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TOP INCOMES AND INCOME INEQUALITY INDICES: A UNIFIED FRAMEWORK BASED ON INEQUALITY INDEX CURVES

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An income inequality index number cannot summarize the complete information in the distribution. We propose a family of inequality index curves, which includes curves generated by popular inequality index numbers (e.g. the top income shares, the Gini coefficient and the Palma ratio). The family has two advantages: (1) The family has an axiomatic foundation. (2) Each curve in the family contains the full information of the distribution. We use the family and micro level data to show that the bottom and middle income people in the U.S. became more equally relatively poor (not just relatively poorer) from 1990 to 2010.

KEYWORDS: income inequality, top income shares, weighted expected utility.
JEL Code: D31, D63, D81.

1. INTRODUCTION

Two popular income inequality measures are the Gini coefficient (or the Gini index) and the top income shares. However, an index number (such as the Gini coefficient and the top income share at a given level), which assigns a single numerical value to the income distribution, cannot summarize the complete information in the income distribution.

Instead of using single index numbers, we propose a family of inequality index curves, which includes many curves generated by popular inequality index numbers (e.g. the top income shares, the Gini coefficient, the single parameter Gini coefficient, the Palma ratio, and the Hoover index). Theoretically, we show that the family has an axiomatic foundation based on the weighted expected utility theory and each curve in the family contains the full information of the income distribution. Empirically, we use the family of inequality index curves to show that the bottom and middle income people in the U.S. became more equally relatively poor (not just relatively poorer) from 1990 to 2010.

1.1. *Motivation for a Unified Framework*

Figure 1 shows the Gini coefficients and top 5% income shares for individual income in the United States from 1980 to 2010. It can be observed that during the period, the Gini coefficients and the top 5% income shares had different patterns. In particular the Gini coefficients are very stable, while top 5% income shares exhibit a significant increase. This demonstrates that two index numbers measure different aspects of the income distribution. Thus, it is desirable to have a unified framework to see full information contained in the income distribution.

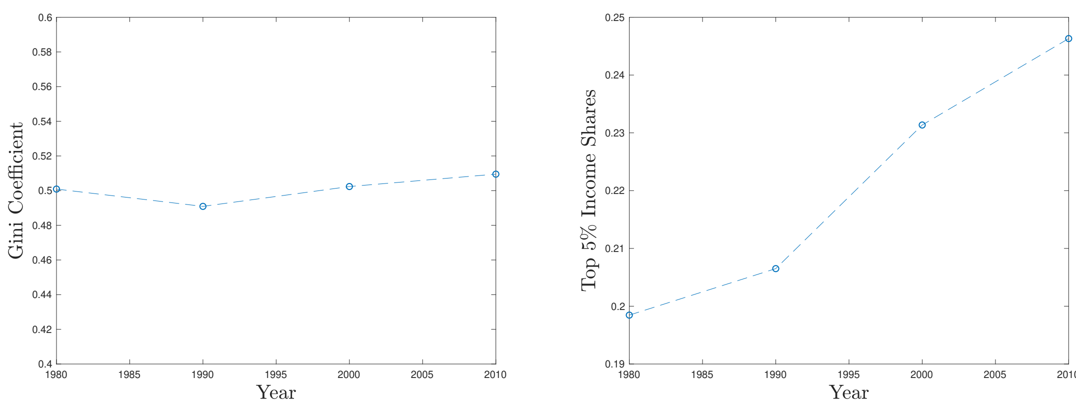


FIGURE 1.— Gini coefficients (left) and top 5% income shares (right) for individual income in the United States from 1980 to 2010. Note the different patterns in the two graphs.

To understand the intuition of our proposed unified framework, we give a motivating example in Table I, which shows that the Gini coefficients can be very similar (in fact both equal to 0.33 in Panel A) when the top income shares are quite different. However, when we consider a curve by truncating the top n samples (i.e. using the low and middle income samples only) and letting the n varying, we see completely different index numbers in Panel B.

Panel A: Income distribution (population size 1000)	Gini coefficient
$(1, 1, \cdots, 1, 500)$	0.33
$(1, 2, \cdots, 999, 1000)$	0.33
Panel B: Income distribution (population size $1000 - n$)	**Gini coefficient**
$(1, 1, \cdots, 1, 1)$	0.00
$(1, 2, \cdots, 999 - n, 1000 - n)$	0.33

TABLE I

In Panel A, two totally different income distributions can have the same Gini coefficient 0.33. In Panel B, we truncate the n top incomes in Panel A, $n \geq 1$, to form two curves. The difference between the two income distributions now becomes clear.

This motives us to introduce inequality index curves (e.g. letting n change in Panel B of Table I) as a more comprehensive way of measuring income inequality, rather than using a single index number to do measurement. Indeed, we shall demonstrate that inequality index curves have an axiomatic foundation, and one can get more information empirically by using the inequality index curves than using the inequality index numbers.

1.2. *Our Contribution*

The contribution of this paper is twofold. Theoretically, we propose a family of relative inequality curves, which includes curves generated by popular income inequality measures, e.g. the top income shares, the Gini coefficient, the single parameter Gini coefficient, the Palma ratio, and the Hoover index. The family has two advantages: (1) The family has an axiomatic foundation based on the weighted expected utility theory; see Theorem 1. (2) Each curve in the family contains the full information of the income distribution;

see Proposition 1.

Empirically, the proposed family also leads to new results. Previous empirical studies based on the top income shares (e.g. Atkinson (2007); Kaplan (2017); Piketty (2003); Saez and Veall (2005); Gabaix et al. (2016)) have shown that the bottom and middle income people in several countries became relatively poorer. Using the family of relative inequality curves and micro level data, we complement their results by showing that the bottom and middle income people in the U.S. became more equally relative poor (not just relative poorer) from 1990 to 2010.

The precise meaning of "more equally relative proor" is as follows: (1) The uniform downward trend of the inequality index curves across the time horizon 1980 to 2010 in Figures 3 (Gini index curves), 5 (Hoover and Palma index curves), and 6 (DW index curve) all show that the bottom and middle income people in U.S. became more relatively equal. (2) However, the bottom and middle income in U.S. became relatively poorer, because the top incomes shares increased steadily from 1990 to 2010; see Figure 4.

1.3. *Literature Review*

Income inequality is one of the central concerns in economic theory and economic policy (see, e.g., Atkinson (1983) and Kaplan (2017)). Our paper is related to three streams of research on income inequality. First, there is a large literature on income inequality indices; see, e.g., Mehran (1976), Weymark (1981), Sen (1976), Porath and Gilboa (1994), and Porath, Gilboa, and Schmeidler (1997). Besides the famous Gini index, popular indices are the Hoover index (Hoover, 1936), the Palma ratio (Palma, 2006), the DW inequality index (Donaldson and Weymark, 1980), and the AKS index (Atkinson, 1970; Kolm, 1969; Sen, 1973).

We complement this stream of literature by studying index curves instead of index numbers.

Secondly, since the pioneering work of Piketty (2003) and Piketty and Saez (2003) there has been a growing interest in using top income shares, especially because in recent decades the shares of top incomes have risen dramatically in France, U.K., U.S., and many other countries; see, e.g., Atkinson (2007), Saez and Veall (2005). Indeed, the decline of the middle class in the U.S. was frequently discussed in the news media (see, e.g., Daugherty (2017), Mason (2015), White (2016)). Besides the top income shares, there are also other measures used in the media, such as Lévy measure (Lévy, 1987a,b) which is defined as the income shares of the middle 60% of the income distribution, and the polarization index in Wolfson (1994) and Foster and Wolfson (2010).

We complement this stream of literature by showing empirically that the low and middle income people are actually more equally relatively poor, not just relatively poorer as found in the existing literature. In addition, the inequality index curves proposed in this paper seem to be a natural metric to study the bottom and middle income, because the curves focus on the bottom and middle income directly.

Finally, there is an active stream of literature on providing axiomatic foundation to income inequality measurement, dating back to the classical papers by Sen (1976), Donaldson and Weymark (1980), and Weymark (1981). The contribution of our axiomatic framework to this stream of literature is twofold:

(i) We use the Pareto criterion in our axioms, which is weaker than the standard Pigou-Dalton transfer principle used in the literature. This relaxation helps us to include the top income shares in our axiomatic framework For a discussion on the limitations of the Pigou-Dalton transfer principle, see Atkinson and Piketty (2007).

(ii) We use the weak-zero independence axiom in our framework to characterize the income inequality measures directly, instead of using the social evaluation function. The weak independence axiom in Chew (1983) was originally proposed to relax the independence axiom in the von Neumann-Morgenstern utility theory; he uses the weak independence axiom to study the measurement of income inequality, via the social evaluation function within the AKS framework (Atkinson, 1970; Kolm, 1969; Sen, 1973). For related work on the weighted expected utility theory, see Fishburn (1981, 1983), and Nakamura (1984, 1985). For further characterizations on social evaluation functions, see the generalized absolute Gini coefficient in Weymark (1981), linear evaluation functions in Porath and Gilboa (1994), min-of-means in Porath, Gilboa, and Schmeidler (1997), and equality mindedness in Yaari (1988). To our best knowledge, this paper is the first one that applies the weak independence axiom to the measurement of income inequality directly, without using the social evaluation function within the AKS framework. In addition, our weak-zero independence axiom is weaker than the weak independence axiom.

The rest of this paper is organized as follows. Section 2 gives the main theoretical results of the paper. An empirical study is given in Section 3. Section 4 discusses the axioms used in justifying the index curves. Section 5 concludes. All proofs are given in the online appendices.

2. INEQUALITY INDEX CURVES

In this section, we shall first provide an axiomatic foundation for a general class of inequality indices for the bottom and middle income. This class is general enough to include many popular income inequality index numbers, such as the Gini coefficient, the DW inequality index coefficient, the top income shares, the Palma ratio, and the Hoover index. We then introduce the inequality index curves generated by the index numbers within this class. Finally, we study two special cases of the curves, namely the Gini curve and the top income shares curve.

2.1. *An Axiomatic Foundation*

For a fixed population size n, the income distribution is represented by an n-dimensional vector $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ with ascending rankings, i.e., $y_1 \leq y_2 \leq \cdots \leq y_n$, where $y_i \in [0, +\infty)$ denotes the income of i-th large individual for $i = 1, 2, \cdots, n$. For notation convenience, we denote by $\mathcal{I} = \{\mathbf{y} \in [0, +\infty)^n \mid 0 \leq y_1 \leq y_2 \leq \cdots \leq y_n < +\infty\}$ the family of possible incomes. Moreover, we write $\mathbf{y_1} > \mathbf{y_2}$ if and only if $y_{1,i} > y_{2,i}$ for $i = 1, 2, \ldots, n$. In addition, we set $y_0 = 0$ throughout the paper.

Definition 1 (Inequality index numbers for the lower and middle income) *For an income distribution* $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ *and a given positive integer* $k \in [1, n]$, *the inequality index number for the lower and middle income* $(y_1, \ldots, y_k)$ *is defined as*

$$(1) \qquad I_{LM}(\mathbf{y}) := \begin{cases} \frac{b_1 y_1 + b_2(y_2 - y_1) + \cdots + b_k(y_k - y_{k-1})}{a_1 y_1 + a_2(y_2 - y_1) + \cdots + a_k(y_k - y_{k-1})}, & D(\mathbf{y}) \neq 0; \\ \frac{b_{k+1}}{a_{k+1}}, & D(\mathbf{y}) = 0, \end{cases}$$

where $a_i > 0$, $b_i > 0$, $\frac{b_{k+1}}{a_{k+1}} > \frac{b_i}{a_i}$ *for* $i = 1, 2, \cdots, k$, *and*

$$D(\mathbf{y}) := a_1 y_1 + a_2(y_2 - y_1) + \cdots + a_k(y_k - y_{k-1}).$$

Our main theoretical result below shows that the inequality index numbers in (1) can be uniquely characterized in an axiomatic framework.

Theorem 1 *(An Axiomatic Foundation) The preference relationship* $\succeq$ *on* $\mathcal{I}$ *satisfies Axiom 1–5 in Section 4 if and only if there exist a positive integer* $k \in [1, n]$ *and positive* $a_i, b_i \in \mathbb{R}_+$ *for* $i = 1, 2, \cdots, k+1$ *with* $\frac{b_{k+1}}{a_{k+1}} > \frac{b_i}{a_i}$ *for* $i = 1, 2, \cdots, k$, *such that for any* $\mathbf{y}_i = (y_1^i, y_2^i, \cdots, y_n^i) \in \mathcal{I}$, $i = 1, 2$,

$$\mathbf{y_1} \succeq \mathbf{y_2} \iff I_{LM}(\mathbf{y_1}) \leq I_{LM}(\mathbf{y_2}).$$

The main difficulties of the proof of Theorem 1 are the characterization of the null income set and how to associate it with the weighted (nonlinear) expected utility theory. The appearance of the null income distribution also distinguishes our proof from the proof in Chew (1983). Similar to **?**, which takes the dual on the linear expected utility theory, we take the dual on the weighted (nonlinear) expected utility theory by viewing the income distribution as its inverse on the domain of the distribution functions and then using the weighted (nonlinear) expected utility theory to characterize its dual version.

The definition of the inequality index I_{LM} is quite general, as it includes many well-known inequality measures for the bottom and middle income (including the overall version), such as the Gini coefficient, the DW inequality index coefficient, top income shares, the Palma ratio, and the Hoover index. Indeed, when $k = n$ our index number I_{LM} contains these well-known inequality indices as special cases; see Table II for a summary.

In addition, consider the well-known AKS index (Atkinson, 1970; Kolm, 1969; Sen, 1973) defined as

$$(2) \qquad I_{AKS}(\mathbf{y}) = 1 - \frac{\Theta(\mathbf{y})}{\mu(\mathbf{y})},$$

where $\Theta(\mathbf{y})$ is the social evaluation function and $\mu(\mathbf{y})$ is the mean of $\mathbf{y}$. If we let the social evaluation function $\Theta(\mathbf{y})$ be the linear functions $\sum_{i=1}^{n} r_i y_i$ (see, e.g., Porath and Gilboa (1994)), then the AKS index (2) becomes a special case of I_{LM} in (1). In particular, two popular AKS indices with linear social evaluation functions are the Gini coefficient and the DW inequality index, which are all special cases of I_{LM} in (1). In general, our index number I_{LM} and an AKS index number with a nonlinear social evaluation function are different.

