (2018).
3. The model in Dai et al. (2015) reduces to that in Ben Tahar, Soner and Touzi (2010) by assuming full tax rebate and symmetric tax rates.
4. Numerical results generated by the penalty method are used as benchmarks to examine the robustness of asymptotic formulas in Cai, Chen and Dai (2018). However, the paper does not address the convergence of the penalty method.
5. The PDE for an optimal stopping problem is a standard variational inequality equation (i.e., with a constraint on solution itself), while the PDE for a singular control problem is a variatonal inequality equation with gradient constraints.
6. Here the subscripts for u stand for partial derivatives.
7. Notice that a classical solution of an equation is not necessarily a state-constraint viscosity solution (see Katsoulake 1994, p497).
8. This can be rigorously proved by applying comparison principle in $b \leq 1$.

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E-Companion to "Penalty Method for Portfolio Selection with Capital Gains Tax"

Appendix EC.1: Proof of Proposition 3.1

EC.1.1. Comparison Principle

For $\xi=(x,y,k)$, we use notation $|\xi|=\sqrt{x^2+y^2+k^2}$.

1. Suppose the assertion $u\leqslant v$ on $\bar{D}$ is not true. Then there exists $\xi_0\in D\cup\Gamma_2$ such that $u(\xi_0)>v(\xi_0)$. Set $a=\frac{1}{3}[u(\xi_0)-v(\xi_0)]$. Let ε be a small positive constant and $q\in(p,1)$ be a constant sufficiently close to p, both of which depend on a and will be determined later. We define

$$w(\xi)=x+(1-\alpha+\varepsilon)y+(\alpha+\varepsilon)k, \qquad \delta=\frac{a}{w^q(\xi_0)}, \qquad g(\xi)=\delta w^q(\xi).$$

Note that $z=x+(1-\alpha)y+\alpha k\geqslant 0$ and $y+k\geqslant \max\{0,-x\}$. Hence,

$$w\geqslant z, \quad w\geqslant \frac{\varepsilon}{2}(|x|+y+k)\geqslant \frac{\varepsilon}{2}|\xi| \quad \forall\,\xi=(x,y,k)\in\bar{D}.$$

This implies, since $u(\xi)-v(\xi)\leqslant O(1)|\xi|^p$ and $q>p$, that there exists $\bar{\xi}\in D\cap\Gamma_2$ such that

$$u(\bar{\xi})-v(\bar{\xi})-2g(\bar{\xi})=\max_{\xi\in\bar{S}}\Big\{u(\xi)-v(\xi)-2g(\xi)\Big\}\geqslant a>0.$$

2. Let $\mathbf{n}=\frac{1}{2}(0,1,1)$. For each positive integer i we define

$$\Phi(\xi,\eta)=|i(\xi-\eta)+\varepsilon\mathbf{n}|^2,\ \ \phi_i(\xi,\eta)=g(\xi)+g(\eta)+\Phi(\xi,\eta),$$
$$\varphi_i(\xi,\eta)=u(\xi)-v(\eta)-\phi_i(\xi,\eta),\ \ (\xi_i,\eta_i)=\underset{(\xi,\eta)\in\bar{D}\times\bar{D}}{\operatorname{argmax}}\ \varphi_i(\xi,\eta).$$

First of all, from $\varphi_i(\xi_i,\eta_i)\geqslant\varphi_i(\bar{\xi},\bar{\xi})\geqslant a-\varepsilon^2$ we obtain

$$u(\xi_i)-[v(\eta_i)+g(\xi_i)+g(\eta_i)+|i(\xi_i-\eta_i)+\varepsilon\mathbf{n}|^2]\geqslant a-\varepsilon^2. \tag{EC.1.1}$$

Again, as $u(\xi)-v(\eta)\leqslant O(1)(|\xi|^p+|\eta|^p)$ and $q>p$, the sequence $\{(\xi_i,\eta_i,|i(\xi_i-\eta_i)+\varepsilon\mathbf{n}|^2)\}_{i=1}^{\infty}$ is a bounded sequence on $\mathbb{R}^7$, with a bound that does not depend on i. Hence along a subsequence of $i\to\infty$, $(\xi_i,\eta_i)\longrightarrow(\hat{\xi},\hat{\xi})$, for some $\hat{\xi}\in\bar{D}$. For notational simplicity, we assume that the whole sequence converges. Now, using $\varphi_i(\xi_i,\eta_i)\geqslant\varphi_i(\hat{\xi},\hat{\xi}+\varepsilon\mathbf{n}/i)$, we obtain

$$|i(\xi_i-\eta_i)+\varepsilon\mathbf{n}|^2\leqslant v\Big(\hat{\xi}+\frac{\varepsilon\mathbf{n}}{i}\Big)-u(\hat{\xi})+g(\hat{\xi})+g\Big(\hat{\xi}+\frac{\varepsilon\mathbf{n}}{i}\Big)+u(\xi_i)-v(\eta_i)-g(\xi_i)-g(\eta_i).$$

Sending $i \to \infty$ we then find that

$$\lim_{i\to\infty} |i(\xi_i - \eta_i) + \varepsilon \mathbf{n}|^2 = 0.$$

Thus, $\eta_i = \xi_i + [\varepsilon\mathbf{n} + o(1)]/i$ for $i \gg 1$. Consequently, as $|\mathbf{n}| < 1$,

$$\eta_i \in D, \qquad i|\xi_i - \eta_i| \leqslant \varepsilon \qquad \forall\, i \gg 1.$$

3. Since $\varphi_i(\cdot,\cdot)$ attains a local maximum at (ξ_i, η_i), by Ishii's lemma, there exist two matrices $M_1 = (m_1^{ij})_{3\times 3}$ and $M_2 = (m_2^{ij})_{3\times 3}$ such that

$$(\Phi_\xi(\xi_i,\eta_i), M_1) \in J^{2+}(u-g)(\xi_i), \quad (-\Phi_\eta(\xi_i,\eta_i), M_2) \in J^{2-}(v+g)(\eta_i), \tag{EC.1.2}$$

$$\begin{pmatrix} M_1 & 0 \\ 0 & -M_2 \end{pmatrix} \leqslant \Big(D^2\Phi(\xi_i,\eta_i)\Big)_{6\times 6} + \frac{1}{4i^2}\Big(D^2\Phi(\xi_i,\eta_i)\Big)^2. \tag{EC.1.3}$$

We can calculate

$$\Phi_\xi = -\Phi_\eta = 2i^2(\xi - \eta) + 2i\varepsilon\mathbf{n}, \quad \Phi_{\xi\xi} = 2i^2 I, \quad \Phi_{\xi\eta} = -2i^2 I, \quad \Phi_{\eta\eta} = 2i^2 I,$$

where I is the 3×3 identity matrix. In particular, denoting $\xi_i = (x_1, y_1, k_1)$ and $\eta_i = (x_2, y_2, k_2)$ and multiplying (EC.1.3) by $(0, y_1, 0, 0, y_2, 0)$ from the left and by $(0, y_1, 0, 0, y_2, 0)^T$ from the right we obtain, when $i \gg 1$,

$$y_1^2 m_1^{22} - y_2^2 m_2^{22} \leqslant (2i^2 + 2i^2)|y_1 - y_2|^2 \leqslant 4i^2|\xi_i - \eta_i|^2 \leqslant 4\varepsilon^2. \tag{EC.1.4}$$

Since u is a subsolution with $\xi_i \in D \cup \Gamma_2$ and v is a supersolution with $\eta_i \in D$, by the definition of viscosity sub/supersolution and (EC.1.2) we obtain

$$F[\xi_i, u(\xi_i), \Phi_\xi(\xi_i,\eta_i) + g_\xi(\xi_i), M_1 + D^2 g(\xi_i)] \geqslant 0,$$
$$F[\eta_i, v(\eta_i), -\Phi_\eta(\xi_i,\eta_i) - g_\eta(\eta_i), M_2 - D^2 g(\eta_i)] \leqslant 0.$$

Taking the difference and using the definition of F we then obtain

$$\begin{aligned}
0 \leqslant\ & \Big\{\frac{\sigma^2}{2}(y_1^2 m_1^{22} - y_2^2 m_2^{22}) + \mu y_1 \Phi_{y_1} + \mu y_2 \Phi_{y_2} + r x_1 \Phi_{x_1} + r x_2 \Phi_{x_2}\Big\} \\
& + \Big\{U^*\Big(g_{x_1}(\xi_i) + \Phi_{x_1}(\xi_i,\eta_i)\Big) - U^*\Big(-g_{x_1}(\eta_i) - \Phi_{x_2}(\xi_i,\eta_i)\Big)\Big\} \\
& + \Big\{\mathscr{L}^1 g(\xi_i) + \mathscr{L}^1 g(\eta_i) - \beta[u(\xi_i) - v(\eta_i) - g(\xi_i) - g(\eta_i)]\Big\} \\
& + \Big\{\lambda z_1[\mathscr{B}^1\phi(\xi_i,\eta_i)]^+ - \lambda z_2[-\mathscr{B}^2\phi(\xi_i,\eta_i)]^+\Big\} \\
& + \Big\{\lambda[\mathscr{S}^2\phi(\xi_i,\eta_i)]^+ - \lambda[-\mathscr{S}^2\phi(\xi_i,\eta_i)]^+\Big\} \\
& = I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Here the superscripts 1 and 2 for $\mathscr{L},\mathscr{B}$ and $\mathscr{S}$ correspond to operations with respect to ξ and η, respectively. Also, $(x_1,y_1,k_1)=\xi_i,(x_2,y_2,k_2)=\eta_i,\ z_1=x_1+(1-\alpha)y_1+\alpha k_1, z_2=x_2+(1-\alpha)y_2+\alpha k_2, w_1=x_1+(1-\alpha+\varepsilon)y_1+(\alpha+\varepsilon)k_1, w_2=x_2+(1-\alpha+\varepsilon)y_2+(\alpha+\varepsilon)k_2$.