I_{LM} with $k=n$	$\{b_i\}_{i=1}^n$	$\{a_i\}_{i=1}^n$
Gini coefficient	$2i-1-n$	n
DW inequality index Donaldson and Weymark (1980)	$i^\delta-(i-1)^\delta-n$ $\delta>1$	n
Top income shares, Atkinson and Piketty (2007)	$\begin{cases}0\,, 0\le i\le i_0\\ 1\,, i_0<i\le n\end{cases}$ $i_0\in\{1,2,\cdots,n\}$	1
Palma ratio, Palma (2006)	$\begin{cases}0\,, 0\le i<[0.9n]\\ 1\,, [0.9n]\le i\le n\end{cases}$	$\begin{cases}0\,, [0.4n]<i\le n\\ 1\,, 0\le i\le [0.4n]\end{cases}$
Hoover index, Hoover (1936)	$\begin{cases}k_0+n\,, y_i\ge\mu\\ k_0-n\,, y_i<\mu\end{cases}$	$2n$

TABLE II

EXISTING INEQUALITY INDEX NUMBERS AS SPECIAL CASES OF THE GENERAL FRAMEWORK I_{LM}. HERE $[\cdot]$ MEANS THE INTEGER PART, $|A|$ IS THE CARDINAL NUMBER OF THE SET A, μ IS THE MEAN OF $(y_1,y_2,\cdots,y_k)$, AND $k_0=|\{i\mid\mu>y_i\}|-|\{i\mid\mu\le y_i\}|$.

2.2. *Income Inequality Index Curves*

For a given inequality number I_{LM} in (1) we can generate an inequality index curve by changing k as follows.

DEFINITION 2 *(Inequality index curve) An inequality index curve $\alpha\mapsto I(0,\alpha)$ induced by I_{LM} is defined as a curve such that $I(0,\alpha)$ represents the income inequality measure I_{LM} for the quantile range $(0,\alpha)$ of an income distribution; in particular, $I(0,\frac{i}{n})$ is the inequality index number for the bottom and middle income $(y_1,y_2,\cdots,y_i)$, $i=1,2,\cdots,n$.*

For example, $I(0,0.9)$ represents the inequality index number for the income quantile range $(0,0.9)$ of an income distribution, and $I(0,1)$ represents the overall inequality index number.

As is discussed before, a single index number cannot summarize the complete information in the income distribution. If the overall inequality index number satisfies the following basic transfer principle

$$(3) \qquad I_{LM}(\mathbf{e}_1) < I_{LM}(\mathbf{e}_2) < \cdots < I_{LM}(\mathbf{e}_n),$$

then the next proposition shows that the inequality index curve contains the complete information of the underlying income distribution, up to a positive affine transformation.[1]

Proposition 1 *(Inequality index curve contains full information). If the overall inequality index with $k = n$ satisfies the requirement (3), then for two income distributions $\mathbf{y}_i = (y_1^i, y_2^i, \cdots, y_n^i) > \mathbf{0}$, $i = 1, 2$, the corresponding inequality curves coincide if and only if $\mathbf{y}_1 = c \cdot \mathbf{y}_2$ for a positive constant $c > 0$.*

Note that the requirement (3) in the proposition is rather weak and is satisfied by many popular inequality measures, such as Gini coefficient, DW inequality index, Theil index, and Hoover index. Indeed, the requirement (3) is weaker than the strict Pigou-Dalton transfer principle; for more discussion about the Pigou-Dalton transfer principle, see Section 4.

2.3. *Two special cases: Gini curve and top income shares curve*

In this subsection, we shall present two special inequality index curves generated by the Gini coefficient and the top income shares, respectively.

[1] Here we define $\mathbf{e}_i = (0, \cdots, 0, \overset{i}{1}, 1, \cdots, 1)$, $i = 1, 2, \cdots, n$, where the first 1 appears in the i-th position. In addition, we denote by $\mathbf{0} = (0, 0, \cdots, 0)$. It is easy to see that any income distribution $\mathbf{y} = (y_1, y_2, \cdots, y_n) \in \mathcal{I}$ can be expressed as $\mathbf{y} = y_1 \cdot \mathbf{e}_1 + (y_2 - y_1) \cdot \mathbf{e}_2 + \cdots + (y_n - y_{n-1}) \cdot \mathbf{e}_n$.

Other inequality index curves induced by (1) will be discussed briefly in Section 3.

The Gini coefficient for the lower and middle income is defined by

$$G(0, \frac{i}{n}) = \frac{y_1 + 3y_2 + \cdots + (2i-1)y_i}{i(y_1 + y_2 + \cdots + y_i)} - 1, \quad 1 \le i \le n.$$

This yields the Gini curve $\alpha \mapsto G(0, \alpha)$, where

$$G(0, \alpha) = \frac{2\int_0^\alpha uq(u)\,du}{\alpha \int_0^\alpha q(u)\,du} - 1,$$

and $q(u)$ is the quantile function of the income distribution.

In addition to the Gini coefficient, the top income shares are also covered by the framework (1). Combining top income shares at all levels, we define the top income shares curve as follows. For a given income distribution $\mathbf{y} = (y_1, y_2, \cdots, y_n)$, top income shares curve is defined as $\{(\frac{i}{n}, S(\frac{i}{n}, 1)) \mid i = 1, 2, \cdots, n\}$, where $S(\alpha, 1)$ is the top income shares of the quantile range $[\alpha, 1]$. The top income shares curve is linked to the famous Lorenz curve $L(\alpha)$ by a linear transformation,[2] i.e., $L(\alpha) = 1 - S(\alpha, 1)$ for any $\alpha \in [0, 1]$. A graphical representation of $G(0, \alpha)$ and $L(\alpha)$ is given in Figure 2.

Theoretically, an inequality index curve is less restrictive than a top income shares curve (or equivalently Lorenz curve). More precisely, an inequality index curve is simply a curve starting at $(0, 0)$ and ending at $(1, I)$, where I is the corresponding overall inequality index, without any other constraints on convexity. However, a top income shares curve must be strictly decreasing and concave on $[0, 1]$. The feature that inequality index curves have less constraints can be helpfu to facilitate the detection of new empirical patterns. For example, the Gini curves in Figure 3 display significant

[2]Lorenz curve is define by $L(\alpha) = \frac{\int_0^\alpha q(u)\,du}{\int_0^1 q(u)\,du}$ for $0 \le \alpha \le 1$, where $q(u)$ is the quantile function of the income distribution. Gini coefficient and Lorenz curve are linked by the relationship $G = 2\int_0^1 (u - L(u))\,du$.

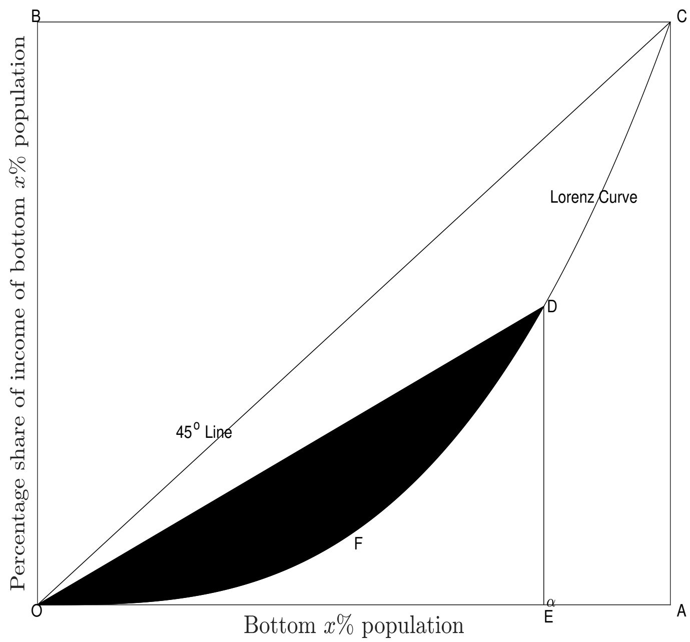


FIGURE 2.— The overall Gini coefficient $G(0,1)$ is given by area OFDC divided by area OCA. Similarly, the Gini coefficient for the bottom and middle income $G(0,\alpha)$ is given by area OFD divided by area ODE. Note that the line OFDC is the Lorenz curve $L(\alpha)$, and $L(\alpha) = 1 - S(\alpha,1)$, where $S(\alpha,1)$ is the top income share of the quantile range $[\alpha,1]$

downward trend from 1980 to 2010, while the top income shares curves in Figure 4 move much less during the same time period, due to the significant shape constraints.

The Gini curve and the income shares curves$S(\alpha,\beta)$ are closely related to each other as shown in the following proposition.

PROPOSITION 2 *(i) Top* $1-\alpha$ *income shares* $S(\alpha,1)$ *can be computed explicitly via the Gini curve,*

$$S(\alpha,1) = 1 - \frac{1-G(0,1)}{\alpha(1-G(0,\alpha))} \exp(-2\int_{\alpha}^{1} \frac{1}{u(1-G(0,u))}\,du), \tag{4}$$

where $G(0,\alpha)$ *is the Gini coefficient for the bottom and middle income.*

(ii) Top $1-\alpha$ *income shares* $S(\alpha,1)$ *can be approximated by*

$$S(\alpha,1)=\frac{1+G(0,1)-\alpha(1+G(0,\alpha))}{1-\alpha G(0,\alpha)}+o(1-\alpha)\ \text{as}\ \alpha\to 1^{-}. \tag{5}$$

Table III suggests that the Approximation formula (5) can be quite accurate.

U.S.A.	Year	$S(99.9\%,1)$	$S(99\%,1)$	$S(95\%,1)$	$S(90\%,1)$
Approximation by (5)	2010	1.29%	10.02%	25.05%	38.02%
True values	2010	1.29%	10.00%	24.50%	36.47%

TABLE III

3. AN EMPIRICAL STUDY OF INEQUALITY INDEX CURVES

Previous empirical studies have shown that in recent decades the shares of top incomes have risen dramatically in France, U.K., U.S., and many other countries; see, e.g., Atkinson (2007), Saez and Veall (2005), Wolfson (1994), Foster and Wolfson (2010); Using the proposed income inequality curves, we shall show in this section that not only the lower and middle income people in U.S. are relatively poorer, but the income inequality among the lower and middle income people has a clear downward trend from 1980 to 2010. More precisely, the lower and middle income people in U.S. become more equally relative poorer.

3.1. *Data Description*

The micro level, individual total income data set used in this paper is from IPUMS-CPS. It is an integrated data set from the Current Population Survey (CPS), which are U.S. household monthly surveys conducted jointly by the the U.S. Census Bureau and the Bureau of Labor Statistics. We choose four years, 1980, 1990, 2000, 2010, for our empirical analysis. Below is a brief description of the dataset downloaded from IPUMS-CPS.

Year	Data Size	Individual Sample Size
1980	236.9M	181488
1990	234.6M	158079
2000	232.3M	133710
2010	239.6M	209802

TABLE IV

Summary of the datasets for year 1980, 1990, 2000, and 2010. The data set also contain weights needed to recovered the population level data. Data Source: IPUMS-CPS

More precisely, first, we use the micro data to generate the income distribution $(y_1, y_2, \cdots, y_n)$ of a particular country in a given year, where n is the population size. Then, by Definition 2, we can generate Gini curves, single parameter Gini curves, Palma curves, and Hoover curves. In the online supplement, Appendix E, we shall describe the detail format of the data and the computer algorithm that we used to extract the data.

3.2. *More equally relative poorer of the lower and middle income people in the U.S.*

Since Gini coefficient is the most popular inequality measure, we first use the Gini curve to measure income inequality. It should be emphasized that our analysis is robust and is also demonstrated by other inequality index curves in the framework (1), as will be shown in Section 3.3.

Figure 3 presents Gini curves for the individual income in U.S. from 1980-2010. Comparing to the overall Gini coefficients (horizontal dash lines) which are virtually unchanged through the years and stable around the level 0.5, the Gini curves (the solid lines) exhibit a consistent downward trend throughout the years. This shows that the lower and middle income people become more equal.

Figure 4 shows that the top income shares curves, $S(\alpha, 1)$ for $\alpha \in [0, 84, 1]$,

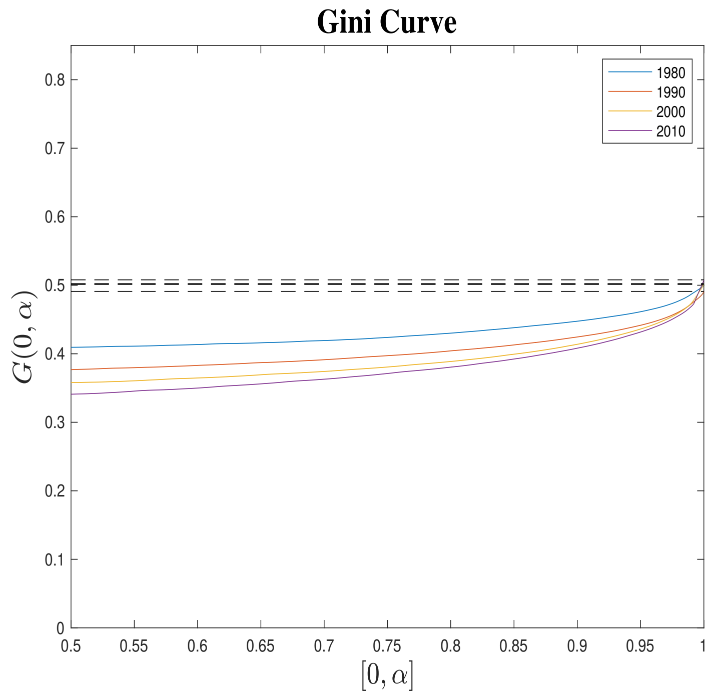
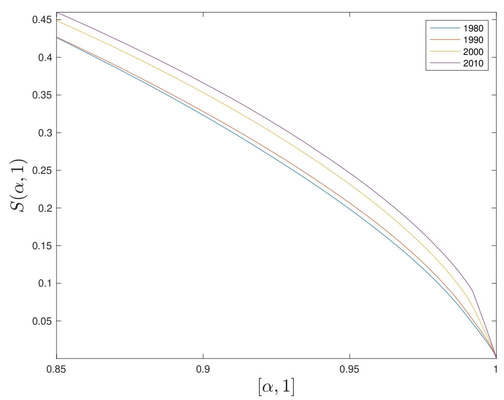


FIGURE 3.— This figure shows the Gini curves of annual individual pre-tax total income for United States. Dashed lines (the overall Gini coefficients) do not show obvious changes from 1980 to 2010, while by the Gini curves up to level 99.54% exhibit a consistent downward trend throughout the years.

have consistent upward trend from 1980 to 2010. For a broader income bracket, $\alpha \in [0.5, 1]$, the top income shares $S(\alpha, 1)$ show consistent upward trend from 1990 to 2010. Equivalently, by the relationship $S(0, \alpha) = 1 - S(\alpha, 1)$, the income shares for the bottom and middle income people have consistent downward trend from 1990 to 2010. Hence, we conclude that the bottom and middle income people in the U.S. become poorer from 1990 to 2010.

Interestingly, this is also consistent with the fact the Gini coefficients are stable from 1990 to 2010. Indeed, recalling the approximation in Proposition 2, $S(\alpha, 1) \approx \frac{1+G(0,1)-\alpha(1+G(0,\alpha))}{1-\alpha G(0,\alpha)}$ when $\alpha \to 1^-$. It is easy to see that top income shares $S(\alpha, 1)$ will increase if the overall Gini coefficient $G(0, 1)$

remains relatively stable but the Gini coefficient for the bottom and middle income $G(0, \alpha)$ decreases, as the function $f(x) = \frac{1+G-\alpha(1+x)}{1-\alpha x}$ is decreasing for a certain range of x.



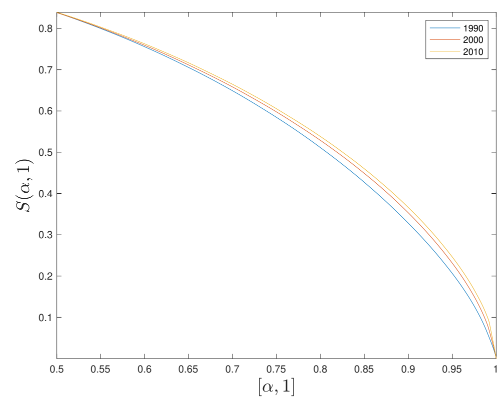


FIGURE 4.— This figure shows the top income shares curves of annual individual pre-tax total income for the U.S. Note that top income shares $S(\alpha, 1)$ for $\alpha \in [0.84, 1]$ show consistent upward trend throughout years 1980 to 2010 and for a broader income bracket, $\alpha \in [0.5, 1]$, the top income shares $S(\alpha, 1)$ show consistent upward trend from 1990 to 2010.

To summarize, we have obtained two conclusions: From 1990 to 2010, (i) the lower and middle income people in the U.S. become more equal; and (ii) the lower and middle income people in the U.S. become relatively poorer. Hence, we are able to conclude that the bottom and middle income people in the U.S. become more equally relatively poor from 1990 to 2010.

3.3. *Robustness Check*

In this section we will show that, in addition to the Gini curve, other inequality index curves induced by the indices covered in the general framework (1), such as Palma ratio, DW inequality index coefficient, and Hoover index (see Table II), also lead to similar conclusions with the Gini coefficient.

3.3.1. *Palma curve, Hoover curve, and DW inequality index curve*

Figure 5 presents the Hoover curve and Palma curve for the U.S. from 1980 to 2010. Note the empirical results are consistent with those for the Gini curve. More precisely, each overall inequality index number does not show a consistent trend, while the corresponding inequality index curve shows a significant and consistent downward trend.

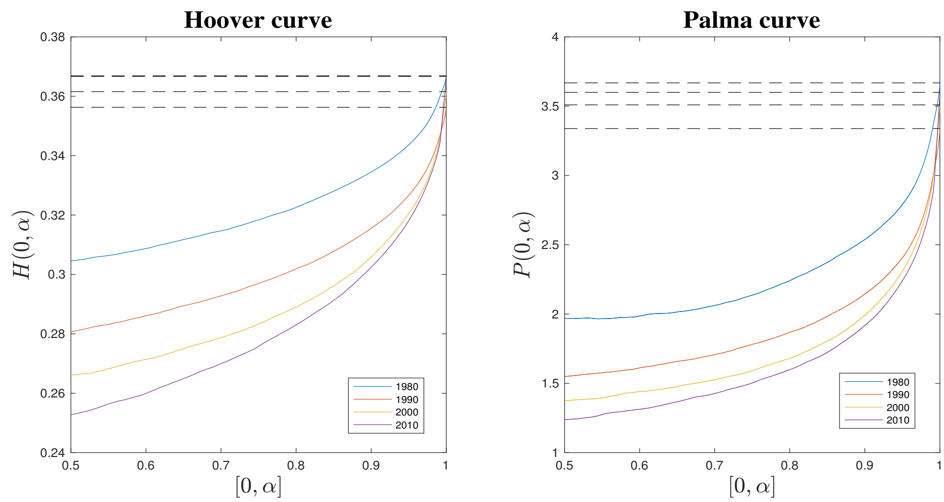


FIGURE 5.— Hoover curves and Palma curves of annual individual pre-tax total income for United States. Similar to the Gini curve, both Hoover curve and Palma cure show a consistent downward trend (up to 99% level).

Figure 6 presents the DW inequality index curve for the U.S. from 1980 to 2010, and the conclusions are similar with those for the Gini curve.

3.3.2. *AKS index with a nonlinear social evaluation function*

As mentioned before, an AKS index becomes a special case of I_{LM} in (1), if the social evaluation function is a linear function. The Gini coefficient and the DW inequality index are two examples of the most popular AKS indices with linear social evaluation function, and they have been studied empirically in the previous sections. Now we consider the Atkinson index

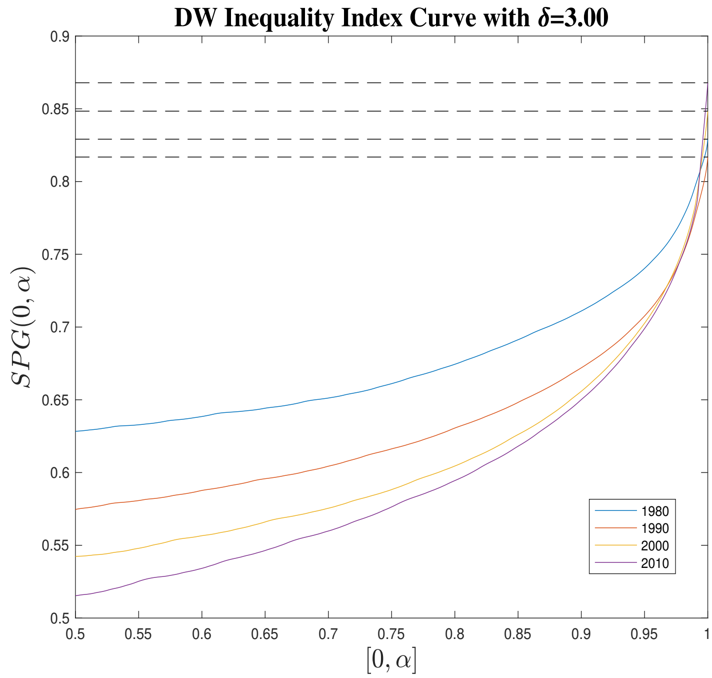


FIGURE 6.— DW inequality index curves of annual individual pre-tax total income for United States. Similar to the Gini curve, the DW inequality index curves show a consistent downward trend (up to 99% level).

number, (Atkinson, 1970), which is perhaps the most famous AKS index with a non-linear social evaluation function and is given by

$$A(y_1, y_2, \cdots, y_n) = \begin{cases} 1 - \frac{1}{\mu}(\frac{1}{n}\sum_{i=1}^{n} y_i^{1-\epsilon})^{\frac{1}{1-\epsilon}}, \text{ for } 0 \leq \epsilon \neq 1; \\ 1 - \frac{1}{\mu}(\Pi_{i=1}^{n} y_i)^{\frac{1}{n}}, \text{ for } \epsilon = 1, \end{cases}$$

where $y_1 \leq y_2 \leq \cdots \leq y_n$, $\mu = \sum_{i=1}^{n} y_i/n$, and ϵ is an inequality aversion parameter. One can use the Atkinson index number at different levels to generate an Atkinson curve.

Figure 7 presents Atkinson curves of annual individual pre-tex total income for the U.S., with inequality aversion parameter $\epsilon = 1.00, 1.40, 1.80, 2.20$ (Atkinson, 1970). Compared with Gini curve, Hoover curve, and Palma curve, Atkinson curves do not seem to yield consistent patterns. Thus, al-

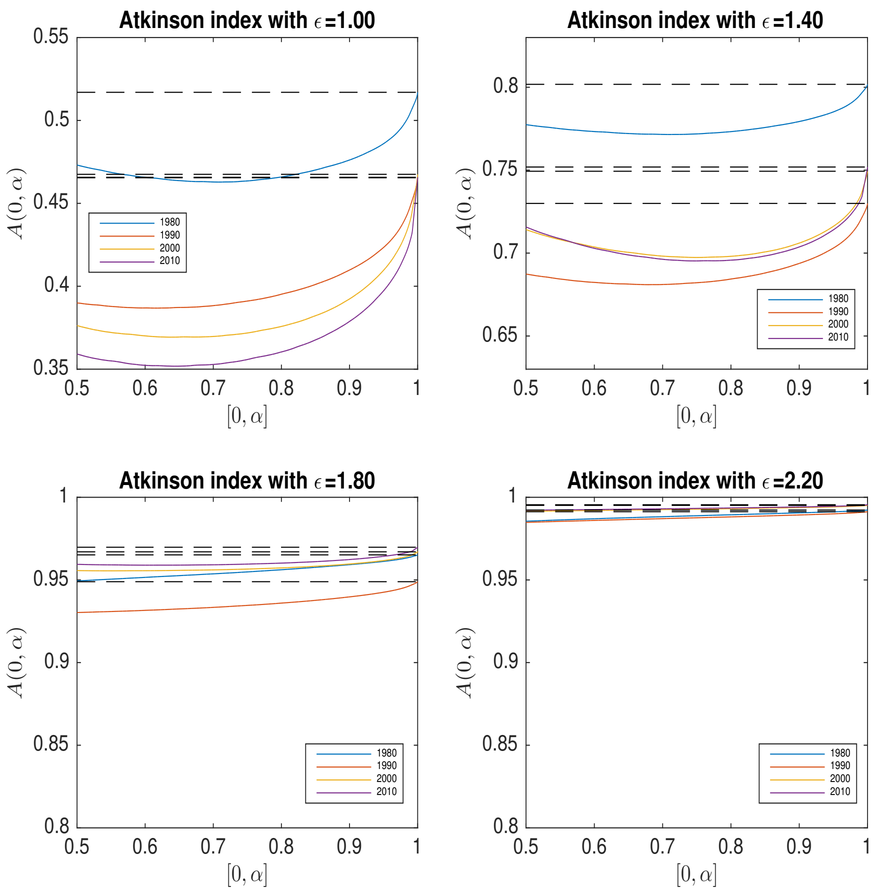


Figure 7.— The Atkinson curves generated by the Atkinson index number for inequality aversion parameter $\epsilon = 1.00, 1.40, 1.80, 2.20$ of annual individual pre-tax total income for the U.S. with same y-axis scales. The Atkinson curves for different inequality aversion parameters do not seem to show consistent patterns. When $\epsilon = 1.00$, the conclusion is the same as that of the Gini coefficient: Atkinson indices for the bottom and middle income decrease significantly as time goes by. However, when $\epsilon = 1.40, 1.80, 2.20$, the Atkinson curves do not present the same conclusion with the case $\epsilon = 1.00$. In particular, when $\epsilon = 1.80, 2.20$, Atkinson curves show few changes, as they almost coincide into one curve.

though our empirical conclusion that the bottom and middle income people in the U.S. become more equally relatively poor from 1990 to 2010 seems to be quite robust for inequality curves within our framework of I_{LM} in (1), indices beyond our framework (such as Atkinson index) may lead to mixed results.

3.4. *Australian Income Data*

We use individual Australian tax return data to generate Gini curves. The data is downloaded from the Australia Tax Office: `https://data.gov.au/data/dataset/62ae540b-01b0-4c2e-a984-b8013884f1ec`, which contain a $1\% - 2\%$ sample of records for each year we studied.

Three Gini curves and top income shares curves from 2009, 2011 and 2013 are presented in Figure 8. Different from the U.S. case, Figure 8 shows that for the Australia the Gini curves for the quantile range $[0, \alpha]$, $\alpha \geq \alpha_0 = 0.946$ are increasing (not decreasing), while the corresponding top income shares are decreasing. The Gini curves for the quantile range $[0, \alpha]$ for $\alpha \in [0.5, \alpha_0]$ show significant downward trend from 2009 to 2010, where $\alpha_0 = 0.946$; while for $\alpha \in [\alpha_0, 1]$, the Gini curves for the quantile range $[0, \alpha]$ have significant upward trend from 2009 to 2010. The overall Gini coefficients (dashed lines) have moved upward from 2009, consistent with the trend found in Kaplan et al. (2018). The top income shares $S(\alpha, 1)$ show significant upward trend from 2009 to 2010, for $\alpha \in [0.5, 1]$.

In addition to the U.S. case, we also conduct an empirical study of the income inequality in Australia, and get s is also drawing attention recently (see, e.g., Kaplan et al. (2018)); see Section 3.4.

4. AXIOMATIC FRAMEWORK FOR THE INEQUALITY INDEX CURVE

In this section we present detailed axioms needed for our main theoretical result Theorem 1. A decision maker's ranking of elements from the family

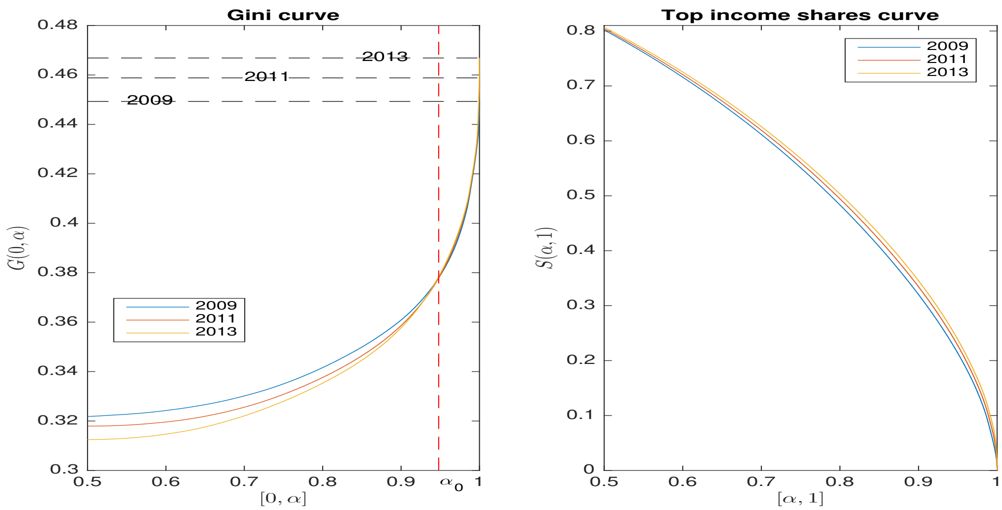


FIGURE 8.— This figure shows the Gini curves and top income shares of annual individual pre-tax total income for Australia. Note that the Gini curves for the quantile range $[0,\alpha]$ for $\alpha \in [0.5,\alpha_0]$ show significant downward trend from 2009 to 2010, where $\alpha_0 = 0.946$; while for $\alpha \in [\alpha_0, 1]$, the Gini curves for the quantile range $[0,\alpha]$ have significant upward trend from 2009 to 2010, including the overall Gini coefficients (dashed lines). In the right figure, the top income shares $S(\alpha,1)$ show significant upward trend from 2009 to 2010, for $\alpha \in [0.5, 1]$.

of income distributions $\mathcal{I}$ can be represented by a preference relationship $\succeq$, which is assumed to satisfy the following axioms.