4. We estimate each term I_j, for $j=1,\cdots,5$.

- First we use (EC.1.4) to estimate

$$\begin{aligned}I_1 &:= \frac{\sigma^2}{2}(y_1^2m_1^{22}-y_2^2m_2^{22})+\mu y_1\Phi_{y_1}+\mu y_2\Phi_{y_2}+rx_1\Phi_{x_1}+rx_2\Phi_{x_2}\\ &\leqslant 2\sigma^2\varepsilon^2+\mu[2i^2(y_1-y_2)^2+2\varepsilon i(y_1-y_2)]+2ri^2(x_1-x_2)^2\\ &\leqslant (2\sigma^2+4\mu+2r)\varepsilon^2.\end{aligned}$$

- Next we notice that $[g_{x_1}(\xi_i)+\Phi_{x_1}(\xi_i,\eta_i)]-[-g_{x_2}(\eta_i)-\Phi_{x_2}(\xi_i,\eta_i)]=g_{x_1}(\xi_i)+g_{x_2}(\eta_i)>0$.

Since $U^*(\cdot)$ is a decreasing function, we obtain

$$I_2:=U^*\big(g_{x_1}(\xi_i)+\Phi_{x_1}(\xi_i,\eta_i)\big)-U^*\big(-g_{x_2}(\eta_i)-\Phi_{x_2}(\xi_i,\eta_i)\big)\leqslant 0.$$

- Set $\bar\alpha=1-\alpha+\varepsilon$. Using $x=w-\bar\alpha y-(\alpha+\varepsilon)k\leqslant w-\bar\alpha y$ and $q\in(0,1)$, we have

$$\begin{aligned}\mathscr{L}g(x,y,k) &= \delta q w^{q-2}\Big(\frac{\sigma^2(q-1)}{2}(\bar\alpha y)^2+(\mu\bar\alpha y+rx)w\Big)-\beta g\\ &\leqslant \delta q w^{q-2}\Big(\frac{\sigma^2(q-1)}{2}\Big[\bar\alpha y-\frac{(\mu-r)w}{\sigma^2(1-q)}\Big]^2+\frac{(\mu-r)^2w^2}{2\sigma^2(1-q)}+rw^2\Big)-\beta g\\ &\leqslant -[\beta-\beta_q]g(w), \qquad \beta_q:=q\Big(r+\frac{(\mu-r)^2}{2\sigma^2(1-q)}\Big).\end{aligned}$$

In view of (EC.1.1) we then derive that

$$\begin{aligned}I_3 &:= \mathscr{L}^1g(\xi_i)+\mathscr{L}^2g(\eta_i)-\beta[u(\xi_i)-v(\eta_i)-g(\xi_i)-g(\eta_i)]\\ &\leqslant -[\beta-\beta_q][g(\xi_i)+g(\eta_i)]-\beta(a-\varepsilon^2).\end{aligned}$$

- Note that, since $z\leqslant w$,

$$z\mathscr{B}g(\xi)=\delta q z w^{q-1}[(1-\alpha+\varepsilon)+(\alpha+\varepsilon)-1]=2\varepsilon\delta q z w^{q-1}\leqslant 2\varepsilon q g(\xi).$$

Hence, using $[s_1+s_2]^+\leqslant s_1^++s_2^+$ and $s_1^+-s_2^+\leqslant(s_1-s_2)^+$ we obtain

$$\begin{aligned}I_4 &:= \lambda z_1[\mathscr{B}^1\phi(\xi_i,\eta_i)]^+-\lambda z_2[-\mathscr{B}^2\phi(\xi_i,\eta_i)]^+\\ &\leqslant \lambda z_1[\mathscr{B}^1g(\xi_i)]^++\lambda z_2[\mathscr{B}^2g(\eta_i)]^+\\ &\quad +\lambda z_2[\mathscr{B}^1\Phi(\xi_i,\eta_i)+\mathscr{B}^2\Phi(\xi_i,\eta_i)]^++\lambda|z_1-z_2|\,|\mathscr{B}^1\Phi(\xi_i,\eta_i)|\\ &\leqslant 2\varepsilon\lambda q[g(\xi_i)+g(\eta_i)]+0+\lambda|\xi_i-\eta_i|[2i^2|\xi_i-\eta_i|+2i\varepsilon]\\ &\leqslant 2\varepsilon\lambda q[g(\xi_i)+g(\eta_i)]+4\lambda\varepsilon^2.\end{aligned}$$

- Finally,

$$\mathscr{S}g(\xi) = \delta q w^{q-1}[(1-\alpha)y + \alpha k - (1-\alpha+\varepsilon)y - (\alpha+\varepsilon)k] \leqslant 0.$$

Hence,

$$\begin{aligned} I_5 &:= \lambda[\mathscr{S}^1\phi]^+ - \lambda[-\mathscr{S}^2\phi]^+ \leqslant \lambda[\mathscr{S}^1\phi + \mathscr{S}^2\phi]^+ \\ &\leqslant \lambda[2i^2|\xi_i - \eta_i|^2 + 2i|(\xi_i - \eta_i)|\varepsilon] \leqslant 4\lambda\varepsilon^2. \end{aligned}$$

In summary, we obtain

$$\begin{aligned} 0 &\leqslant I_1 + I_2 + I_3 + I_4 + I_5 \\ &\leqslant -\Big(\beta - \beta_q - 2\lambda q\varepsilon\Big)[g(\xi_i) + g(\eta_i)] - \Big(\beta a - [2\sigma^2 + 4\mu + 2r + 8\lambda + \beta]\varepsilon^2\Big). \end{aligned}$$

Hence, taking $q \in (p,1)$ such that $\beta_q < \beta$ and then taking ε sufficiently small, we obtain a contradiction, which indicates that $u \leqslant v$ in $\bar{D}$. Thus the comparison principle holds.

EC.1.2. Continuity of Value Function $V(\mathbf{x};\lambda)$

To prove part (ii), we first show that the value function $V(\mathbf{x};\lambda)$ is continuous.

Denote $z_t \equiv x_t + (1-\alpha)y_t + \alpha k_t$. We define

$$\tau \equiv \tau(\pi) = \inf\{t \geqslant 0 \mid z_t = 0\}. \tag{EC.1.5}$$

In the sequel, with same sub and/or superscripts, the variable z always relates (x,y,k) via $z = x + (1-\alpha)y + \alpha k$.

The proof of the upper-semi-continuity is tricky, and the following lemma plays a crucial role.

Lemma EC.1.1. *For any $(x,y,k) \in \bar{D}$ and $\delta > 0$, we have*

$$V_\lambda(x+\delta, y, k) \leq V_\lambda(x,y,k) + C\delta^p. \tag{EC.1.6}$$

$V_\lambda(x,y,k)$ is continuous with respect to x.

Basic Properties Before we prove the lemma, it is worth noticing the following:
(i). In $\bar{D}$, $V_\lambda(x,y,k)$ is non-decreasing with respect to x, y, and k.
(ii). $0 \leq V_\lambda(x,y,k) \leq V_*(x,y,k) \leq Cz^p$ for some positive constant $C > 0$ that is independent of $(x,y,k) \in \bar{D}$; here V_* is the value function with singular control. So V_λ is finite everywhere.
(iii). $V_\lambda(ax,ay,ak) = a^p V_\lambda(x,y,k)$, for any positive constant $a > 0$.

Proof of Lemma EC.1.1 Due to the above property (iii), (EC.1.6) is apparently true for $(x,y,k)\equiv(0,0,0)$. Consider $(x,y,k)\in\bar{D}\setminus(0,0,0)$. We define $\{(x_{1t},y_{1t},k_{1t})\}_{t\geq 0}$ by the sdes in (2.1) with initial value $(x_{10},y_{10},k_{10})=(x+\delta,y,k)$ and an admissible strategy $\pi:=\{l_{1t},m_{1t},c_{1t}\}_{t\geq 0}\in\mathcal{A}_\lambda$. Set

$$\tau_1=\inf\{t>0\mid z_{1t}=0\}.$$

It is easy to see that $y_{1t}\geqslant 0, k_{1t}\geqslant 0$, and $z_{1t}>0$ for $t\in[0,\tau_1)$.