AXIOM 1 (Weak Ordering) *$\succeq$ is a transitive and complete ordering on $\mathcal{I}$.*

This axiom is standard in the expected utility theory. Specifically, the transition means that if $\mathbf{y}_1 \succeq \mathbf{y}_2$ and $\mathbf{y}_2 \succeq \mathbf{y}_3$, then we must have $\mathbf{y}_2 \succeq \mathbf{y}_3$; and the completeness means that for $\mathbf{y}_1$ and $\mathbf{y}_2$, we have $\mathbf{y}_1 \succeq \mathbf{y}_2$ or $\mathbf{y}_2 \succeq \mathbf{y}_1$ or both.

AXIOM 2 (Pareto Ranking) *$\mathbf{y} \succeq \mathbf{0}$ for any $\mathbf{y} \in \mathcal{I}$ and there exists at least one $\mathbf{y} \in \mathcal{I}$ such that $\mathbf{y} \succ \mathbf{0}$.[3] Moreover, if there exists $i \geq 2$ such that $\mathbf{e}_i \succ \mathbf{0}$,*

[3]Here the notation $\succ$ means strictly preference relationship, i.e., $\mathbf{y}_1 \succ \mathbf{y}_2$ indicates

then $\mathbf{e}_j \succ \mathbf{0}$ *for any* $1 \leq j \leq i \leq n$.

The first part of this axiom is $\mathbf{y} \succeq \mathbf{0}$ for any $\mathbf{y} \in \mathcal{I}$, which means "a little is always better than nothing" and indicates that it is always preferable to make an individual better off without making anyone else worse off, consistent with the classical Pareto criterion. Here the requirement that at least one $\mathbf{y} \in \mathcal{I}$ exists such that $\mathbf{y} \succ \mathbf{0}$ means the preference relationship $\succeq$ is non-trivial, which is a very common assumption in the decision theory.

The second part indicates that if $\mathbf{e}_i \succ \mathbf{0}$, then we have $\mathbf{e}_j \succ \mathbf{0}$ for any $j \leq i$. This means that the income base with more non-zero elements is not considered to be worse than that with less non-zero elements, which is intuitive and consistent with the recognition of the social justice. Moreover, together with the scale invariance axiom and the assumption that "a little is always better than nothing'', the second part of the Pareto Ranking axiom complies with, but is strictly weaker than, the classic Pigou-Dalton transfer principle. As a result, almost all existing inequality measures satisfy this axiom. In particular, the top income share, one of the most commonly used inequality measures in the recent top incomes research, satisfies the Pareto Ranking axiom under the assumption that "a little is always better than nothing''.

To our best knowledge, we are the first one to use the Pareto ranking axiom, as a way to relax the Pigou-Dalton transfer principle, which is widely used in the inequality index literature. Here is the motivation. Recall that the principle says that the measurement of inequality should be reduced by a progressive transfer, i.e., a certain amount of wealth transferred from a richer to a poorer individual without affecting their relative rankings should reduce the inequality. Supports for the Pigou-Dalton transfer principle are given in Amiel and Cowell (2002), Gaertner and Namezie (2003), Amiel, Cowell,

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$\mathbf{y}_1 \succeq \mathbf{y}_2$ and $\mathbf{y}_2 \not\succeq \mathbf{y}_1$.

and Slottje (2004), Amiel, Cowell, and Gaertner (2012), and Bosmans et al. (2018). However, Atkinson and Piketty (2007) discuss the drawback of the principle. For example, if we transform one dollar from the richest person in the three person distribution $(1, 1, 4)$ to another person, then the distribution becomes $(1, 2, 3)$ after the transfer. While the transfer reduces the income gap between 4 and 1, it also increases inequality between the first two persons, transforming the gap from 0 to 1, a very large relative increase. Thus, although a progressive transfer unambiguously reduces inequality between the individuals involved in the transfer, it is far from being obvious that everyone else would agree that inequality on the whole has declined as a result.

Therefore, we relax the Pigou-Dalton transfer principle and start instead from the viewpoint of Pareto criterion. We postulate that it is always preferable to make any one individual better off without making at least one individual worse off; more precisely, for any two basic income distributions $\mathbf{e}_i = (0, 0, \cdots, 0, \overset{i}{1}, 1, \cdots, 1)$ and $\mathbf{e}_{i+1} = (0, 0, \cdots, 0, \overset{i+1}{1}, 1, \cdots, 1)$, the first one is always considered to be not worse than the second according to the Pareto criterion.

Axiom 3 (Weak-Zero Independence) *For any* $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{I}$ *with* $\mathbf{y}_1 \sim \mathbf{y}_2$[4] *and any* $r \geq 0$*, there exists* $l \geq 0$ *such that for all* $\mathbf{y} \in \mathcal{I}$*,* $r \cdot \mathbf{y}_1 + \mathbf{y} \sim l \cdot \mathbf{y}_2 + \mathbf{y}$.

The above weak-zero independence axiom is different from the weak independence axiom in Chew (1983)'s, as he requires $r > 0$ and $l > 0$. Essentially, the weak-zero independence axiom is used here to characterize a relative inequality measure.[5] In particular, this axiom also plays a key role

[4]Here the notation $\sim$ means the equivalent preference relationship, i.e., $\mathbf{y}_1 \sim \mathbf{y}_2$ indicates $\mathbf{y}_1 \succeq \mathbf{y}_2$ and $\mathbf{y}_2 \succeq \mathbf{y}_1$.

[5]Using the weak independence axiom allows the relative representation of the measure and can also be useful for the characterization of the generalized relative Gini coefficient

in “dropping” top incomes from technical point of view; see the Lemma B.2 in the appendix.

The weak independence axiom in Chew (1983) was originally proposed to relax the independence axiom in the von Neumann-Morgenstern utility theory; he uses the weak independence axiom to study the measurement of income inequality, via the social evaluation function within the AKS framework. We use the weak-zero independence axiom in our framework to characterize the income inequality measures directly, instead of using the social evaluation function.

Axiom 4 (Betweenness) *For any* $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{I}$ *with* $\mathbf{y}_1 \succeq \mathbf{y}_2$*, then* $\mathbf{y}_1 \succeq r \cdot \mathbf{y}_1 + l \cdot \mathbf{y}_2 \succeq \mathbf{y}_2$ *for any* $(r, l) \in [0, +\infty) \times [0, +\infty) - \{(0,0)\}$.

The betweenness axiom is also standard in the expected utility theory, indicating that any positive combinations of two economies with different inequality extents must lie between these two economies.

Axiom 5 (Continuity) *For any* $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \nsim \mathbf{0}$ *with* $\mathbf{y}_1 \succeq \mathbf{y}_2$ *and* $\mathbf{y}_2 \succeq \mathbf{y}_3$*, there exists* $\alpha \in [0,1]$ *such that* $\mathbf{y_2} \sim \alpha \cdot \mathbf{y}_1 + (1-\alpha) \cdot \mathbf{y}_3$.

Axiom 5 endows the preference relationship with the property of continuity, which is also standard in expected utility theory.

5. CONCLUSIONS

We propose a family of inequality index curves, which includes curves generated by popular inequality index numbers (e.g. the top income shares, the Gini coefficient, the single parameter Gini coefficient, the Palma ratio, and the Hoover index). The family has two advantages: (1) The family has an axiomatic foundation based on the weighted expected utility theory.

in Weymark (1981).

(2) Each curve in the family contains the full information of the income distribution. The previous empirical studies based on the top income shares have shown that the lower and middle income people become relatively poorer in recent decades in several countries. Using the family of inequality index curves, we complement the previous empirical studies by showing that the lower and middle income people in the U.S. became more equally relatively poor (not just relatively poorer) from 1990 to 2010.

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APPENDIX A: PROOF OF PROPOSITION 1

PROOF: Since the overall inequality index satisfies the following transfer principle $I_{LM}(\mathbf{e}_1) < I_{LM}(\mathbf{e}_2) < \cdots < I_{LM}(\mathbf{e}_n)$, by direct calculation we have

$$\text{(A.1)} \quad \frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < \frac{b_n}{a_n}.$$

For any two income distributions: $\mathbf{y}_1 = (y_1^1, y_2^1, \cdots, y_n^1) > \mathbf{0}$ and $\mathbf{y}_2 = (y_1^2, y_2^2, \cdots, y_n^2) > \mathbf{0}$, if their inequality index curves are the same, then by the definition of the inequality index curve we have,

$$\text{(A.2)} \quad I_{LM}(y_1^1, y_2^1, \cdots, y_i^1) = I_{LM}(y_1^2, y_2^2, \cdots, y_i^2), \text{ for } i = 1, 2, \cdots, n.$$

Therefore, by (A.2) we have that for $i = 1, 2, \cdots, n$,

$$\text{(A.3)} \quad \frac{\sum_{j=1}^{i} b_j(y_j^1 - y_{j-1}^1)}{\sum_{j=1}^{i} a_j(y_j^1 - y_{j-1}^1)} = \frac{\sum_{j=1}^{i} b_j(y_j^2 - y_{j-1}^2)}{\sum_{j=1}^{i} a_j(y_j^2 - y_{j-1}^2)},$$

where $y_0^1 = y_0^2 = 0$.

When $i = 1$, (A.3) is clearly true. When $i = 2$, (A.3) becomes

$$\text{(A.4)} \quad \frac{b_1 y_1^1 + b_2(y_2^1 - y_1^1)}{a_1 y_1^1 + a_2(y_2^1 - y_1^1)} = \frac{b_1 y_1^2 + b_2(y_2^2 - y_1^2)}{a_1 y_1^2 + a_2(y_2^2 - y_1^2)},$$

which yields

$$\text{(A.5)} \quad (a_1 b_2 - a_2 b_1) y_1^2 y_2^1 = (a_1 b_2 - a_2 b_1) y_1^1 y_2^2.$$

By the inequality (A.1) we know that $a_1 b_2 - a_2 b_1 \neq 0$. Hence equation (A.5) reduces to $y_1^2 y_2^1 = y_1^1 y_2^2$. Since $\mathbf{y}_1 > 0$ and $\mathbf{y}_2 > 0$, we further have

$$\text{(A.6)} \quad \frac{y_1^1}{y_1^2} = \frac{y_2^1}{y_2^2}.$$

With the initial condition (A.6), we will prove the result by induction. Suppose that there exist a positive integer $2 \leq m \leq n-1$ and a constant $c > 0$ such that

$$\text{(A.7)} \quad \frac{y_1^1}{y_1^2} = \frac{y_2^1}{y_2^2} = \cdots = \frac{y_m^1}{y_m^2} = c.$$

When $j = m+1$, with the help of (A.7), equation (A.3) becomes

$$\text{(A.8)} \quad \frac{\sum_{j=1}^{m}(b_j - b_{j+1})y_j^1 + b_{m+1}y_{m+1}^1}{\sum_{j=1}^{m}(a_j - a_{j+1})y_j^1 + a_{m+1}y_{m+1}^1} = \frac{\sum_{j=1}^{m}(b_j - b_{j+1})y_j^1 + cb_{m+1}y_{m+1}^2}{\sum_{j=1}^{m}(a_j - a_{j+1})y_j^1 + ca_{m+1}y_{m+1}^2}.$$

Cross multiplication of (A.8) gives

$$\text{(A.9)} \quad (Ab_{m+1} - Ba_{m+1})(y_{m+1} - cz_{m+1}) = 0,$$

where $A = \sum_{j=1}^{m}(a_j - a_{j+1})y_j^1$ and $B = \sum_{j=1}^{m}(b_j - b_{j+1})y_j^1$. We expand the first term in equation (A.9) as

$$\text{(A.10)} \quad Ab_{m+1} - Ba_{m+1} = \sum_{j=1}^{m}((a_j - a_{j+1})b_{m+1} - (b_j - b_{j+1})a_{m+1})y_j.$$

By assumption (A.1), we have

$$(a_j - a_{j+1})b_{m+1} > (a_j - a_{j+1}\frac{b_j}{a_j})a_{m+1} > (b_j - b_{j+1})a_{m+1},$$

thus the coefficients of y_j in equation (A.10) are all positive for $j = 1, 2, \cdots, m$, and then the left-hand side of (A.10) are positive. Therefore, by equation (A.9) we know $y_{m+1} = cz_{m+1}$. Now by the induction assumption, we obtain

$$\frac{y_1^1}{y_1^2} = \frac{y_2^1}{y_2^2} = \cdots = \frac{y_n^1}{y_n^2} = c,$$

from which we conclude that $\mathbf{y}_1 = c \cdot \mathbf{y}_2$ for some positive constant $c > 0$. *Q.E.D.*

APPENDIX B: SOME LEMMAS AND PROPOSITIONS

PROPOSITION B.1 *Suppose the preference relationship satisfies Axiom 1. Then for any* $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{I}$ *with* $\mathbf{y}_1 \succeq \mathbf{y}_2$, $\mathbf{y}_1 \succeq r \cdot \mathbf{y}_1 + l \cdot \mathbf{y}_2 \succeq \mathbf{y}_2$ *for any* $(r, l) \in [0, +\infty) \times [0, +\infty) - \{(0, 0)\}$ *if and only if* $r_1 \cdot \mathbf{y}_1 + l_1 \cdot \mathbf{y}_2 \succeq r_2 \cdot \mathbf{y}_1 + l_2 \cdot \mathbf{y}_2$ *for any* $(r_i, l_i) \in [0, +\infty) \times [0, +\infty) - \{(0, 0)\}$ *for* $i = 1, 2$ *with* $r_1 l_2 \geq r_2 l_1$.

PROOF: Necessity. Suppose $\mathbf{y}_1 \succeq \mathbf{y}_2$ for two income distributions $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{I}$, then we have

$$r_1 \cdot \mathbf{y}_1 + l_1 \cdot \mathbf{y}_2 \succeq \mathbf{y}_2, \text{ for } (r_1, l_1) \in [0, +\infty) \times [0, +\infty) - \{(0, 0)\}. \tag{B.1}$$

On the other hand, because $(r_1, l_1) \in [0, +\infty) \times [0, +\infty) - \{(0, 0)\}$, we have $r_1 > 0$ or $l_1 > 0$. Without loss of generality we assume $r_1 > 0$, then we can rewrite

$$r_2 \cdot \mathbf{y}_1 + l_2 \cdot \mathbf{y}_2 = \frac{r_2}{r_1} \cdot (r_1 \cdot \mathbf{y}_1 + l_1 \cdot \mathbf{y}_2) + \frac{r_1 l_2 - r_2 l_1}{r_1} \cdot \mathbf{y}_2. \tag{B.2}$$

Note that $r_2 > 0$ or $l_2 > 0$. Thus,

$$\left(\frac{r_2}{r_1}, \frac{r_1 l_2 - r_2 l_1}{r_1}\right) \in [0, +\infty) \times [0, +\infty) - \{(0, 0)\}. \tag{B.3}$$

A combination of (B.1), (B.2), and (B.3) gives

$$r_1 \cdot \mathbf{y}_1 + l_1 \cdot \mathbf{y}_2 \succeq r_2 \cdot \mathbf{y}_1 + l_2 \cdot \mathbf{y}_2,$$

which completes the proof of the necessity.

Sufficiency. For any $\mathbf{y}_1 \succeq \mathbf{y}_2$, let $r_1 = 1$, $l_1 = 0$, $r_2 = r \geq 0$, and $l_2 = l \geq 0$, where $(r, l) \in [0, +\infty) \times [0, +\infty) - \{(0, 0)\}$. Then we see that r_1, r_2, l_1, l_2 satisfy the condition $r_1 l_2 - r_2 l_1 \geq 0$ and $(r_i, l_i) \in [0, +\infty) \times [0, +\infty) - \{(0, 0)\}$, $i = 1, 2$. Therefore, we have

$$\mathbf{y}_1 \succeq r \cdot \mathbf{y}_1 + l \cdot \mathbf{y}_2, \text{ for } r, l \in [0, +\infty) \times [0, +\infty) - \{(0, 0)\}. \tag{B.4}$$

For another inequality, we let $r_1 = r \geq 0$, $l_1 = l \geq 0$, $r_2 = 0$, and $l_2 = 1$, where $(r, l) \in [0, +\infty) \times [0, +\infty) - \{(0, 0)\}$, and we see that r_1, r_2, l_1, l_2 satisfy the condition $r_1 l_2 - r_2 l_1 \geq 0$ and $(r_i, l_i) \in [0, +\infty) \times [0, +\infty) - \{(0,0)\}$, $i = 1, 2$, then

$$r \cdot \mathbf{y}_1 + l \cdot \mathbf{y}_2 \succ \mathbf{y_2}, \text{ for any } (r,l) \in [0,+\infty) \times [0,+\infty) - \{(0,0)\}. \tag{B.5}$$

The proof is terminated by combining two inequalities (B.4) and (B.5). *Q.E.D.*

LEMMA B.1 *The betweenness axiom (Axiom 4) means the scale independence of the inequality index. More precisely, for any income distribution* $\mathbf{y} \in \mathcal{I}$ *and* $c > 0$*, we have* $c \cdot \mathbf{y} \sim \mathbf{y}$*.*

PROOF: For any income distribution $\mathbf{y} \in \mathcal{I}$ and $c > 0$, let $\mathbf{y}_1 \sim \mathbf{y}$ and $\mathbf{y}_2 \sim \mathbf{y}$, and choose $r > 0$ and $l > 0$ such that $r + l = c$. Then by Axiom 4, we have $\mathbf{y} \succeq r \cdot \mathbf{y} + l \cdot \mathbf{y} \succeq \mathbf{y}$, which indicates $\mathbf{y} \sim c \cdot \mathbf{y}$, for any $c > 0$. *Q.E.D.*

Before proceeding further, we need to characterize an important kind of income distributions, i.e., the null income distribution, which will play a key role in our axiomatic framework. The appearance of the null income distribution is also the most significant difference between our axiomatic framework and the one in Chew (1983).

DEFINITION 3 (Null Income Distributions) *An income distribution* $\mathbf{y}_0 \in \mathcal{I}$ *is called a null income distribution with respect to the preference relationship* $\succeq$*, if* $\mathbf{y}_0 + \mathbf{y} \sim \mathbf{y}$ *holds for any* $\mathbf{y} \in \mathcal{I}$*.*

The null income distribution arises naturally in the theory of the income inequality. For example, the zero income distribution $\mathbf{0}$ is a null income distribution for any inequality index. Another example is the translation invariance inequality measure I in Weymark (1981), which satisfies $I(\mathbf{y} + c \cdot \mathbf{e}_1) = I(\mathbf{y})$ for any $c \geq 0$; hence in this case $c \cdot \mathbf{e}_1$ for $c > 0$ is a null income distribution. We will denote by $\mathcal{N}(\succeq)$ the null income set with respect to the preference relationship $\succeq$.

LEMMA B.2 (Closeness of the Null Income Set) *Axiom 3 implies the closeness of the null income set with respect to the preference relationship* $\sim$*. More precisely, for any income distribution* $\mathbf{y} \sim \mathbf{y}_0$ *with* $\mathbf{y}_0 \in \mathcal{N}(\succeq)$*, we have* $\mathbf{y} \in \mathcal{N}(\succeq)$*.*

PROOF: Consider any income distribution $\mathbf{y}$ that is equivalent to a null income distribution $\mathbf{y}_0$, i.e., $\mathbf{y} \sim \mathbf{y}_0$, where $\mathbf{y}_0 \in \mathcal{N}(\succeq)$. By Axiom 3, for $r = 1$ there exists an $l \geq 0$ such that

(B.6) $$\mathbf{y} + \mathbf{y}_1 \sim l \cdot \mathbf{y}_0 + \mathbf{y}_1, \text{ for any } \mathbf{y}_1 \in \mathcal{I}.$$

Since $\mathbf{y}_0$ is a null income distribution, (B.6) reduces to

(B.7) $$\mathbf{y} + \mathbf{y}_1 \sim \mathbf{y}_1, \quad \text{for any } \mathbf{y}_1 \in \mathcal{I},$$

which leads to the desired result $\mathbf{y} \in \mathcal{N}(\succeq)$. *Q.E.D.*

LEMMA B.3 *Suppose the preference relationship* $\succeq$ *satisfies Axiom 1, Axiom 2, Axiom 3, and Axiom 4. Then*

$$P_{\min} = \mathcal{N}(\succeq),$$
$$\max\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} \in P_{\max},$$
$$\min\{\mathbf{e}_i \mid \mathbf{e}_i \notin \mathcal{N}(\succeq), i = 1, 2, \cdots, n\} \in P_{\text{smin}},$$

where $P_{\max}$ *and* $P_{\min}$ *denote the maximal and minimal elements in the family* $\mathcal{I}$ *of income distributions, respectively,* P_{smin} *denotes the minimal element* *in the family of non-null income distributions*[6]*, i.e.,*

$$P_{\max} := \{\mathbf{y}_0 \in \mathcal{I} \mid \mathbf{y}_0 \succeq \mathbf{y} \textit{ for all } \mathbf{y} \in \mathcal{I}\},$$
$$P_{\min} := \{\mathbf{y}_0 \in \mathcal{I} \mid \mathbf{y} \succeq \mathbf{y}_0 \textit{ for all } \mathbf{y} \in \mathcal{I}\},$$
$$P_{\text{smin}} := \{\mathbf{y}_0 \notin \mathcal{N}(\succeq) \mid \mathbf{y} \succeq \mathbf{y}_0 \textit{ for all } \mathbf{y} \notin \mathcal{N}(\succeq)\},$$

and $\max\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ *and* $\min\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ *stand for the maximal and minimal income distributions in* $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ *under the preference relationship* $\succeq$*, respectively.*

Proof: Before proving $P_{\min} = \mathcal{N}(\succeq)$, we point out a simple fact that all the null income distributions are equivalent. Indeed, for any two null income distributions $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{N}(\succeq)$, we have $\mathbf{y}_1 + \mathbf{y}_2 \sim \mathbf{y}_1 \sim \mathbf{y}_2$ by the definition of the null income distribution. By Axiom 2 we have $\mathbf{y} \succeq \mathbf{0}$ for any $\mathbf{y} \in \mathcal{I}$, thus $\mathbf{y} \succeq \mathbf{y}_0$ for any $\mathbf{y}_0 \in \mathcal{N}(\succeq)$, which indicates $\mathcal{N}(\succeq) \subseteq P_{\min}$. For any $\mathbf{y} \in P_{\min}$, by definition we know $\mathbf{0} \succeq \mathbf{y}$, which yields $\mathbf{y} \sim \mathbf{0}$ by combining with the first part of Axiom 2. We then obtain $P_{\min} \subseteq \mathcal{N}(\succeq)$, and thus $\mathcal{N} = P_{\min}$.

If $\{\mathbf{e}_1, \mathbf{e}_2 \cdots, \mathbf{e}_n\} \subseteq \mathcal{N}(\succeq)$, then for any $\mathbf{y} = (y_1, y_2, \cdots, y_n) \in \mathcal{I}$ we have

$$\mathbf{y} = y_1 \cdot \mathbf{e}_1 + (y_2 - y_1) \cdot \mathbf{e}_2 + \cdots + (y_n - y_{n-1})\mathbf{e}_n \sim \mathbf{0}. \tag{B.8}$$

Hence, for any $\mathbf{y}_1 \in \mathcal{I}$ and $\mathbf{y}_2 \in \mathcal{I}$, we have $\mathbf{y}_1 \sim \mathbf{y}_2 \sim \mathbf{0}$ by (B.8), which contradicts the first part of the Axiom 2: There is at least one $\mathbf{y} \in \mathcal{I}$ such that $\mathbf{y} \succ \mathbf{0}$. Thus, there exists at least an $\mathbf{e}_i$ such that $\mathbf{e}_i \notin \mathcal{N}(\succeq)$.

Now we have the following two cases.

The first case: $\{\mathbf{e}_i\} = (\mathcal{N}(\succeq) \cap \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\})^c$, which means that there is only one basic element $\mathbf{e}_i$ that is not a null income distribution. By Axiom

[6] Here "smin" is short for the "second minimal", since the "first minimal" has already been denoted by $P_{\min}$, which is just the null income set, as we will see in the proof.

2 we obtain $i = 1$. By the definition of the null income distribution we have

$$\text{(B.9)} \quad \sum_{i=1}^{n} r_i \mathbf{e}_i \sim \mathbf{e}_1 \text{ or } \sum_{i=1}^{n} r_i \mathbf{e}_i \sim \mathbf{e}_2 \sim \mathbf{0}, \text{ for any } r_i \geq 0, \ i = 1, 2, \cdots, n,$$

via Lemma B.1. Recall that any income distribution $\mathbf{y} \in \mathcal{I}$ can be written as a linear combination of income basis $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$. Hence, $\mathbf{y} \sim \mathbf{e}_1$ or $\mathbf{y} \sim \mathbf{e}_2 \sim \mathbf{0}$ by (B.9). By the assumption, $\mathbf{e}_1$ is not the null income distribution, thus by Lemma B.2 and Axiom 2 we know $\mathbf{e}_1 \succ \mathbf{0}$. Indeed, by Axiom 2 we know $\mathbf{e}_1 \succeq \mathbf{0}$, but the case $\mathbf{e}_1 \sim \mathbf{0}$ is impossible according to Lemma B.2, since any income distribution equivalent to the null income distribution must be a null income distribution too. Therefore, we obtain

$$\max\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} = \max\{\mathbf{e}_1, \mathbf{e}_2\} = \mathbf{e}_1 \in P_{\max},$$
$$\min\{\mathbf{e}_i \mid \mathbf{e}_i \notin \mathcal{N}(\succeq), i = 1, 2, \cdots, n\} = \min\{\mathbf{e}_1, \mathbf{e}_2\} = \mathbf{e}_1 \in P_{\text{smin}}.$$

The second case: $\{\mathbf{e}_i\} \subset (\mathcal{N}(\succeq) \cap \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\})^c$, which means that there are at least two or more basic income elements that are not null income distributions. Axiom 2 further indicates that

$$\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{i_0}\} = (\mathcal{N}(\succeq) \cap \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\})^c,$$

where $i_0 \geq 2$ is an integer, meaning that in the family of the income basis, $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{i_0}$ are not null income distributions, and $\mathbf{e}_{i_0+1}, \mathbf{e}_{i_0+2}, \cdots, \mathbf{e}_n$ are all null income distributions.

By the definition of the null income distribution, we have

$$\text{(B.10)} \quad \sum_{i=1}^{n} r_i \cdot \mathbf{e}_i \sim \sum_{i=1}^{i_0} r_i \cdot \mathbf{e}_i, \text{ for any } r_i \geq 0, i = 1, 2, \cdots, i_0.$$

On the other hand, by Lemma B.1 and Axiom 4 we have

$$\text{(B.11)} \quad \max\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{i_0}\} \succeq \sum_{i=1}^{i_0} r_i \cdot \mathbf{e}_i, \text{ for any } r_i \geq 0, i = 1, 2, \cdots, i_0,$$

and if there is at least one $r_i > 0$ for $i = 1, 2, \cdots, i_0$, we have

$$\text{(B.12)} \quad \sum_{i=1}^{i_0} r_i \cdot \mathbf{e}_i \succeq \min\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{i_0}\}.$$

Therefore, combining (B.11) and (B.12) yields that

$$\max\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} = \max\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{i_0}\} \in P_{\text{max}},$$
$$\min\{\mathbf{e}_i \mid \mathbf{e}_i \notin \mathcal{N}(\succeq), i = 1, 2, \cdots, n\} = \min\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{i_0}\} \in P_{\text{smin}}.$$

The proof is terminated by combining the above two cases.

Q.E.D.

LEMMA B.4 (Ratio Consistency, Chew (1983)) *Suppose the preference relationship $\succeq$ satisfies Axioms 3 and 4, and there exist $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \in \mathcal{I}$ and $r_1 > 0, r_2 > 0, l_1 > 0, l_2 > 0$ with $\mathbf{y}_1 \sim \mathbf{y}_2 \nsim \mathbf{y}_3$, such that*

$$r_i \cdot \mathbf{y}_1 + \mathbf{y}_3 \sim l_i \cdot \mathbf{y}_2 + \mathbf{y}_3 \text{ holds for } i = 1, 2.$$

Then we have $\frac{l_1}{r_1} = \frac{l_2}{r_2}$.