Next we consider the sde system, for $(x_{3t},y_{3t},k_{3t},x_{4t},y_{4t},k_{4t})$,

$$\begin{cases} dx_{3t}=(rx_{3t}-c_{1t}\mathbb{1}_{\{t<\tau_1\}})dt-l_{2t}z_{3t}dt+[(1-\alpha)y_{3t}+\alpha k_{3t}]m_{1t}dt,\\ dx_{4t}=rx_{4t}dt-l_{2t}z_{4t}dt+[(1-\alpha)y_{4t}+\alpha k_{4t}]m_{1t}dt,\\ dy_{3t}=(\mu-m_{1t})y_{3t}dt+\sigma y_{3t}d\mathcal{B}_t+l_{2t}z_{3t}dt,\\ dy_{4t}=(\mu-m_{1t})y_{4t}dt+\sigma y_{4t}d\mathcal{B}_t+l_{2t}z_{4t}dt,\\ dk_{3t}=l_{2t}z_{3t}dt-m_{1t}k_{3t}dt,\\ dk_{4t}=l_{2t}z_{4t}dt-m_{1t}k_{4t}dt+rk_{4t}dt,\end{cases}\tag{EC.1.7}$$

with initial values $(x_{30},y_{30},k_{30})=(x,y,k)$ and $(x_{40},y_{40},k_{40})=(\delta,0,0)$, where $z_{it}=x_{it}+(1-\alpha)y_{it}+\alpha k_{it}\quad\forall\, i=2,3,4$, $(x_{2t},y_{2t},k_{2t})=(x_{3t},y_{3t},k_{3t})+(x_{4t},y_{4t},k_{4t})$, and

$$l_{2t}=\begin{cases}\dfrac{l_{1t}z_{1t}}{z_{2t}} & \text{if}\ \ t<\tau_1,\ \min\limits_{s\in[0,t]}z_{3s}>0,\ \&\ \min\limits_{s\in[0,t]}z_{4s}>0,\\ 0 & \text{otherwise}.\end{cases}$$

Denote by $[0,\bar{\tau})$ the maximal existence interval of the sde system (EC.1.7). Since $z_{20}=z+\delta>0$, we have $\bar{\tau}>0$. Note that if a solution exists in $[0,T)$ and $\inf_{t\in[0,T)}z_{2t}>0$, then the solution can be locally extended beyond $[0,T)$. We now define

$$\hat{\tau}=\sup\left\{t<\bar{\tau}\,\middle|\, t<\tau_1,\min_{s\in[0,t]}z_{3s}>0,\min_{s\in[0,t]}z_{4s}>0\right\}.$$

Clearly, by the definition of $\hat{\tau}$,

$$z_{3t}>0,\quad z_{4t}>0,\quad z_{2t}=z_{3t}+z_{4t}>0\qquad\forall\, t\in[0,\hat{\tau}).$$

Consequently, by the differential equations for k_{3t} and k_{4t} we have

$$k_{3t}\geqslant 0,\quad k_{4t}\geqslant 0,\quad k_{2t}\geqslant 0\quad\forall\, t\in[0,\hat{\tau}).$$

Next, using $l_{2t}z_{2t} = l_{1t}z_{1t}$ for each $t \in [0,\hat{\tau})$, a direct computation gives, for $t \in [0,\hat{\tau})$,

$$\begin{cases} dx_{2t} = (rx_{2t} - c_{1t})dt - l_{1t}z_{1t}dt + [(1-\alpha)y_{2t} + \alpha k_{2t}]m_{1t}dt, \\ dy_{2t} = y_{2t}[(\mu - m_{1t})dt + \sigma d\mathcal{B}_t] + l_{1t}z_{1t}dt, \\ dk_{2t} = l_{1t}z_{1t}dt - m_{1t}k_{2t}dt + rk_{4t}dt, \\ (x_{20}, y_{20}, k_{20}) = (x+\delta, y, k). \end{cases}$$

Comparing the equations satisfied by (x_{1t}, y_{1t}, k_{1t}) and (x_{2t}, y_{2t}, k_{2t}), we obtain

$$\begin{array}{ll} y_{2t} = y_{1t} & \forall t \in [0,\hat{\tau}), \\ k_{2t} = k_{1t} + r\int_0^t k_{4s}e^{-\int_s^t m_\theta d\theta}ds \geqslant k_{1t} & \forall t \in [0,\hat{\tau}), \\ x_{2t} = x_{1t} + \alpha\int_0^t e^{r(t-s)}(k_{2s} - k_{1s})m_s ds \geqslant x_{1t} & \forall t \in [0,\hat{\tau}), \\ z_{2t} \geqslant z_{1t} & \forall t \in [0,\hat{\tau}), \\ 0 \leqslant l_{2t} \leqslant l_{1t} \leqslant \lambda & \forall t \in [0,\hat{\tau}). \end{array} \tag{EC.1.8}$$

The boundedness of $\{l_{2t}\}_{t\in[0,\hat{\tau})}$ implies that $(z_{3\hat{\tau}}, z_{4\hat{\tau}}) := \lim_{t\nearrow\hat{\tau}}(z_{3t}, z_{4t})$ exists if $\hat{\tau} < \infty$.

(i) If $\hat{\tau} = \tau_1$, we have, by definition, $l_{2t} = 0$ for all $t \geq \hat{\tau}$.

(ii) If $\hat{\tau} < \tau_1$ and $\min\{z_{3\hat{\tau}}, z_{4\hat{\tau}}\} = 0$, by definition, we also have $l_{2t} = 0$ for all $t \geq \hat{\tau}$.

(iii) The case $\hat{\tau} < \tau_1$ and $\min\{z_{3\hat{\tau}}, z_{4\hat{\tau}}\} > 0$ is impossible since the solution can be extended beyond $[0,\hat{\tau}]$ and $\min\limits_{t\in[0,\hat{\tau}+\varepsilon]} \min\{z_{3t}, z_{4t}\} > 0$ for some small positive $\varepsilon > 0$, contradicting the definition of $\hat{\tau}$.

Thus, we have $0 \leqslant l_{2t} \leqslant l_{1t} \leqslant \lambda$ for all $t \in [0,\infty)$. Moreover, $\hat{\tau} = \tau_1$ or $\hat{\tau} < \tau_1$ & $\min\{z_{3\hat{\tau}}, z_{4\hat{\tau}}\} = 0$.

Noting that $z_{3t} > 0$ for $t \in [0,\hat{\tau})$ and $\{c_{1t}\mathbb{1}_{\{t<\tau_1\}}, l_{2t}, m_{1t}\}$ is an admissible strategy starting from $(x_{30}, y_{30}, k_{30}) = (x, y, k)$, we obtain

$$V_\lambda(x,y,k) \geqslant \mathbb{E}\left[\int_0^{\hat{\tau}} e^{-\beta t}U(c_{1t})dt\right].$$

Consequently,

$$\begin{aligned} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta t}U(c_{1t})dt\right] - V_\lambda(x,y,k) \leqslant \mathbb{E}\left[\int_{\hat{\tau}}^{\tau_1} e^{-\beta t}U(c_{1t})dt\right] &\leq \mathbb{E}\Big[e^{-\beta\hat{\tau}}\mathbb{1}_{\{\hat{\tau}<\tau_1\}}V_\lambda(x_{\hat{\tau}}, y_{\hat{\tau}}, k_{\hat{\tau}})\Big] \\ &\leq C\ \mathbb{E}\Big[e^{-\beta\hat{\tau}}\mathbb{1}_{\{\hat{\tau}<\tau_1\}}z_{\hat{\tau}}^p\Big], \end{aligned} \tag{EC.1.9}$$

where we have used property (ii) of the value function.

To continue, we need the following lemma whose proof will be left at the end of this proof.

Lemma EC.1.2. *If $\hat\tau<\infty$, then $z_{4\hat\tau}>0$. Consequently, if $\hat\tau<\tau_1$, then $z_{3\hat\tau}=0$ and $0<z_{\hat\tau}\leqslant z_{2\hat\tau}=z_{4\hat\tau}$ almost surely.*

Applying Lemma EC.1.2 to (EC.1.9), we get

$$\mathbb{E}\left[\int_0^{\tau_1} e^{-\beta t}U(c_{1t})dt\right]-V_\lambda(x,y,k)\le\ C\ \mathbb{E}[e^{-\beta\hat\tau}\mathbb{1}_{\{\hat\tau<\infty\}}z_{4\hat\tau}^p].$$

Set $\Pi_t:=y_{4t}/z_{4t}$. For $t\in[0,\hat\tau]\cap[0,\infty)$, we have

$$dz_{4t}=z_{4t}[rdt+(1-\alpha)(\mu-r)\Pi_t dt+(1-\alpha)\sigma\Pi_t d\mathcal{B}_t].$$

Thus, by Itô Lemma,

$$\begin{aligned}\frac{d(e^{-\beta t}z_{4t}^p)}{e^{-\beta t}z_{4t}^p}&=\left\{-\beta+pr+p(1-\alpha)(\mu-r)\Pi_t-\frac{p(1-p)}{2}(1-\alpha)^2\sigma^2\Pi_t^2\right\}dt+p(1-\alpha)\sigma\Pi_t d\mathcal{B}_t\\&\le\left\{-\beta+pr+\frac{p(\mu-r)^2}{2(1-p)\sigma^2}\right\}dt+p(1-\alpha)\sigma\Pi_t d\mathcal{B}_t\le p(1-\alpha)\sigma\Pi_t d\mathcal{B}_t;\end{aligned}$$

here we have used the assumption $\beta>\beta_p$. For each positive integer n, define $\hat\tau_n=\sup\{t<\hat\tau|\ \sup_{s\in[0,t)}\Pi_s<n\}$. Then $\{e^{-\beta\,(t\wedge\hat\tau_n)}z_{4\,t\wedge\hat\tau_n}^p\}_{t\ge0}$ is a supermartingale. Hence,

$$\mathbb{E}[e^{-\beta\,(T\wedge\hat\tau_n)}z_{4\,T\wedge\hat\tau_n}^p]\le\mathbb{E}[z_{40}^p]=\delta^p\qquad\forall\ T>0.$$

Sending n and T to infinity, we obtain by Fatou's lemma that $\mathbb{E}[e^{-\beta\hat\tau}\mathbb{1}_{\{\hat\tau<\infty\}}z_{4\hat\tau}^p]\le\delta^p$. Thus,

$$\mathbb{E}\left[\int_0^{\tau_1}U(c_{1t})dt\right]-V_\lambda(x,y,k)\leqslant C\delta^p.$$

Letting $\pi=\{(c_{1t},l_{1t},m_{1t})\}_{t\ge0}$ go through all admissible strategies, we obtain (EC.1.6).

*Proof of **Lemma EC.1.2**.* We use a contradiction argument by assuming that (there is a sample at which) $\hat\tau<\infty$ and $z_{4\hat\tau}=0$.

Set $\zeta_t=z_{4t}y_{3t}-z_{3t}y_{4t}$. A direct calculation gives $\zeta_0=\delta y$ and for $t\ge0$,

$$d\zeta_t=\zeta_t[(r+\mu-m_{1t})dt+\sigma d\mathcal{B}_t]+y_{4t}(\alpha rk_{3t}+c_{1t})dt.$$

Since $y_{4t}\ge0,k_{3t}\ge0$, and $c_{1t}\ge0$ for $t\in[0,\hat\tau)$, we have $\zeta_t\geqslant0$ for $t\in[0,\hat\tau]$. Consequently, $0\leqslant\zeta_{\hat\tau}=-z_{3\hat\tau}y_{4\hat\tau}$; hence $z_{3\hat\tau}y_{4\hat\tau}=0$. We now consider two cases: (i) $y_{4\hat\tau}=0$ and (ii) $z_{3\hat\tau}=0$.

(i) Suppose $y_{4\hat\tau}=0$. From the differential equation for y_{4t} we derive that $y_{4t}=0$ for all $t\in[0,\hat\tau]$. This implies that $z_{4t}=\delta e^{-rt}$ for all $t\in[0,\hat\tau]$, contradicting $z_{4\hat\tau}=0$.

(ii) Suppose $z_{3\hat\tau}=0$. Then we obtain from $0\leqslant z_{\hat\tau}\leqslant z_{2\hat\tau}=z_{3\hat\tau}+z_{4\hat\tau}=0$ that $z_{\hat\tau}=z_{2\hat\tau}=0$. This implies from (EC.1.8) that $\int_0^{\hat\tau}rk_{4s}e^{-\int_s^{\hat\tau}m_\theta d\theta}ds=0$. Hence, $k_{4t}=0$ for all $t\in[0,\hat\tau)$.

From the differential equation for k_{4t} we find that $l_{2t}z_{4t}=0$ for all $t\in[0,\hat\tau)$, from which we derive that $y_{4t}=0$ and $z_{4t}=\delta e^{rt}$ for all $t\in[0,\hat\tau]$, contradicting again $z_{4\hat\tau}=0$.

The above contradiction implies that $z_{4\hat\tau}>0$ when $\hat\tau<\infty$. This completes the proof. Q.E.D.

Lower-Semi-Continuity Let $(x,y,k)\in\bar D$ be fixed and $\{(\ell_t,m_t,c_t)\}$ be an admissible strategy. Let $\{(x_t,y_t,k_t)\}$ be the solution of (2.1). Set $\tau=\min\{t\geq 0\mid z_t=0\}$.

Let $(\tilde x,\tilde y,\tilde k)\in\bar D$ be arbitrary. Let $\{(\tilde x_t,\tilde y_t,\tilde k_t)\}$ be the solution of sdes in (2.1) with strategy $\{(c_t,\ell_t,m_t)\}$ and initial value $(\tilde x_0,\tilde y_0,\tilde k_0)=(\tilde x,\tilde y,\tilde k)$. Set $\tilde\tau=\min\{t\geqslant 0|\tilde z_t=0\}$. By continuous dependence of sde with respect to initial value we have

$$\lim_{\bar D\ni(\tilde x,\tilde y,\tilde k)\to(x,y,k)}(\tilde x_t,\tilde y_t,\tilde k_t,\tilde z_t)=(x_t,y_t,k_t,z_t)\quad\forall\, t\geqslant 0.$$

Since $z_t>0$ for each $t\in[0,\tau)$, we derive that $\liminf_{\bar D\ni(\tilde x,\tilde y,\tilde k)\to(x,y,k)}\tilde\tau\geqslant\tau$. Consequently

$$\liminf_{\bar D\ni(\tilde x,\tilde y,\tilde k)\to(x,y,k)}V_\lambda(\tilde x,\tilde y,\tilde k)\geqslant\liminf_{\bar D\ni(\tilde x,\tilde y,\tilde k)\to(x,y,k)}\mathbb{E}\left[\int_0^{\tau\wedge\tilde\tau}U(c_t)dt\right]=\mathbb{E}\left[\int_0^{\tau}U(c_t)dt\right]$$

by the dominated convergence theorem. Upon trying all admissible strategies, we obtain

$$\liminf_{\bar D\ni(\tilde x,\tilde y,\tilde k)\to(x,y,k)}V_\lambda(\tilde x,\tilde y,\tilde k)\geqslant V_\lambda(x,y,k).$$

Thus, V_λ is lower-semicontinuous in $\bar D$.

Upper-Semi-Continuity Let $(x,y,k)\in\bar D$ be fixed. If $z=x+(1-\alpha)y+\alpha k=0$, then

$$\limsup_{\bar D\ni(\tilde x,\tilde y,\tilde k)\to(x,y,k)}V_\lambda(\tilde x,\tilde y,\tilde k)\leqslant\limsup_{\bar D\ni(\tilde x,\tilde y,\tilde k)\to(x,y,k)}C|\tilde z|^p=C|z|^p=0=V_\lambda(x,y,k).$$

Next we consider the case $z>0$. Let $\{(x_t,y_t,k_t)\}_{t\geq 0}$ be the solution of (2.1) with initial value $(x_0,y_0,k_0)=(x,y,k)$ and admissible strategy $\ell_t\equiv\lambda, c_t\equiv 0, m_t\equiv 0$, i.e. the solution of

$$\begin{cases}dx_t=(rx_t-\lambda z_t)dt,\\ dy_t=y_t[\mu dt+\sigma d\mathcal{B}_t]+\lambda z_t dt,\\ dk_t=\lambda z_t dt.\end{cases}$$

We define

$$\begin{aligned}\tau_2&=\sup\Big\{t>0\mid z<2z_s<4z\ \forall\, s\in[0,t]\Big\},\\ \tau_\delta&=\sup\Big\{t>0\,\Big|\,|(2\mu-\sigma^2)s+2\sigma\mathcal{B}_s|<\delta\ \forall\, s\in[0,t]\}.\end{aligned}$$

Since $z > 0$, we have $\tau_2 > 0$. When $t \in [0, \tau_2]$, $z/2 \leqslant z_t \leqslant 2z$. For any $t > 0$ and $\delta > 0$,

$$V_\lambda(x,y,k) \geqslant \mathbb{E}\left[V_\lambda(x_{t\wedge\tau\wedge\tau_\delta}, y_{t\wedge\tau\wedge\tau_\delta}, k_{t\wedge\tau\wedge\tau_\delta})e^{-\beta(t\wedge\tau_2\wedge\tau_\delta)}\right] .$$

Note that when $t \in [0, \tau_2 \wedge \tau_\delta]$, using $|(\mu - \sigma^2/2)s + \sigma\mathcal{B}_s| \leqslant \delta/2$ and $z/2 \leqslant z_s \leqslant 2z$ for each $s \in [0, t]$, we obtain estimates, setting $\eta = \lambda z/2$,

$$\begin{aligned}
k_t &= k + \int_0^t \lambda z_s ds \geqslant k + \eta t \geqslant (k + \eta t)e^{-\delta},\\
x_t &= e^{rt}\Big[x - \int_0^t \lambda z_s e^{-rs} ds\Big] \geqslant e^{rt}[x - 4\eta t],\\
y_t &\geq y e^{(\mu-\sigma^2/2)t+\sigma\mathcal{B}_t} + \eta\int_0^t e^{(\mu-\sigma^2/2)(t-s)+\sigma(\mathcal{B}_t-\mathcal{B}_s)} ds \geqslant (y + \eta t)e^{-\delta}.
\end{aligned}$$

Thus, when t and δ are small enough such that $(x - 4\eta t)e^{rt} + (1-\alpha)(y + \eta t)e^{-\delta} + \alpha(k + \eta t)e^{-\delta} > 0$, we have

$$\begin{aligned}
V_\lambda(x,y,k) &\geqslant \mathbb{P}(\tau_2 \wedge \tau_\delta \geq t)e^{-\beta t} V_\lambda(e^{rt}(x - 4\eta t), (y + \eta t)e^{-\delta}, (k + \eta t)e^{-\delta})\\
&= \mathbb{P}(\tau_2 \wedge \tau_\delta \geq t)e^{-\beta t - \delta p} V_\lambda(e^{rt+\delta}(x - 4\eta t), y + \eta t, k + \eta t)\\
&\geqslant \mathbb{P}(\tau_2 \wedge \tau_\delta \geqslant t)e^{-\beta t - \delta p}\Big\{V_\lambda(x + \eta t, y + \eta t, k + \eta t) - C|x + \eta t - e^{rt+\delta}(x - 4\eta t)|^p\Big\};
\end{aligned}$$

where the last equation is obtained by (EC.1.6). Thus, for arbitrary small enough $t > 0$ and $\delta > 0$,

$$V_\lambda(x + \eta t, y + \eta t, k + \eta t) \leqslant C|x + \eta t - e^{rt+\delta}(x - 4\eta t)|^p + \frac{e^{\beta t + \delta p} V(x,y,k)}{\mathbb{P}(\tau_2 \wedge \tau_\delta > t)}.$$

Hence, by monotonicity of V_λ,

$$\begin{aligned}
\limsup_{\bar{D}\ni(\tilde{x},\tilde{y},\tilde{k})\to(x,y,k)} V_\lambda(\tilde{x},\tilde{y},\tilde{k}) &\leqslant \limsup_{t\searrow 0} V_\lambda(x + \eta t, y + \eta t, k + \eta t)\\
&\leqslant \lim_{\delta\searrow 0}\lim_{t\searrow 0}\left\{C|x + \eta t - e^{rt+\delta}(x - 4\eta t)|^p + \frac{e^{\beta t+\delta p} V(x,y,k)}{\mathbb{P}(\tau_2 \wedge \tau_\delta > t)}\right\}\\
&= \lim_{\delta\searrow 0}\left\{C|x - xe^{\delta}|^p + V_\lambda(x,y,k)e^{\delta p}\right\} = V_\lambda(x,y,k);
\end{aligned}$$

here we use the fact that $\mathbb{P}(\tau_2 \wedge \tau_\delta > 0) = 1$. Thus, V_λ is upper-semi-continuous in $\bar{D}$.