LEMMA B.5 (Characterization of Null Income Set under Preference Relationship $\succeq$) *Suppose the preference relationship $\succeq$ satisfies Axioms 1-4. Then $\mathcal{N}(\succeq) = \text{conv}\{\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} \cap \mathcal{N}(\succeq)\}$, where $\text{conv}\{A\}$ denotes the convex cone of A; more precisely, for $X = \{x_1, x_2, \cdots, x_n\}$ the convex cone $\text{conv}(X)$ is defined to be the set $\{\sum_{j=1}^n t_j x_j \mid x_j \in X, \sum_{j=1}^n t_j = 1, t_j \in [0, 1], j = 1, 2, \cdots, n\}$.*

PROOF: Since any positive convex combination of the null income elements in $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is sill a null income distribution, we infer

$$\text{(B.13)} \quad \text{conv}\{\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} \cap \mathcal{N}(\succeq)\} \subseteq \mathcal{N}(\succeq).$$

For another direction, we know by definition of the null income distribution,

$$\text{(B.14)}\quad \mathbf{y}_0 + \mathbf{y} \sim \mathbf{y}, \text{ for any } \mathbf{y} \in \mathcal{I},$$

for any $\mathbf{y}_0 \in \mathcal{N}(\succeq)$ and $\mathbf{y} \in \mathcal{I}$. On the other hand, any income distribution can be written as a linear combination of the income basis, hence we can write

$$\text{(B.15)}\quad \mathbf{y}_0 = a_1 \cdot \mathbf{e}_1 + a_2 \cdot \mathbf{e}_2 + \cdots + a_n \cdot \mathbf{e}_n,$$

$$\text{(B.16)}\quad \mathbf{y} = b_1 \cdot \mathbf{e}_1 + b_2 \cdot \mathbf{e}_2 + \cdots + b_n \cdot \mathbf{e}_n,$$

where $a_i, b_i \geq 0$ for $i = 1, 2, \cdots, n$. Substituting (B.15) and (B.16) into (B.14) yields

$$\text{(B.17)}\quad \sum_{i=1}^{n} (a_i + b_i) \cdot \mathbf{e}_i \sim \sum_{i=1}^{n} b_i \cdot \mathbf{e}_i, \text{ for any } b_i \geq 0, \ i = 1, 2, \cdots, n.$$

It remains to determine the coefficients set $\{a_1, a_2, \cdots, a_n\}$ via (B.17). For notation convenience, we set $N = \{i \mid \mathbf{e}_i \in \mathcal{N}(\succeq)\}$; note that the complement of set N is $N^c = \{i \mid \mathbf{e}_i \notin \mathcal{N}(\succeq)\}$. With the notations of N and N^c, (B.17) can be rewritten as

$$\text{(B.18)}\quad \sum_{i \in N^c} (a_i + b_i) \cdot \mathbf{e}_i \sim \sum_{i \in N^c} b_i \cdot \mathbf{e}_i, \text{ for any } b_i \geq 0, \ i = 1, 2, \cdots, n.$$

By Lemma B.3, we know that there exists an integer $1 \leq i_0 \leq n$ such that $\mathbf{e}_{i_0} \in P_{\max}$. It is straightforward to see $\mathbf{e}_{i_0} \notin \mathcal{N}(\succeq)$, as the preference relationship $\succeq$ is not a trivial one according to Axiom 2. Also by Lemma B.3, there exists $1 \leq j_0 \leq n$ such that $\mathbf{e}_{j_0} \in P_{\text{smin}}$. Now we have the following two cases:

The first case: $\mathbf{e}_{i_0} \sim \mathbf{e}_{j_0} \succ \mathbf{0}$. Then for any $\mathbf{y} \in \mathcal{I}$, we have either $\mathbf{y} \sim \mathbf{e}_{i_0}$ or $\mathbf{y} \sim \mathbf{0}$. Owing to $\mathbf{e}_{i_0} \notin \mathcal{N}(\succeq)$, we infer by Axiom 4

$$\text{(B.19)}\quad \mathcal{N}(\succeq) \subseteq \text{conv}\{\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} \cap \mathcal{N}(\succeq)\}.$$

The second case: $\mathbf{e}_{i_0} \succ \mathbf{e}_{j_0} \succ \mathbf{0}$, which allows us to consider the income distribution

$$\text{(B.20)} \quad S_p = (2-p)\cdot \mathbf{e}_{i_0} + (p-1)\cdot \mathbf{e}_{j_0}, \text{ for } 1 \le p \le 2.$$

By Axiom 4 and Proposition B.1, we have

$$\text{(B.21)} \quad S_{p_1} \succeq S_{p_2} \iff 2 \ge p_2 \ge p_1 \ge 1.$$

For any income basis $\mathbf{e}_i$ with $i \in N^c$, by Axiom 5 there is a unique p such that $\mathbf{e}_i \sim S_p$, and we denote such a correspondence by $p = \psi(i)$, i.e.,

$$\text{(B.22)} \quad \mathbf{e}_i \sim S_{\psi(i)}, \text{ for } i = 1, 2, \cdots, n.$$

Therefore, by Lemma B.4 and (B.22), we obtain that for any income distribution $r_1 \cdot \mathbf{e}_1 + r_2 \cdot \mathbf{e}_2 + \cdots + r_n \cdot \mathbf{e}_n$ with $r_i \ge 0$, $i = 1, 2, \cdots, n$, there exist $\alpha(i) > 0$ for all $i \in N^c$ such that

$$\text{(B.23)} \quad \sum_{i \in N^c} r_i \cdot \mathbf{e}_i \sim \sum_{i \in N^c} r_i \alpha(i) \cdot S_{\psi(i)}.$$

Substituting (B.18) into (B.23), we have

$$\text{(B.24)} \quad \sum_{i \in N^c} (a_i + b_i)\alpha(i) \cdot S_{\psi(i)} \sim \sum_{i \in N^c} b_i \alpha(i) \cdot S_{\psi(i)},$$

for any $b_i \ge 0$, $i = 1, 2, \cdots, n$. Thus, it remains to determine the coefficients $\{a_i \mid i \in N^c\}$ via (B.24).

Consider the case that there exists at least one $b_i > 0$ for $i \in N^c$. In this case, we know that

$$\text{(B.25)} \quad \sum_{i \in N^c} b_i \alpha(i) > 0.$$

Therefore, by Lemma B.1, the condition (B.25), and the strictly decreasing of S_p in (B.21), (B.24) becomes

$$\text{(B.26)} \quad \frac{\sum_{i \in N^c} (a_i + b_i)\alpha(i)\psi(i)}{\sum_{i \in N^c} (a_i + b_i)\alpha(i)} = \frac{\sum_{i \in N^c} b_i \alpha(i)\psi(i)}{\sum_{i \in N^c} b_i \alpha(i)}.$$

If $a_i = 0$ for all $i \in N^c$, then (B.26) automatically holds; otherwise, there exists an $i \in N^c$ such that $a_i > 0$. By rearranging (B.26), we have

$$\text{(B.27)} \quad \frac{\sum_{i\in N^c} a_i\alpha(i)\psi(i)}{\sum_{i\in N^c} a_i\alpha(i)} = \frac{\sum_{i\in N^c} b_i\alpha(i)\psi(i)}{\sum_{i\in N^c} b_i\alpha(i)}.$$

Note that equation (B.27) holds for any $b_i \geq 0$ and at least one of them is positive. We choose $(b_1, b_2 \cdots, b_n) = (1, 0, \cdots, 0)$, $(0, 1, 0, \cdots, 0)$, $\cdots$, $(0, 0, \cdots, 0, 1)$ and substitute them into (B.27), yielding

$$\text{(B.28)} \quad \psi(i) = \psi(j), \text{ for any } i, j \in N^c,$$

which means that $\psi(i)$ is a constant for all $i \in N^c$.

On the other hand, by the definition of the income distribution $\mathbf{e}_{i_0}$, $\mathbf{e}_{j_0}$, and (B.20), we find that

$$\text{(B.29)} \quad i_0, j_0 \in N^c \text{ and } \mathbf{e}_{i_0} \sim S_1, \mathbf{e}_{j_0} \sim S_2.$$

Therefore, $\psi(i_0) = 1$ and $\psi(j_0) = 2$ by the definition of the function ψ in (B.22), and (B.29), leading to a contradiction to (B.28). Hence, we conclude that if there exists at least one $b_i > 0$ for $i \in N^c$, then

$$\text{(B.30)} \quad a_i = 0, \text{ for any } i \in N^c,$$

which indicates that

(B.31)

$$\mathcal{N}(\succeq) \subseteq \{\sum_{i\in N} a_i{\cdot}\mathbf{e}_i \mid a_i \geq 0 \text{ for all } i \in N\} = \text{conv}\{\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} \cap \mathcal{N}(\succeq)\}.$$

The desired result then follows by combining (B.13), (B.19), and (B.31).

Q.E.D.

LEMMA B.6 (Tail Consecutiveness) *Suppose the preference relationship $\succeq$ satisfies Axiom 1, Axiom 2 and Axiom 3. Then there exists an integer $k \in \{1, 2, \cdots, n\}$ such that*

$$\{\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \cdots, \mathbf{e}_n, \mathbf{e}_{n+1}\} = \mathcal{N}(\succeq) \cap \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{e}_{n+1}\},$$

where we set $\mathbf{e}_{n+1} := \mathbf{0}$ *for the notation convenience.*

PROOF: For $\mathbf{e}_i \notin \mathcal{N}(\succeq)$ with $2 \leq i \leq n$, we know that $\mathbf{e}_i \succ \mathbf{0}$ by Axiom 2, and the axiom also implies that

(B.32) $\mathbf{e}_j \succ \mathbf{0}, \text{ for any } j \leq i.$

Therefore, by (B.32) we know that there exists an integer $1 \leq k \leq n+1$ such that

(B.33) $\mathbf{e}_j \sim \mathbf{0}, \text{ for any } k+1 \leq j \leq n+1$

and

(B.34) $\mathbf{e}_i \succ \mathbf{0}, \text{ for any } \ 1 \leq i \leq k.$

Since $\mathbf{0}$ is a null income distribution, we obtain by Lemma B.2 and equation (B.33) that $\mathbf{e}_j$ for $k+1 \leq j \leq n+1$ is also a null income distribution. Thus we have

(B.35) $\{\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \cdots, \mathbf{e}_n, \mathbf{e}_{n+1}\} \subseteq \mathcal{N}(\succeq) \cap \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{e}_{n+1}\}.$

Note that all the null income distributions are equivalent; indeed, for any two null income distributions $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{N}(\succeq)$, we have $\mathbf{y}_1 + \mathbf{y}_2 \sim \mathbf{y}_1 \sim \mathbf{y}_2$ by the definition of the null income distribution. Therefore, by (B.34) we know that $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_k$ are not null income distributions, since they are not equivalent to zero income distribution $\mathbf{0}$. Thus we have

(B.36) $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_k\} \subseteq \{\mathcal{N}(\succeq) \cap \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{e}_{n+1}\}\}^c.$

Combining (B.35) and (B.36) yields the desired result. *Q.E.D.*

LEMMA B.7 (Characterization of Null Income Set under Inequality Index (1)) *For the inequality index* I_{LM} *for the bottom and middle income* (1)*, if* *there exists some* $\mathbf{y}_0 = (y_1^0, y_2^0, \cdots, y_n^0) \in \mathcal{I}$*, such that* $I_{LM}(\mathbf{y}_0 + \mathbf{y}) = I_{LM}(\mathbf{y})$ *holds for any* $\mathbf{y} \in \mathcal{I}$*, then* $a_1 y_1^0 + a_2(y_2^0 - y_1^0) + \cdots + a_k(y_k^0 - y_{k-1}^0) = 0$*, which is equivalent to* $y_1^0 = y_2^0 = \cdots = y_k^0$*.*

PROOF: By the condition given in the lemma, we assume that there exists a $\mathbf{y}_0 = (y_1^0, y_2^0, \cdots, y_n^0)$ such that

$$I_{LM}(\mathbf{y}_0 + \mathbf{y}) = I_{LM}(\mathbf{y}), \text{ for any } \mathbf{y} \in \mathcal{I}. \tag{B.37}$$

For notation convenience, we denote $\mathbf{y} = (y_1, y_2, \cdots, y_n)$.

The first case is that $I_{LM}(\mathbf{y}) = \frac{b_{k+1}}{a_{k+1}}$, which, by definition of the inequality index I_{LM}, means

$$\sum_{i=1}^{k} a_i(y_i - y_{i-1}) = 0, \text{ with } y_0 = 0. \tag{B.38}$$

By (B.37) we know $I_{LM}(\mathbf{y}_0 + \mathbf{y}) = \frac{b_{k+1}}{a_{k+1}}$, which, by definition of the inequality index I_{LM}, means

$$\sum_{i=1}^{k} a_i(y_i - y_{i-1} + y_i^0 - y_{i-1}^0) = 0, \text{ with } y_0 = y_0^0 = 0. \tag{B.39}$$

Combining (B.38) and (B.39) yields

$$\sum_{i=1}^{k} a_i(y_i^0 - y_{i-1}^0) = 0, \text{ with } y_0^0 = 0. \tag{B.40}$$

Since $a_i > 0$ and $y_i - y_{i-1} \geq 0$ for $i = 1, 2, \cdots, k$, (B.40) yields $y_1^0 = y_2^0 = \cdots = y_k^0$.

The second case is that $I_{LM}(\mathbf{y}) = \frac{b_1 y_1 + b_2(y_2 - y_1) + \cdots + b_k(y_k - y_{k-1})}{a_1 y_1 + a_2(y_2 - y_1) + \cdots + a_k(y_k - y_{k-1})}$, which by definition means $a_1 y_1 + a_2(y_2 - y_1) + \cdots + a_k(y_k - y_{k-1}) \neq 0$. Hence, (B.37) becomes

$$\frac{\sum_{i=1}^{k} b_i(y_i - y_{i-1} + y_i^0 - y_{i-1}^0)}{\sum_{i=1}^{k} a_i(y_i - y_{i-1} + y_i^0 - y_{i-1}^0)} = \frac{\sum_{i=1}^{k} b_i(y_i - y_{i-1})}{\sum_{i=1}^{k} a_i(y_i - y_{i-1})}, \tag{B.41}$$

for any $(y_1, y_2, \cdots, y_n)$ with $a_1 y_1 + a_2(y_2 - y_1) + \cdots + a_k(y_k - y_{k-1}) \neq 0$.

Cross multiplication of (B.42) gives

$$\text{(B.42)} \quad \sum_{i=1}^{k} b_i(y_i^0 - y_{i-1}^0) = \frac{\sum_{i=1}^{k} b_i(y_i - y_{i-1})}{\sum_{i=1}^{k} a_i(y_i - y_{i-1})} \sum_{i=1}^{k} a_i(y_i^0 - y_{i-1}^0).$$

Because (B.42) holds for any $(y_1, y_2, \cdots, y_n)$ with $a_1 y_1 + a_2(y_2 - y_1) + \cdots + a_k(y_k - y_{k-1}) \neq 0$, we obtain the conclusion that $\sum_{i=1}^{k} a_i(y_i^0 - y_{i-1}^0) = \sum_{i=1}^{k} b_i(y_i^0 - y_{i-1}^0) = 0$, which further implies $y_1^0 = y_2^0 = \cdots = y_k^0$.

The proof is terminated by combining the above two cases. *Q.E.D.*

APPENDIX C: PROOF OF THEOREM 1

PROOF: Sufficiency. Now we suppose the preference relationship $\succeq$ is defined by (1), i.e.,

$$\text{(C.1)} \quad \mathbf{y}_1 \succeq \mathbf{y}_2 \text{ if and only if } I_{LM}(\mathbf{y}_1) \leq I_{LM}(\mathbf{y}_2).$$

To check that the preference relationship satisfies the weak ordering axiom (Axiom 1), we need to verify two things: The first one is transitivity, i.e., $\mathbf{y}_1 \succeq \mathbf{y}_2$ and $\mathbf{y}_2 \succeq \mathbf{y}_3$ implies $\mathbf{y}_1 \succeq \mathbf{y}_3$. This apparently holds by the definition of the preference relationship (C.1).