This completes the proof of continuity.

Appendix EC.2: Proof of Theorem 3.1

EC.2.1. Convergence of Value Function

We introduce a formula for the liquidation value z_t of the portfolio at time $t \geqslant 0$. Denote by $\pi_t = y_t/S_t$ the number of stock shares in the portfolio. Then $y_t = \pi_t S_t$ and from the equations for dy_t and dk_t in (2.1) we find that

$$y_{t-}dM_t = dL_t - S_t d\pi_t, \qquad k_{t-}dM_t = dL_t - dk_t. \tag{EC.2.1}$$

Substituting these expressions into the dx_t equation in (2.1) gives

$$dx_t = (rx_t - c_t)dt - (1-\alpha)S_t d\pi_t - \alpha dk_t.$$

Using integrating factor e^{-rt} and initial value $x_{0-} = x$ we obtain

$$x_t = xe^{rt} - e^{rt}\int_{0-}^{t} e^{-rs}\Big[c_s ds + (1-\alpha)S_s d\pi_s + \alpha dk_s\Big].$$

Also, since $k_{0-} = k$ and $\pi_{0-} = y_{0-}/S_0 = y_{0-} = y$, we have

$$z_t = x_t + (1-\alpha)S_t\pi_t + \alpha k_t = x_t + (1-\alpha)S_t\Big(y + \int_{0-}^{t} d\pi_s\Big) + \alpha\Big(k + \int_{0-}^{t} dk_s\Big).$$

Thus, we obtain the following **decomposition formula**:

$$z_t = z_t^{\mathbf{x},C} - e^{rt}\Big\{(1-\alpha)\eta_t^{\pi} + \alpha\eta_t^{k}\Big\}, \tag{EC.2.2}$$

where

$$z_t^{\mathbf{x},C} = xe^{rt} + (1-\alpha)yS_t + \alpha k - \int_0^t e^{r(t-s)}c_s ds, \tag{EC.2.3}$$

$$\eta_t^{\pi} = \int_{0-}^{t} \Big[S_s e^{-rs} - S_t e^{-rt}\Big] d\pi_s, \tag{EC.2.4}$$

$$\eta_t^{k} = \int_{0-}^{t} \Big[e^{-rs} - e^{-rt}\Big] dk_s. \tag{EC.2.5}$$

It is easy to see from (EC.2.1) that $\{\pi_t, k_t\}$ depends only on (y, k, L, M); in particular, it does not depend on (x, C).

Let $\mathbf{x} = (x, y, k) \in \bar{D}$ be an arbitrarily fixed point and ε be an arbitrarily fixed small positive constant. We want to show that there exists a positive and finite constant λ_ε such that

$$V_*(\mathbf{x}) \geqslant V(\mathbf{x}, \lambda) \geqslant V_*(\mathbf{x}) - 4\varepsilon \qquad \forall\, \lambda \geqslant \lambda_\varepsilon. \tag{EC.2.6}$$

Here λ_ε may depend on variable $\mathbf{x}$. Since $\mathcal{A}_\lambda \subset \mathcal{A}$, the first inequality is trivial. We then focus on the second inequality. We divide the proof into five steps.

Step 1. Use Continuity of V_*.

Since V_* is assumed to be continuous on $\bar{D}$, there exists $\delta > 0$ such that

$$|V_*(\hat{\mathbf{x}}) - V_*(\mathbf{x})| \leqslant \varepsilon \qquad \forall \hat{\mathbf{x}} \in \bar{D},\ |\hat{\mathbf{x}} - \mathbf{x}| \leqslant 2\delta.$$

Set $z = x + (1-\alpha)y + \alpha k$. If $z < 2\delta$, then for $\hat{\mathbf{x}} = (-(1-\alpha)y - \alpha z, y, k)$, the corresponding portfolio is instantly liquidated so $V_*(\hat{\mathbf{x}}) = 0$; see Ben Tahar, Soner and Touzi (2007, 2010). This implies from $|V_*(\hat{\mathbf{x}}) - V_*(\mathbf{x})| \leqslant \varepsilon$ that $V_*(\mathbf{x}) \leqslant \varepsilon$, so (EC.2.6) holds, since $V(\mathbf{x}, \lambda) \geqslant \int_0^\infty \mathbf{U}(0,t)dt = 0$. In the sequel, we examine the general case $z \geqslant 2\delta$.

Set $\mathbf{x}_\delta = (x - \delta, y, k)$. We shall construct a finite rate investment-consumption strategy $\mathcal{S}^*$ for the regular control problem with initial portfolio $\mathbf{x}$ from an ε-suboptimal strategy $\mathcal{S}$ for the singular control problem with initial portfolio $\mathbf{x}_\delta$. Basically $\mathcal{S}^*$ is a regularization of the singular control $\mathcal{S}$, where the extra δ amount of initial cash is put into the cash account to make sure that the consumption strategy in $\mathcal{S}^*$ can follow the consumption strategy in $\mathcal{S}$ for a long enough time, so the value function $V(\mathbf{x}, \lambda)$ is close to the value function $V_*(\mathbf{x}_\delta)$.

Since $z \geqslant 2\delta$, we have $\mathbf{x}_\delta \in \bar{D}$ and $V_*(\mathbf{x}_\delta) \geqslant V_*(\mathbf{x}) - \varepsilon$. There exists a strategy $\mathcal{S} = (C, L, M) \in \mathcal{A}$, with $C = \{c_t\}, L = \{L_t\}$, and $M = \{M_t\}$, such that

$$V_*(\mathbf{x}_\delta) \leqslant \mathbb{E}\Big[\int_0^{\tau_{\mathbf{x}_\delta}} \mathbf{U}(c_t, t)dt\Big] + \varepsilon,$$

where $\tau_{\mathbf{x}_\delta}$ is the first time at which the liquidation value of the portfolio starting from $\mathbf{x}_\delta$ with strategy $\mathcal{S}$ becomes zero. We define, for each $n \geqslant \max\{1, 5/\delta\}$,

$$\tau^n = \sup\Big\{t \in [0,n] \,\Big|\, t < \tau_{\mathbf{x}_\delta}, L_t \leqslant n,\ M_t \leqslant n,\ \frac{\max_{0\leqslant s\leqslant t} S_s}{\min_{0\leqslant s\leqslant t} S_s} \leqslant n\Big\}. \tag{EC.2.7}$$

Since almost surely $\tau^n \to \tau_{\mathbf{x}_\delta}$ as $n \to \infty$, by Fatou's Lemma, there exists an $n \geqslant 1 + y + k + 3r$ such that,

$$\mathbb{E}\Big[\int_0^{\tau_{\mathbf{x}_\delta}} \mathbf{U}(c_t, t)dt\Big] \leqslant \mathbb{E}\Big[\int_0^{\tau^n} \mathbf{U}(c_t, t)dt\Big] + \varepsilon.$$

Also, by integrability, there exists $\delta_1 > 0$ such that

$$\mathbb{E}\Big[\int_0^{\tau^n} \mathbf{U}(c_t, t)dt\Big] \leqslant \mathbb{E}\Big[\mathbf{1}_{\hat{\Omega}} \int_0^{\tau^n} \mathbf{U}(c_t, t)dt\Big] + \varepsilon \ \text{ as long as } \ \mathbb{P}(\hat{\Omega}) \geqslant 1 - \delta_1.$$

In conclusion, writing τ^n as τ for notational simplicity, we derive that

$$\mathbb{E}\Big[\mathbf{1}_{\hat{\Omega}}\int_0^\tau \mathbf{U}(c_t,t)dt\Big] \geqslant V_*(\mathbf{x}) - 4\varepsilon \qquad \text{if} \;\; \mathbb{P}(\hat{\Omega}) \geqslant 1-\delta_1. \tag{EC.2.8}$$

We shall construct a strategy $\mathcal{S}^* = (C^*, L^*, M^*)$ in $\mathcal{A}_{\lambda_\varepsilon}$ for some positive λ_ε such that $c_t^* = c_t$ for all $t > 0$ and $\tau_{\mathbf{x}}^* \geqslant \tau$ on a set $\hat{\Omega}$ of measure at least $1-\delta_1$, where $\tau_{\mathbf{x}}^*$ is the first time at which the liquidation value of the portfolio starting from $\mathbf{x}$ with strategy $\mathcal{S}^*$ becomes zero. Then (EC.2.6) follows from (EC.2.8).

For convenience, we modify (C, L, M) to a new strategy, which for notational simplicity we still denote by (C, L, M), such that

$$(L_t, M_t, c_t) = (L_\tau, M_\tau, 0) \quad \forall\, t \geqslant \tau.$$

Also, for convenience, we make the natural extension

$$(L_t, M_t, c_t, S_t, \pi_t, y_t, k_t) = (0, 0, 0, e^{rt}, y, ye^{rt}, k) \quad \forall\, t < 0.$$

We denote by (π_t, y_t, k_t) the solution of (EC.2.1) subject to the initial value $(\pi_{0-}, y_{0-}, k_{0-}) = (y, y, k)$. Since $S_0 = 1$, we have $y_t = \pi_t S_t$. In addition,

$$(\pi_t, k_t) = (\pi_\tau, k_\tau) \quad \forall\, t \geqslant \tau.$$

Sine $(C, L, M) \in \mathcal{A}$ is associated with the initial portfolio $\mathbf{x}_\delta$, we have, by the formula (EC.2.2),

$$z_t^{\mathbf{x}_\delta, C} - (1-\alpha)\eta_t^\pi - \alpha\eta_t^k \geqslant 0 \qquad \forall\, t \geqslant 0. \tag{EC.2.9}$$

Note that if we start from the initial portfolio $\mathbf{x}$ with the same consumption strategy, then

$$z_t^{\mathbf{x}, C} = z_t^{\mathbf{x}_\delta, C} + \delta e^{rt}. \tag{EC.2.10}$$

With the same consumption strategy $C^* = C$ (up to τ), we shall use the extra wealth δe^{rt} to balance the deviation of the liquidation value of the portfolio with a new investment strategy (L^*, M^*) from that of the reference portfolio with strategy (L, M).

Remark EC.2.1. Once ε is given, one finds a constant δ and a sub-optimal strategy that depend on ε. Then we find constants n and δ_1 depending on ε. Theoretically, these constants can be estimated explicitly by using the theory of probability. In the sequel, ε, n, δ, and δ_1 are all fixed and $\tau = \tau^n$ defined in (EC.2.7).

Step 2. Remove Wash-Sale.

We call a sale from stock holding $y_{t-} > 0$ to $y_t = 0$ a **Wash-Sale**, i.e., the sale at which $dM_t = 1$. Since a regular control can never make stock holding diminish in finite time, we shall modify the sell strategy M to $\tilde{M}$ where $\tilde{M}$ does not contain any Wash-Sale.

Let $\tilde{h} \in (0, 1/3)$ be a small positive constant to be determined. Set $\tilde{M} = (1 - \tilde{h})M$. We define $\{\tilde{\pi}_t\}$ and $\{\tilde{k}_t\}$ as the solution of (EC.2.1) with M replaced by $\tilde{M}$ and L unchanged; more precisely, $(\{k_t\}, \{\pi_t\}, \{\tilde{k}_t\}, \{\tilde{\pi}_t\})$ are solutions of the following SDEs:

$$\begin{aligned} d\pi_t &= \frac{dL_t}{S_t} - \pi_{t-} dM_t, \quad d\tilde{\pi}_t = \frac{dL_t}{S_t} - \tilde{\pi}_{t-} d\tilde{M}_t \ \ \forall\, t \geqslant 0, \\ dk_t &= dL_t - k_{t-} dM_t, \quad d\tilde{k}_t = dL_t - \tilde{k}_{t-} d\tilde{M}_t \ \forall\, t \geqslant 0, \\ \pi_t &= y, k_t = k, \quad \tilde{\pi}_t = y, \tilde{k}_t = k \qquad\qquad \forall\, t < 0. \end{aligned} \tag{EC.2.11}$$

By comparison (the strategy $\tilde{M}$ sells no more than that of the strategy M), we see that $0 \leqslant k_t \leqslant \tilde{k}_t, 0 \leqslant \pi_t \leqslant \tilde{\pi}_t$. Taking the difference of the equations for $d\pi_t$ and $d\tilde{\pi}_t$ we obtain

$$d(\tilde{\pi}_t - \pi_t) = -(\tilde{\pi}_{t-} - \pi_{t-}) d\tilde{M}_t + \tilde{h} \pi_{t-} dM_t.$$

After integration, we obtain for each $t \in \mathbb{R}$,

$$\int_{-\infty}^{t} [\tilde{\pi}_{s-} - \pi_{s-}] d\tilde{M}_s = \tilde{h} \int_{-\infty}^{t} \pi_{s-} dM_s + (\pi_t - \tilde{\pi}_t) \leqslant \tilde{h} \int_{-\infty}^{t} \pi_{s-} dM_s.$$

Consequently,

$$\int_{-\infty}^{t} \Big| d(\tilde{\pi}_s - \pi_s) \Big| \leqslant \int_{-\infty}^{t} (\tilde{\pi}_{s-} - \pi_{s-}) d\tilde{M}_s + \tilde{h} \int_{-\infty}^{t} \pi_{s-} dM_s \leqslant 2\tilde{h} \int_{-\infty}^{t} \pi_{s-} dM_s.$$

We now find an upper bound of π_t. Since $d\pi_t \leqslant dL_t / S_t$, after integration, we have

$$0 \leqslant \pi_t \leqslant y + \int_{[0,t]} \frac{dL_s}{S_s} \leqslant \frac{y + L_\tau}{\min_{0 \leqslant s \leqslant \tau} S_s}.$$

Thus,

$$\int_{-\infty}^{\infty} \Big| d(\tilde{\pi}_s - \pi_s) \Big| \leqslant 2\tilde{h} \frac{y + L_\tau}{\min_{0 \leqslant s \leqslant \tau} S_s} M_\tau \leqslant \frac{4\tilde{h} n^2}{\min_{0 \leqslant s \leqslant \tau} S_s}.$$

Consequently, by the definition of η_t^π in (EC.2.4) we obtain

$$\sup_{t \in \mathbb{R}} \Big| \eta_t^\pi - \eta_t^{\tilde{\pi}} \Big| \leqslant \max_{0 \leqslant s \leqslant \tau} S_s \int_{\mathbb{R}} \Big| d(\pi_s - \tilde{\pi}_s) \Big| \leqslant 4\tilde{h} n^2 \frac{\max_{0 \leqslant s \leqslant \tau} S_s}{\min_{0 \leqslant s \leqslant \tau} S_s} \leqslant 4\tilde{h} n^3. \tag{EC.2.12}$$

In a similar manner, for η_t^k defined in (EC.2.5) we can show that

$$\sup_{t\in\mathbb{R}}\left|\eta_t^k-\eta_t^{\tilde{k}}\right|\leqslant\int_{\mathbb{R}}\left|dk_s-d\tilde{k}_s\right|\leqslant 2\tilde{h}[y+L_\tau]M_\tau\leqslant 4\tilde{h}n^2. \tag{EC.2.13}$$

With $\tilde{M}_t=(1-\tilde{h})M_t$, we would like to solve the linear ODE $dX_t=-X_{t-}d\tilde{M}_t$. If $\Delta\tilde{M}_t:=\tilde{M}_t-\tilde{M}_{t-}>0$, a direct computation gives $X_t=(1-\Delta\tilde{M}_t)X_{t-}=e^{\ln(1-\Delta\tilde{M}_t)}X_{t-}$. Taking care of countably many jumps of $\tilde{M}$, we derive that the solution of $dX_t=-X_{t-}d\tilde{M}_t$ is given by $X_t=X_{0-}e^{-\hat{M}_t}$ where $\hat{M}_t$ is defined as follows: $\hat{M}_t=0$ for $t<0$; for $t\geqslant 0$,

$$\hat{M}_t=\tilde{M}_t+\sum_{s\leqslant t,\,\Delta M_s>0}\left\{\ln\frac{1}{1-\Delta\tilde{M}_s}-\Delta\tilde{M}_s\right\}\quad\forall\, t\geqslant 0.$$

We claim that the function is bounded. Note that for $\zeta\in[0,1)$, we have

$$\ln\frac{1}{1-\zeta}-\zeta=\sum_{m=2}^{\infty}\frac{\zeta^m}{m}\leqslant\zeta\ln\frac{1}{1-\zeta}.$$

Since $\Delta\tilde{M}_t=(1-\tilde{h})\Delta M_t\leqslant 1-\tilde{h}$, applying the above estimate for $\zeta=\Delta\tilde{M}_s\leqslant 1-\tilde{h}$ we find that

$$\hat{M}_t\leqslant\tilde{M}_t+\ln\frac{1}{\tilde{h}}\sum_{s\leqslant t,\,\Delta M_s>0}\Delta\tilde{M}_s\leqslant\left[1+\ln\frac{1}{\tilde{h}}\right]M_t\leqslant 2|\ln\tilde{h}|M_t\quad\forall\, t\geqslant 0.$$

We use the notation $dM_t=M_t-M_{t-}$ when $M_t>M_{t-}$. Direct differentiation gives

$$de^{-\hat{M}_t}=-e^{-\hat{M}_{t-}}d\tilde{M}_t.$$

Also, approximating $\tilde{M}_t$ by functions of finitely many jumps, one can derive that the solutions $(\tilde{\pi},\tilde{k})$ of the linear equations in (EC.2.11) are given by

$$\tilde{k}_t=e^{-\hat{M}_t}\Big(y+\int_{-\infty}^{t}e^{\hat{M}_{s-}}dL_s\Big),\qquad\tilde{\pi}_t=e^{-\hat{M}_t}\Big(k+\int_{-\infty}^{t}e^{\hat{M}_{s-}}\frac{dL_s}{S_s}\Big)\quad\forall\, t\in\mathbb{R}. \tag{EC.2.14}$$

Step 3. Construct Regular Investment Strategy.