The second one is completeness, i.e., for any two $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{I}$, we have $\mathbf{y}_1 \succeq \mathbf{y}_2$ or $\mathbf{y}_2 \succeq \mathbf{y}_1$ or both. This is true, because for two numbers $I_{LM}(\mathbf{y}_1)$ and $I_{LM}(\mathbf{y}_2)$ we must have $I_{LM}(\mathbf{y}_1) \leq I_{LM}(\mathbf{y}_2)$ or $I_{LM}(\mathbf{y}_2) \leq I_{LM}(\mathbf{y}_1)$ or both.

To verify Axiom 2, we will first check that $\mathbf{y} \succeq \mathbf{0}$ for any $\mathbf{y} \in \mathcal{I}$. Indeed, by the definition of the inequality index for the bottom and middle income, if $I_{LM}(\mathbf{y}_0) = \frac{b_1 y_1 + b_2(y_2 - y_1) + \cdots + b_k(y_k - y_{k-1})}{a_1 y_1 + a_2(y_2 - y_1) + \cdots + a_k(y_k - y_{k-1})}$ for some income distribution $\mathbf{y}_0 \in \mathcal{I}$, then we obtain that

$$\text{(C.2)} \quad I_{LM}(\mathbf{y}_0) \leq \max\{\frac{b_1}{a_1}, \frac{b_2}{a_2}, \cdots, \frac{b_k}{a_k}\},$$

because $\frac{b_i}{a_i} > 0$ and $y_i - y_{i-1} \geq 0$ for $i = 1, 2, \cdots, k$. Therefore, by (C.2) and (1), we know that,

(C.3)
$$I_{LM}(\mathbf{y}) \leq \max\{\frac{b_1}{a_1}, \frac{b_2}{a_2}, \cdots, \frac{b_k}{a_k}, \frac{b_{k+1}}{a_{k+1}}\} = \frac{b_{k+1}}{a_{k+1}} = I_{LM}(\mathbf{0}), \quad \text{for any } \mathbf{y} \in \mathcal{I}.$$

Thus, by (C.1) and (C.3) we have $\mathbf{y} \succeq \mathbf{0}$ for any $\mathbf{y} \in \mathcal{I}$. Note that $I_{LM}(\mathbf{e}_i) = \frac{b_i}{a_i}$ for $i = 1, 2, \cdots, k$, which is strictly less than $I_{LM}(\mathbf{0})$. Hence, by (C.1) we have $\mathbf{e}_k \succ \mathbf{0}$. Therefore, there exists at least one $\mathbf{y}$ such that $\mathbf{y} \succ \mathbf{0}$.

To verify the second part of Axiom 2, we suppose that there exists $i \geq 2$ such that $\mathbf{e}_i \succ \mathbf{0}$, then by (C.1) we know that it is equivalent to $I_{LM}(\mathbf{e}_i) = \frac{b_i}{a_i} < \frac{b_{k+1}}{a_{k+1}}$. Now for any $j \leq i$, by a direct calculation we obtain $I_{LM}(\mathbf{e}_j) = \frac{b_j}{a_j} < \frac{b_{k+1}}{a_{k+1}}$, which means $\mathbf{e}_j \succ \mathbf{0}$ by (C.1). Hence Axiom 2 is verified.

To verify the weak-zero independence axiom (Axiom 3), we only need to focus on the case $r > 0$, because we may simply choose $l = 0$ for $r = 0$ to ensure that the weak-zero independence axiom holds. There are two cases:

Case 1: $\mathbf{y}_1 \in \mathcal{N}(\succeq)$, which, by Lemma B.7, means $\sum_{i=1}^{k} a_k(y_k - y_{k-1}) = 0$ with $y_0 = 0$. Letting $l = 0$, we have $I_{LM}(r \cdot \mathbf{y}_1 + \mathbf{y}_3) = I_{LM}(l \cdot \mathbf{y}_2 + \mathbf{y}_3)$ for any $\mathbf{y}_3 \in \mathcal{I}$, which by (C.1) yields $r \cdot \mathbf{y}_1 + \mathbf{y}_3 \sim l \cdot \mathbf{y}_2 + \mathbf{y}_3$ for any $\mathbf{y}_3 \in \mathcal{I}$.

Case 2: $\mathbf{y}_1 \notin \mathcal{N}(\succeq)$. For notation convenience we denote

$$I_1(\mathbf{y}) = b_1 y_1 + b_2(y_2 - y_1) + \cdots + b_k(y_k - y_{k-1}),$$
$$I_2(\mathbf{y}) = a_1 y_1 + a_2(y_2 - y_1) + \cdots + a_k(y_k - y_{k-1}).$$

Hence, $I_{LM}(\mathbf{y}) = \frac{I_1(\mathbf{y})}{I_2(\mathbf{y})}$ for $\mathbf{y} \in \mathcal{I}$ with $I_2(\mathbf{y}) \neq 0$. The assumption $\mathbf{y}_1 \notin \mathcal{N}(\succeq)$ implies $I_1(\mathbf{y}_1) \neq 0$ by Lemma B.7. For any $\mathbf{y}_2 \in \mathcal{I}$ that satisfies

$$\text{(C.4)} \quad I_{LM}(\mathbf{y}_1) = I_{LM}(\mathbf{y}_2),$$

we assert that $I_2(\mathbf{y}_1) \neq 0$. Indeed, if $I_2(\mathbf{y}_2) = 0$, then $I_{LM}(\mathbf{y}_2) = \frac{b_{k+1}}{a_{k+1}} > I_{LM}(\mathbf{y}_1)$, as $\frac{b_{k+1}}{a_{k+1}} > \frac{b_i}{a_i}$ for any $i = 1, 2, \cdots, k$, which contradicts (C.4).

To find an l such that $r \cdot \mathbf{y}_1 + \mathbf{y}_3 \sim l \cdot \mathbf{y}_2 + \mathbf{y}_3$ with $\mathbf{y}_1 \sim \mathbf{y}_2$ and $r > 0$, we need to solve the following equation for l:

$$\text{(C.5)} \qquad \frac{I_1(r \cdot \mathbf{y}_1 + \mathbf{y}_3)}{I_2(r \cdot \mathbf{y}_1 + \mathbf{y}_3)} = \frac{I_1(l \cdot \mathbf{y}_2 + \mathbf{y}_3)}{I_2(l \cdot \mathbf{y}_2 + \mathbf{y}_3)}, \text{ with } \frac{I_1(\mathbf{y}_1)}{I_2(\mathbf{y}_1)} = \frac{I_1(\mathbf{y}_2)}{I_2(\mathbf{y}_2)}.$$

By the linearity of I_1 and I_2, (C.5) becomes

$$\text{(C.6)} \qquad \frac{rI_1(\mathbf{y}_1) + I_1(\mathbf{y}_3)}{rI_2(\mathbf{y}_1) + I_2(\mathbf{y}_3)} = \frac{lI_1(\mathbf{y}_2) + I_1(\mathbf{y}_3)}{lI_2(\mathbf{y}_2) + I_2(\mathbf{y}_3)}, \text{ with } \frac{I_1(\mathbf{y}_1)}{I_2(\mathbf{y}_1)} = \frac{I_1(\mathbf{y}_2)}{I_2(\mathbf{y}_2)}.$$

Solving (C.6) for l yields

$$\text{(C.7)} \qquad l = r\frac{I_2(\mathbf{y}_1)}{I_2(\mathbf{y}_2)},$$

which is independent of $\mathbf{y}_3$. Thus, (C.7) gives the coefficient l such that, for any $\mathbf{y}_1 \sim \mathbf{y}_2$ and $r > 0$, $r \cdot \mathbf{y}_1 + \mathbf{y}_3 \sim l \cdot \mathbf{y}_2 + \mathbf{y}_3$ holds for any $\mathbf{y}_3 \in \mathcal{I}$. Therefore, Axiom 3 is verified.

To verify the betweenness axiom (Axiom 4). Suppose there are two income distributions $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{I}$ with $\mathbf{y}_1 \succeq \mathbf{y}_2$, which is equivalent to $I_{LM}(\mathbf{y}_1) \leq I_{LM}(\mathbf{y}_2)$. There are four cases to be discussed.

Case 1: $I_2(\mathbf{y}_1) \neq 0$ and $I_2(\mathbf{y}_2) \neq 0$. By the definition of the inequality index I_{LM}, we have

$$\text{(C.8)} \qquad \frac{I_1(\mathbf{y}_1)}{I_2(\mathbf{y}_1)} \leq I_{LM}(r \cdot \mathbf{y}_1 + l \cdot \mathbf{y}_2) = \frac{rI_1(\mathbf{y}_1) + lI_1(\mathbf{y}_2)}{rl_2(\mathbf{y}_1) + lI_2(\mathbf{y}_2)} \leq \frac{I_1(\mathbf{y}_2)}{I_2(\mathbf{y}_2)},$$

for any $(r, l) \in [0, +\infty) \times [0, +\infty) - \{(0, 0)\}$. Hence, by (C.8) we obtain that

$$I_{LM}(\mathbf{y}_1) \leq I_{LM}(r \cdot \mathbf{y}_1 + l \cdot \mathbf{y}_2) \leq I_{LM}(\mathbf{y}_2),$$

which by (C.1) is equivalent to $\mathbf{y}_1 \succeq r \cdot \mathbf{y}_1 + l \cdot \mathbf{y}_2 \succeq \mathbf{y}_2$, for any $(r, l) \in [0, +\infty) \times [0, +\infty) - \{(0, 0)\}$.

Case 2: $I_2(\mathbf{y}_1) \neq 0$ and $I_2(\mathbf{y}_2) = 0$. For $(r, l) \in [0, +\infty) \times [0, +\infty) - \{(0, 0)\}$, note that

$$\text{(C.9)} \qquad I_{LM}(r \cdot \mathbf{y}_1 + l \cdot \mathbf{y}_2) = \begin{cases} I_{LM}(\mathbf{y}_1), & \text{if } r > 0, \\ \frac{b_{k+1}}{a_{k+1}}, & \text{if } r = 0. \end{cases}$$

A combination of $I_{LM}(\mathbf{y}_1) < \frac{b_{k+1}}{a_{k+1}}$ and (C.9) implies $I_{LM}(\mathbf{y}_1) \leq I_{LM}(r \cdot \mathbf{y}_1 + l \cdot \mathbf{y}_2) \leq I_{LM}(\mathbf{y}_2)$. Therefore, we have $\mathbf{y}_1 \succeq r \cdot \mathbf{y}_1 + l \cdot \mathbf{y}_2 \succeq \mathbf{y}_2$ by (C.1).

Case 3: $I_2(\mathbf{y}_1) = 0$ and $I_2(\mathbf{y}_2) = 0$. By the definition of the inequality index I_{LM}, we know that

$$\text{(C.10)} \quad I_{LM}(\mathbf{y}_1) = I_{LM}(\mathbf{y}_2) = \frac{b_{k+1}}{a_{k+1}}.$$

On the other hand, for $(r, l) \in [0, +\infty) \times [0, +\infty) - \{(0, 0)\}$, we obtain that

$$\text{(C.11)} \quad I_{LM}(r \cdot \mathbf{y}_1 + l \cdot \mathbf{y}_2) = \frac{b_{k+1}}{a_{k+1}}.$$

Combining (C.10) and (C.11) leads to $I_{LM}(\mathbf{y}_1) = I_{LM}(r \cdot \mathbf{y}_1 + l \cdot \mathbf{y}_2) = I_{LM}(\mathbf{y}_2)$, which by (C.1) indicates $\mathbf{y}_1 \succeq r \cdot \mathbf{y}_1 + l \cdot \mathbf{y}_2 \succeq \mathbf{y}_2$.

Case 4: $I_2(\mathbf{y}_1) = 0$ and $I_2(\mathbf{y}_2) \neq 0$. Since $I_2(\mathbf{y}_1) = 0$ means $I_{LM}(\mathbf{y}_1) = \frac{b_{k+1}}{a_{k+1}}$ and $I_2(\mathbf{y}_2) \neq 0$ means $I_{LM}(\mathbf{y}_2) < \frac{b_{k+1}}{a_{k+1}}$, we get $I_{LM}(\mathbf{y}_1) > I_{LM}(\mathbf{y}_2)$, which contradicts the assumption $I_{LM}(\mathbf{y}_1) \leq I_{LM}(\mathbf{y}_2)$. Thus, this case is impossible.

At last we shall verify the continuity axiom (Axiom 5). For three income distributions $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ with $I_{LM}(\mathbf{y}_1), I_{LM}(\mathbf{y}_2), I_{LM}(\mathbf{y}_2) < I_{LM}(\mathbf{0})$ and

$$\text{(C.12)} \quad I_{LM}(\mathbf{y}_1) \leq I_{LM}(\mathbf{y}_2) \leq I_{LM}(\mathbf{y}_3),$$

by the definition and linearity of I_{LM} we have

$$\text{(C.13)} \quad I_{LM}(\alpha \cdot \mathbf{y}_1 + (1-\alpha) \cdot \mathbf{y}_3) = \frac{\alpha I_1(\mathbf{y}_1) + (1-\alpha) I_1(\mathbf{y}_3)}{\alpha I_2(\mathbf{y}_1) + (1-\alpha) I_2(\mathbf{y}_3)},$$

which is a function of $\alpha \in [0, 1]$; for notation convenience we denote this function by $f(\alpha)$. The derivative of $f(\alpha)$ is

$$\text{(C.14)} \quad f'(\alpha) = \frac{I_1(\mathbf{y}_1) I_2(\mathbf{y}_3) - I_1(\mathbf{y}_3) I_2(\mathbf{y}_1)}{(\alpha I_2(\mathbf{y}_1) + (1-\alpha) I_2(\mathbf{y}_3))^2} < 0,$$

which is strictly negative, since $I_{LM}(\mathbf{y}_1) < I_{LM}(\mathbf{y}_3)$. Therefore, the function $f(\alpha)$ (equation (C.13)) is a strictly decreasing function of $\alpha \in [0, 1]$ by

(C.14). Noting the fact that $f(0) = I_{LM}(\mathbf{y}_3)$, $f(1) = I_{LM}(\mathbf{y}_1)$, and the inequality (C.12), we know that there exists a unique $\alpha \in [0,1]$ such that

$$I_{LM}(\mathbf{y}_2) = I_{LM}(\alpha \cdot \mathbf{y}_1 + (1-\alpha) \cdot \mathbf{y}_3),$$

which by (C.1) means $\mathbf{y}_2 \sim \alpha \cdot \mathbf{y}_1 + (1-\alpha) \cdot \mathbf{y}_3$.