Let $h\in(0,1/3)$ be a small constant to be fixed. Set $t_i=ih$ for $i\in\mathbb{Z}$. We now define a regular investment strategy L^* and M^*. To make sure that (L^*,M^*) is adapted, we define (L^*,M^*) on $[t_i,t_{i+1}]$ via information on time interval $[t_{i-1},t_i]$. We define L^* by $L_t^*=0$ for all $t\leqslant 0$ and

$$L_t^*=\begin{cases}L_{t_i-h} & \text{if}\ \ t\in(t_i,t_i+\frac{2h}{5}],\\[2ex] L_{t_i-h}+(L_{t_i}-L_{t_i-h})\frac{5t-5t_i-2h}{h} & \text{if}\ \ t\in(t_i+\frac{2h}{5},t_i+\frac{3h}{5}],\\[2ex] L_{t_i} & \text{if}\ \ t\in(t_i+\frac{3h}{5},t_{i+1}],\end{cases}\qquad\forall\, i\in\mathbb{Z},i\geqslant 0.$$

One can check that L_t^* is increasing and Lipschitz continuous in t:

$$\sup_{t\in\mathbb{R}} \frac{dL_t^*}{dt} = \max_{i\geqslant 0} \frac{5(L_{t_i} - L_{t_i-h})}{h} \leqslant \frac{5L_\tau}{h} \leqslant \frac{5n}{h}.$$

Also,

$$L^*_{t_{i+1}} = L_{t_i} \quad \forall\, i \in \mathbb{Z}.$$

We will construct M^* such that $M^*_{t_i} = \hat{M}_{t_{i-1}}$ and $k^*_{t_i} = \tilde{k}_{t_{i-1}}$. For this purpose, we define, for each integer $i \geqslant -1$,

$$\bar{M}_i = \ln \frac{\int_{(t_i,t_{i+1}]} e^{\hat{M}_{s-}} dL_s}{\int_{(t_i,t_{i+1}]} dL_s} \text{ if } \; L_{t_{i+1}} - L_{t_i} > 0, \qquad \bar{M}_i = \hat{M}_{t_i+h/2} \; \text{ otherwise.}$$

Since $\hat{M}_t$ is increasing in t, we have $\hat{M}_{t_i} \leqslant \bar{M}_i \leqslant \hat{M}_{t_{i+1}}$. Also,

$$\int_{(t_i,t_{i+1}]} e^{\hat{M}_{s-}} dL_s = e^{\bar{M}_i} \int_{(t_i,t_{i+1}]} dL_s = \int_{t_i}^{t_{i+1}} e^{\bar{M}_i} dL^*_{s+h}.$$

Now we define $M^*_t = 0$ for all $t \leqslant 0$ and

$$M_t^* = \begin{cases} \hat{M}_{t_{i-1}} + (\bar{M}_{i-1} - \hat{M}_{t_{i-1}}) \frac{3t-3t_i}{h} & \text{if } \; t \in (t_i, t_i + \frac{h}{3}], \\ \bar{M}_{i-1} & \text{if } \; t \in (t_i + \frac{h}{3}, t_i + \frac{2h}{3}], \\ \bar{M}_{i-1} + (\hat{M}_{t_i} - \bar{M}_{i-1}) \frac{3t-3t_i-2h}{h} & \text{if } \; t \in (t_i + \frac{2h}{3}, t_{i+1}], \end{cases} \qquad \forall\, i \in \mathbb{Z}, i \geqslant 0.$$

Then, as a function of t, M_t^* is increasing and Lipschitz continuous:

$$\sup_{t\in\mathbb{R}} \frac{dM_t^*}{dt} \leqslant \frac{3\hat{M}_\tau}{h} \leqslant \frac{6n|\ln \tilde{h}|}{h}; \qquad \frac{dL_t^*}{dt}\frac{dM_t^*}{dt} = 0 \quad \forall\, t \in \mathbb{R}.$$

In addition, for $i \in \mathbb{Z}$,

$$M^*_{t_{i+1}} = \hat{M}_{t_i}, \qquad \int_{t_i}^{t_{i+1}} e^{M_s^*} dL_s^* = e^{\bar{M}_{i-1}} \int_{t_i}^{t_{i+1}} dL_s^* = \int_{(t_i,t_{i+1}]} e^{\hat{M}_{(s-h)-}} dL_{s-h}. \tag{EC.2.15}$$

Note that both $\{M_t^*\}_{t\geqslant 0}$ and $\{L_t^*\}_{t\geqslant 0}$ are adapted processes, since for $t \in [t_i, t_{i+1}]$, both M_t^* and L_t^* depend only on $\{M_s, L_s\}$ with $s \in [t_{i-1}, t_i]$, which are $\mathcal{F}_{t_i}$ measurable.

Finally, we define $\{\pi_t^*, k_t^*\}$ as the solution of

$$d\pi_t^* = \frac{dL_t^*}{S_t} - \pi_t^* dM_t^*, \quad dk_t^* = dL_t^* - k_t^* dM_t^* \quad \forall\, t \geqslant 0,$$
$$\pi_t^* = y, k_t^* = k \quad \forall\, t < 0.$$

Then by variation of constants, we derive that

$$k_t^* = e^{-M_t^*}\Big(y + \int_{-\infty}^t e^{M_s^*} dL_s^*\Big), \qquad \pi_t^* = e^{-M_t^*}\Big(k + \int_{-\infty}^t e^{M_s^*} \frac{dL_s^*}{S_s}\Big) \quad \forall\, t \in \mathbb{R}. \tag{EC.2.16}$$

In view of (EC.2.14), (EC.2.15), and (EC.2.16), we find that

$$k_{t_i}^* = \tilde{k}_{t_{i-1}} \quad \forall\, i \in \mathbb{Z}.$$

Also,

$$\begin{aligned}
\tilde{k}_t \leqslant y + L_t \leqslant 2n, \ \ \int_{\mathbb{R}} \Big|d\tilde{k}_t\Big| \leqslant \int_{\mathbb{R}} dL_t + 2n \int_{\mathbb{R}} d\tilde{M}_t \leqslant 3n^2,\\
k_t^* \leqslant y + L_t^* \leqslant 2n, \ \ \int_{\mathbb{R}} \Big|dk_t^*\Big| \leqslant \int_{\mathbb{R}} dL_t^* + 2n \int_{\mathbb{R}} dM_t^* \leqslant 5|\ln \tilde{h}|n^2.
\end{aligned}$$

Similarly, we can derive that

$$\max\{\tilde{\pi}_t, \pi_t^*\} \leqslant 2n^2, \quad \int_{\mathbb{R}} \Big|d\tilde{\pi}_t\Big| \leqslant 3n^3, \quad \int_{\mathbb{R}} \Big|d\pi_t^*\Big| \leqslant 5|\ln \tilde{h}|n^3.$$

Finally we estimate $\pi_{t_{i+1}}^* - \tilde{\pi}_{t_i}$. Using (EC.2.14), (EC.2.15) and (EC.2.16) we derive that

$$\begin{aligned}
\pi_{t_{i+1}}^* - \tilde{\pi}_{t_i} &= e^{-\hat{M}_{t_i}} \sum_{j=-\infty}^{i} \int_{t_j}^{t_{j+1}} \Big(\frac{e^{M_s^*} dL_s^*}{S_s} - \frac{e^{\hat{M}_{s-h}} dL_{s-h}}{S_{s-h}}\Big)\\
&= e^{-\hat{M}_{t_i}} \sum_{j=-\infty}^{i} \int_{t_j}^{t_{j+1}} \Big\{\Big(\frac{1}{S_s} - \frac{1}{S_{t_j}}\Big) e^{M_s^*} dL_s^* + \Big(\frac{1}{S_{t_j}} - \frac{1}{S_{s-h}}\Big) e^{\hat{M}_{s-h}} dL_{s-h}\Big\}.
\end{aligned}$$

For $\omega \in \Omega$, denote the norm of continuity of stock price by

$$\rho(\omega, h) := \max_{0 \leqslant t_1, t_2 \leqslant \tau, |t_2 - t_1| \leqslant h} \max\Big\{\Big|\frac{1}{S_{t_2}(\omega)} - \frac{1}{S_{t_1}(\omega)}\Big|, \Big|S_{t_1}(\omega) e^{-rt_1} - S_{t_2}(\omega) e^{-rt_2}\Big|\Big\}.$$

Then we have

$$|\pi_{t_{i+1}}^* - \tilde{\pi}_{t_i}| \leqslant \rho(\omega, h)\Big(\int_{-\infty}^{t_{i+1}} dL_s^* + \int_{-\infty}^{t_i} dL_s\Big) \leqslant 2n\rho(\omega, h) \quad \forall\, i \in \mathbb{Z}. \tag{EC.2.17}$$

Step 4. Construct the Regular Strategy in $\mathcal{A}_\lambda$.

Consider the strategy $\mathcal{S}_1 = (C, L^*, M^*)$. We denote by $\{x_t^*, y_t^*, k_t^*\}$ the solution of (2.1) with initial data $(x_0^*, y_0^*, k_0^*) = \mathbf{x}$ and strategy $\mathcal{S}_1$. We set $z_t^* = x_t^* + (1-\alpha) y_t^* + \alpha k_t^*$ and

$$\tau^* = \sup\{t \geqslant 0 \mid z_s^* \geqslant z_s + \tfrac{1}{n} \ \ \forall\, s \in [0, t]\}.$$

Modify $\mathcal{S}_1$ for $t > \tau^*$ by no consumption and no trading. We then obtain an admissible strategy in $\mathcal{A}_\lambda$ with $\lambda = 6n^2|\ln\tilde{h}|/h$ (note that $z_t^* \geqslant \frac{1}{n}$ for all $t \in [0,\tau^*]$ and $dL_t^* = 0$ and $dM_t^* = 0$ for $t > \tau^*$). This gives the estimate, since $\mathbf{U} \geqslant 0$,

$$V(\mathbf{x},\lambda) \geqslant \mathbb{E}\Big[\int_0^{\tau^*} \mathbf{U}(c_t,t)dt\Big].$$

Let us estimate τ^*. We can apply formula (EC.2.2) and the estimates (EC.2.9) and (EC.2.10) to obtain

$$\begin{aligned} z_t^* &= z^{\mathbf{x}_\delta,C} + e^{rt}\Big\{\delta - (1-\alpha)\eta_t^{\pi^*} - \alpha\eta_t^{k^*}\Big\} \\ &= z_t + e^{rt}\Big\{\delta - (1-\alpha)[\eta_t^{\pi^*} - \eta_t^{\pi}] - \alpha[\eta_t^{k^*} - \eta_t^{k}]\Big\}. \end{aligned} \tag{EC.2.18}$$

Now we estimate the right-hand side of the inequality.

- First of all, we have

$$\eta_t^{k^*} - \eta_t^{\tilde{k}} = \int_{-\infty}^{t} [e^{-rs} - e^{-rt}]d(k_s^* - \tilde{k}_s) = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &:= \int_{-\infty}^{t-h} [e^{-r(s+h)} - e^{-rt}]d(k_{s+h}^* - \tilde{k}_s), \\ I_2 &:= \int_{t-h}^{t} [e^{-rt} - e^{-rs}]d\tilde{k}_s + \int_{-\infty}^{t-h} [e^{-r(s+h)} - e^{-rs}]d\tilde{k}_s. \end{aligned}$$

Since $1 - e^{-z} \leqslant z$ for $z > 0$, we have

$$|I_2| \leqslant [1 - e^{-rh}]\int_{\mathbb{R}} e^{-rs}|d\tilde{k}_s| \leqslant rh\int_{\mathbb{R}} |dk_s|.$$

Next, since $k_{t_{j+1}}^* = \tilde{k}_{t_j}$ for every integer j, we have, for each integer i,

$$\begin{aligned} &\Big|\int_{t_i}^{t_{i+1}} [e^{-r(s+h)} - e^{-rt}]d(k_{s+h}^* - \tilde{k}_s)\Big| \\ &= \Big|\int_{t_i}^{t_{i+1}} [e^{-r(s+h)} - e^{-rt_{i+1}}]d(k_{s+h}^* - \tilde{k}_s)\Big| \\ &\leqslant rh\int_{t_i}^{t_{i+1}} \Big(|dk_{s+h}^*| + |d\tilde{k}_s|\Big). \end{aligned}$$

This estimate also holds if the upper limit t_{i+1} of the integral is replaced by t for $t \in [t_i, t_{i+1}]$. Hence,

$$|I_2| + |I_1| \leqslant rh\int_{\mathbb{R}} \Big(2|dk_s| + |dk_s^*|\Big) \leqslant 11rh|\ln\tilde{h}|n^2.$$

Finally, using (EC.2.13) and $n \geqslant 1+3r$, we obtain

$$\begin{aligned}\sup_{t\in\mathbb{R}}\left|\eta_t^{k^*}-\eta_t^{k}\right| &\leqslant \sup_{t\in\mathbb{R}}\left|\eta_t^{k^*}-\eta_t^{\tilde{k}}\right|+\sup_{t\in\mathbb{R}}\left|\eta_t^{\tilde{k}}-\eta_t^{k}\right|\\ &\leqslant 4[\tilde{h}+h|\ln\tilde{h}|]n^3 \quad \forall\, t\in\mathbb{R}.\end{aligned}$$

- Now we estimate

$$\begin{aligned}\eta_t^{\pi^*}-\eta_t^{\tilde{\pi}} &= \int_{-\infty}^{t}\Big[S_s e^{-rs}-S_t e^{-rt}\Big]d[\pi_s^*-\tilde{\pi}_s]=J_1+J_2,\\ J_1 &:= \int_{-\infty}^{t-h}\Big[S_{s+h}e^{-r(s+h)}-S_t e^{-rt}\Big]d[\pi_{s+h}^*-\tilde{\pi}_s],\\ J_2 &:= \int_{t-h}^{t}\Big[S_t e^{-rt}-S_s e^{-rs}\Big]d\tilde{\pi}_s+\int_{-\infty}^{t-h}\Big[S_{s+h}e^{-r(s+h)}-S_s e^{-rs}\Big]d\tilde{\pi}_s.\end{aligned}$$

The term J_2 can be estimated in the same way as before:

$$|J_2|\leqslant \rho(\omega,h)\int_{\mathbb{R}}|d\tilde{\pi}_t|\leqslant 3n^3\rho(\omega,h).$$

To estimate J_1, let $h_1=Nh$ where $N\geqslant 3$ is an integer. Without loss of generality, we assume that $S_t=S_\tau e^{-r(t-\tau)}$ when $t\geqslant\tau$. We compute

$$\begin{aligned}&\Big|\int_{t_{i-N}}^{t_i}\Big[S_{s+h}e^{-r(s+h)}-S_t e^{-rt}\Big]d[\pi_{s+h}^*-\tilde{\pi}_s]\Big|\\ &\leqslant \Big|\int_{t_{i-N}}^{t_i}\Big[S_{s+h}e^{-r(s+h)}-S_{t_i}e^{-rs_{t_i}}\Big]d[\pi_{s+h}^*-\tilde{\pi}_s]\Big|\\ &\quad+\Big|\Big[S_{t_i}e^{-rs_{t_i}}-S_t e^{-rt}\Big][\pi_{s+h}^*-\tilde{\pi}_s]\Big|_{s=t_{i-N}}^{s=t_i}\Big|\\ &\leqslant \rho(\omega,h_1)\int_{t_{i-N}}^{t_i}\Big(|d\pi_{t+h}^*|+|\tilde{\pi}_t|\Big)+4n\rho(\omega,h)\max_{0\leqslant s\leqslant\tau}\{S_s e^{-rs}\},\end{aligned}$$

by the definition of ρ and the estimate (EC.2.17). Hence, we obtain

$$\begin{aligned}|J_1| &\leqslant \rho(\omega,h_1)\int_{\mathbb{R}}\Big(|d\tilde{\pi}_t|+|\mathrm{d}\pi_t^*|\Big)+\Big(\frac{\tau}{h_1}+1\Big)4n\rho(\omega,h)\max_{0\leqslant s\leqslant\tau}\{S_s e^{-rs}\}\\ &\leqslant 8n^3\Big(|\ln\tilde{h}|\rho(\omega,h_1)+\frac{\rho(\omega,h)}{h_1}\Big).\end{aligned}$$

Summarizing the estimate and using (EC.2.12) we then obtain

$$\sup_{t\in\mathbb{R}}\left|\eta_t^{\pi^*}-\eta_t^{\pi}\right|\leqslant 8n^3\Big\{\frac{\tilde{h}}{2}+\rho(\omega,h_1)|\ln\tilde{h}|+\frac{\rho(\omega,h)}{h_1}\Big\}.$$

Altogether, we then obtain

$$z_t^*\geqslant z_t+e^{rt}\Big(\delta-8n^3\Big[\tilde{h}+h|\ln\tilde{h}|+\rho(\omega,h_1)|\ln\tilde{h}|+\frac{\rho(\omega,h)}{h_1}\Big]\Big).$$

Step 5. Complete the Proof.

We now fix

$$\tilde{h} = \min\left\{\frac{1}{3}, \frac{\delta}{40n^3}\right\}.$$

Since each Brownian sample curve is continuous, there exists $h_1 \in (0, 1/3)$ such that

$$\mathbb{P}(\Omega_1) \geqslant 1 - \frac{\delta_1}{2} \quad \text{where} \quad \Omega_1 := \left\{\omega \in \Omega \mid \rho(\omega, h_1) \leqslant \frac{\delta}{40n^3|\ln \tilde{h}|}\right\}.$$

Also, there exists $h_\varepsilon > 0$ such that $h_\varepsilon < \delta/(40n^3|\ln \tilde{h}|)$, $h_1/h_\varepsilon > 3$, and

$$\mathbb{P}(\Omega_2) \geqslant 1 - \frac{\delta_1}{2} \quad \text{where} \quad \Omega_2 := \left\{\omega \in \Omega \mid \rho(\omega, h_\varepsilon) \leqslant \frac{\delta h_1}{40n^3}\right\}.$$

Thus, set

$$\lambda_\varepsilon = \frac{6n^2|\ln \tilde{h}|}{h_\varepsilon}.$$

With $h = h_\varepsilon$, we have

$$z_t^*(\omega) \geqslant z_t(\omega) + e^{rt}\left(\delta - \frac{\delta}{5} - \frac{\delta}{5} - \frac{\delta}{5} - \frac{\delta}{5}\right) \geqslant z_t(\omega) + \frac{\delta}{5} \quad \forall\, t \in [0, \tau(\omega)],\ \omega \in \Omega_1 \cap \Omega_2.$$

Consequently, since $\frac{\delta}{5} \geqslant \frac{1}{n}$, we see that $\tau^* \geqslant \tau$ on $\Omega_1 \cap \Omega_2$, so

$$V(\mathbf{x}, \lambda_\varepsilon) \geqslant \mathbb{E}\left[\mathbf{1}_{\Omega_1 \cap \Omega_2} \int_0^\tau U(c_t, t)dt\right] \geqslant V_*(\mathbf{x}) - 4\varepsilon,$$

by (EC.2.8). Since $V(\mathbf{x}, \lambda)$ is increasing in λ, we thus obtain (EC.2.6). Sending $\varepsilon \searrow 0$ we then complete the proof of Theorem 3.1.