Necessity. By Lemma B.3, there exist $1 \leq i_0, j_0 \leq n$ such that $\mathbf{e}_{i_0} \in P_{\max}$ and $\mathbf{e}_{j_0} \in P_{\text{smin}}$, which allows us to define the income distribution (B.20). Hence, for any $\mathbf{y} = (y_1, y_2, \cdots, y_n) \in \mathcal{I}$, by defining $y_0 = 0$ and by Lemma B.4, we know that there exist $\{\alpha(i) \geq 0\}_{i=1}^{n}$ such that

$$\mathbf{y} = \sum_{i=1}^{n} (y_i - y_{i-1}) \cdot \mathbf{e}_i \sim \sum_{i=1}^{n} (y_i - y_{i-1}) \alpha(i) \cdot S_{\psi(i)}. \tag{C.15}$$

Before proceeding, we claim that

$$\mathcal{N}(\succeq) = \{(y_1, y_2, \cdots, y_n) \mid \sum_{i=1}^{n} \alpha(i)(y_i - y_{i-1}) = 0\}. \tag{C.16}$$

Indeed, for any $\mathbf{y} \in \mathcal{N}(\succeq)$ we have

$$\begin{aligned} \sum_{i=1}^{n} \alpha(i)(y_i - y_{i-1}) &= \sum_{i \in N} \alpha(i)(y_i - y_{i-1}) + \sum_{i \in N^c} \alpha(i)(y_i - y_{i-1}) \\ &= \sum_{i \in N^c} \alpha(i)(y_i - y_{i-1}), \end{aligned}$$

where the last equality holds because $\alpha(i) = 0$ for $i \in N$ by the definition of the null income distributions. On the other hand, by Lemma B.5 we know that $\mathbf{y} = \sum_{i \in N} a_i \mathbf{e}_i$ for $a_i \geq 0$, $i = 1, 2, \cdots, n$, yielding $\sum_{i \in N^c} \alpha(i)(y_i - y_{i-1}) = 0$. Therefore, we have

$$\mathcal{N}(\succeq) \subseteq \{\mathbf{y} \mid \sum_{i=1}^{n} \alpha(i)(y_i - y_{i-1}) = 0\}. \tag{C.17}$$

For the opposite direction, suppose that there is some $\mathbf{y} \in \mathcal{I}$ such that

$$\sum_{i=1}^{n} \alpha(i)(y_i - y_{i-1}) = 0. \tag{C.18}$$

Recalling that $\alpha(i) = 0$ for $i \in N$, (C.18) leads to

$$\sum_{i\in N^c} \alpha(i)(y_i - y_{i-1}) = 0 \iff y_i - y_{i-1} = 0, \text{ for } i \in N^c.$$

Hence, there exist $a_i \geq 0$, $i \in N$ such that $\mathbf{y} = \sum_{i\in N} a_i \mathbf{e}_i$. Therefore,

$$\text{(C.19)} \quad \{(y_1, y_2, \cdots, y_n) \mid \sum_{i=1}^{n} \alpha(i)(y_i - y_{i-1}) = 0\} \subseteq \mathcal{N}(\succeq).$$

Combining (C.17) and (C.19) yields (C.16).

The right-hand side of (C.15) is $\sum_{i=1}^{n}(y_i - y_{i-1})\alpha(i)S_{\psi(i)}$ for $\mathbf{y} = (y_1, y_2, \cdots, y_n)$, which can be expanded as

$$\begin{aligned}&(0, \cdots, 0, \sum_{i=1}^{n}(y_i - y_{i-1})\alpha(i)(1 - \psi(i)), \cdots,\\ &\sum_{i=1}^{n}(y_i - y_{i-1})\alpha(i)(1 - \psi(i)), \sum_{i=1}^{n}(y_i - y_{i-1})\alpha(i), \cdots, \sum_{i=1}^{n}(y_i - y_{i-1})\alpha(i)).\end{aligned}$$

If $\mathbf{y} \notin \mathcal{N}(\succeq)$, then by (C.16) we know that $\sum_{i=1}^{n} \alpha(i)(y_i - y_{i-1}) \neq 0$, therefore, the above distribution can be rewritten as

$$\text{(C.20)} \quad \mathbf{y} \sim \sum_{i=1}^{n}(y_i - y_{i-1})\alpha(i) \cdot S_{\psi(i)} \sim S_p,$$

where

$$\text{(C.21)} \quad p = \sum_{i=1}^{n}(y_i - y_{i-1})\alpha(i)\psi(i) / \sum_{i=1}^{n}(y_i - y_{i-1})\alpha(i).$$

By (C.20) and (C.21), we known that for any two income distributions $\mathbf{y}_i = (y_1^i, y_2^i, \cdots, y_n^i) \in \mathcal{I} - \mathcal{N}(\succeq)$, $i = 1, 2$, we have

$$\text{(C.22)} \quad \mathbf{y}_1 \succeq \mathbf{y}_2 \iff S_{p(\mathbf{y}_1)} \succeq S_{p(\mathbf{y}_2)} \iff p(\mathbf{y}_1) \leq p(\mathbf{y}_2),$$

in which p_j, $j = 1, 2$ are defined by (C.21), i.e.,

$$\text{(C.23)} \quad p(\mathbf{y}_j) = \sum_{i=1}^{n}(y_i^j - y_{i-1}^j)\alpha(i)\psi(i) / \sum_{i=1}^{n}(y_i^j - y_{i-1}^j)\alpha(i).$$

By Lemma B.6, there exists a $k \in [1, n]$, such that

$$\mathbf{e}_i \notin \mathcal{N}(\succeq), \text{ for } i = 1, 2, \cdots, k,$$
$$\mathbf{e}_i \in \mathcal{N}(\succeq), \text{ for } i = k+1, k+2, \cdots, n.$$

Thus, we have

$$\text{(C.24)} \quad \alpha(i) > 0\,, \text{for any } 1 \le i \le k;\ \alpha(i) = 0\,, \text{for any } k+1 \le i \le n.$$

Substituting (C.24) into (C.23), we have

$$\text{(C.25)} \quad p(\mathbf{y}_j) = \sum_{i=1}^{k} (y_i^j - y_{i-1}^j)\alpha(i)\psi(i) / \sum_{i=1}^{k} (y_i^j - y_{i-1}^j)\alpha(i).$$

Therefore, we can define an inequality index $I_{LM}(\mathbf{y}) = p(\mathbf{y})$ for any income distribution $\mathbf{y} \notin \mathcal{N}(\succeq)$.

It remains to find $I_{LM}(\mathbf{y})$ for the income distribution $\mathbf{y} \in \mathcal{N}(\succeq)$ such that (C.1) holds. If $\mathbf{y} \in \mathcal{N}(\succeq)$, then $\mathbf{y} \sim \mathbf{0}$ as all the null income distributions are equivalent. Together with Axiom 2 that $\mathbf{y} \succeq \mathbf{0}$ for any $\mathbf{y} \in \mathcal{I}$, we obtain

$$\text{(C.26)} \quad \mathbf{y_1} \succ \mathbf{y}_2, \text{ for any } \mathbf{y}_1 \notin \mathcal{N}(\succeq) \text{ and } \mathbf{y}_2 \in \mathcal{N}(\succeq),$$

thus $I_{LM}(\mathbf{y}_1) < I_{LM}(\mathbf{y}_2)$ for any $\mathbf{y}_1 \notin \mathcal{N}(\succeq)$ and $\mathbf{y}_2 \in \mathcal{N}(\succeq)$. As $I_{LM}(\mathbf{y}_2)$ are all the same for any $\mathbf{y}_2 \in \mathcal{N}(\succeq)$, by (C.26) we need to define $I_{LM}(\mathbf{y}_2)$ to be strictly larger than all $I_{LM}(\mathbf{y}_1)$ for $\mathbf{y}_1 \notin \mathcal{N}(\succeq)$.

Therefore, in view of (C.25) we let

$$\begin{cases} a_i = \alpha(i), & \text{for } i = 1, 2, \cdots, k, \\ b_i = \alpha(i)\psi(i), & \text{for } i = 1, 2, \cdots, k, \end{cases}$$

and define

$$\text{(C.27)} \quad I_{LM}(\mathbf{y}) = \begin{cases} \frac{b_1 y_1 + b_2(y_2 - y_1) + \cdots + b_k(y_k - y_{k-1})}{a_1 y_1 + a_2(y_2 - y_1) + \cdots + a_k(y_k - y_{k-1})}\,, & I_2(\mathbf{y}) \neq 0; \\ \frac{b_{k+1}}{a_{k+1}}\,, & I_2(\mathbf{y}) = 0, \end{cases}$$

where $\frac{b_{k+1}}{a_{k+1}} > \frac{b_i}{a_i}$ for $i = 1, 2, \cdots, k$. The inequality index (C.27) satisfies (C.1). This completes the proof. *Q.E.D.*

APPENDIX D: PROOF OF PROPOSITION 2

PROOF: (i) Let $q(u)$ be the quantile function and $L(u)$ be the Lorenz curve of an income distribution. Recall the definition of the Gini coefficient for the lower and middle income,

$$G(0,\alpha) = 1 - \frac{2\int_0^\alpha L(u)\,du}{\alpha L(\alpha)}, \text{ for any } 0 < \alpha \leq 1, \tag{D.1}$$

where $G(0,\alpha)$ is the Gini coefficient for the quantile range $[0,\alpha]$ of the income distribution. Rearranging (D.1) yields

$$\alpha L(\alpha)(1 - G(0,\alpha)) = 2\int_0^\alpha L(u)\,du, \text{ for any } 0 \leq \alpha \leq 1. \tag{D.2}$$

Differentiating both sides of equation (D.2) with respect to α gives

(D.3)

$$\alpha(1-G(0,\alpha))L'(\alpha) + L(\alpha)(\alpha(1-G(0,\alpha)))' = 2L(\alpha), \text{ for any } 0 < \alpha < 1.$$

Rearranging (D.3) leads to

$$\frac{L'(\alpha)}{L(\alpha)} = \frac{2 - (\alpha(1 - G(0,\alpha))')}{\alpha(1 - G(0,\alpha))}, \text{ for any } 0 < \alpha < 1. \tag{D.4}$$

Integrating both sides of equation (D.4) for α from v to 1 gives

$$S(v,1) = 1 - \frac{1 - G(0,1)}{v(1 - G(0,v))} \exp\left(-2\int_v^1 \frac{1}{u(1 - G(0,u))}\,du\right),$$

with $0 < v < 1$. This completes the proof of part (i).

(ii) Rearranging (D.1) leads to

$$2\int_\alpha^1 L(u)\,du = 1 - G(0,1) - (1 - G(0,\alpha))\alpha L(\alpha). \tag{D.5}$$

On the other hand, by the definition of the Gini coefficient,

$$G(\alpha,1) = 1 - 2\int_0^1 L^{\alpha,1}(u)\,du = 1 - \frac{2(\int_\alpha^1 L(u)\,du - (1-\alpha)L(\alpha))}{(1-\alpha)(1 - L(\alpha))}, \tag{D.6}$$

where $G(\alpha, 1)$ is the Gini coefficient for the quantile range $[\alpha, 1]$ and $L^{\alpha,1}(u)$ is the Lorenz curve for the quantile range $[\alpha, 1]$. Substituting (D.6) into (D.5) yields

$$\text{(D.7)} \quad L(\alpha) = \frac{\alpha - G(0,1) + G^{\alpha,1}(1-\alpha)}{1 - \alpha G(0,\alpha) + G^{\alpha,1}(1-\alpha)}.$$

Since $S(\alpha, 1) = 1 - L(\alpha)$, we have by (D.7)

$$\begin{aligned} S(\alpha,1) - \frac{1 + G(0,1) - \alpha(1 + G(0,\alpha))}{1 - \alpha G(0,\alpha)} \\ = \frac{(1-\alpha)(\alpha - 1 + \alpha G(0,\alpha) - G)G(\alpha,1)}{(1 - \alpha G(0,\alpha))(1 - \alpha G(0,\alpha) + (1-\alpha)G(\alpha,1))}. \end{aligned}$$

The desired result then follows by noticing that the right-hand side of the above equation is $o(1-\alpha)$ as $\alpha \to 1$. *Q.E.D.*

APPENDIX E: DATA DESCRIPTION AND THE ALGORITHM TO GET THE REQUIRED DATA

An example of data we extract from IPUMS-CPS dataset is given in Table V.

YEAR	SERIAL	MONTH	ASECWTH	PERNUM	ASECWT	INCTOT
2010	1	March	485.9900	1	485.9900	13992
2010	2	March	531.7100	1	531.7100	12000
2010	3	March	474.4000	1	474.4000	8657
2010	3	March	474.4000	2	474.4000	26157
2010	4	March	486.6500	1	486.6500	44000
2010	4	March	486.6500	2	486.6500	6000
2010	5	March	474.4000	1	474.4000	13600

TABLE V

IPUMS-CPS INDIVIDUAL AND FAMILY TOTAL INCOME MICRO DATA EXAMPLE. DATA SOURCES: IPUMS-CPS

In the table, the variable "SERIAL" is an identifying number uniquely assigned to each household in a given survey of a particular month and year. The combination of “YEAR”, “MONTH”, and “SERIAL” provides a unique identifier for every household in IPUMS-CPS. The individual records in the same household are assigned the same serial number. “PERNUM” enumerates all persons within each household consecutively (starting with “1”), in the same order in which they are listed in the original CPS data. “ASECWTH” and “ASECWT” are household and individual level weights, respectively, which should be used to construct population level data. “INCTOT” represents the individual total income, indicating each respondent’s total pre-tax personal income from all sources for the previous calendar year; the income amounts are expressed as they were reported to the interviewer. In particular, “INCTOT” and “ASECWT” are the two main variables that we will use to generate the population income distribution for the corresponding calendar year.

Next, we shall show how to generate income distribution from a given data set as in Table V. Since we consider individual incomes in this paper, we shall only use “INCTOT” and “ASECWT” in the dataset (see e.g., Table V). Below is the algorithm to generate the income distribution $\mathbf{Y}$ with two variables “INCTOT” and “ASECWT”.

As a numerical example, if INCTOT $= [13992, 12000]$ and ASECWT $= [485.9900, 531.7100]$, then $\mathbf{Y} = [13992, \ldots, 13992, 12000, \ldots, 12000]$, with the numbers of 13992 and 12000 being 486 and 532, respectively.

Algorithm 1: Generating the Income Distribution

Input: INCTOT, ASECWT, $\mathbf{Y}$;

1. Initializing $\mathbf{Y} = [\ ]$;
   $N \leftarrow \text{length(INCTOT)}$;
   $i \leftarrow 1$;
2. **while** $i \leq N$; **do**
   $\mathbf{Y} \leftarrow [\mathbf{Y}, \text{INCTOT}(i) \times \text{ones}(1, \text{round}(\text{ASECWT}(i)))]$;
   $i = i + 1$;
   end
3. $\mathbf{Y} \leftarrow \text{sort}(\mathbf{Y})$;

Output: Y

% length($\mathbf{x}$) returns the size of the vector $\mathbf{x}$;

% ones(m, n) returns an m-by-n matrix of ones;

% round(x) returns the nearest integer of x;

% sort($\mathbf{x}$) sorts the elements of the vector $\mathbf{x}$ in an ascending order.