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Exhaustible Resources with Adjustment Costs: Spot and Futures Prices

Abstract

We propose an equilibrium model of exhaustible resources in a comprehensive setting, including exploration, stochastic demand, and stochastic reserve, by extending the models in Pindyck (1980) and Anderson, Kellogg, and Salant (2018). The production adjustments can be quite general, e.g., sudden or gradual changes of the instantaneous production level. The model has three advantages: (1) It can generate low elasticity of production but high elasticity of exploration with respect to prices for general exhaustible resources (e.g., coal, natural gas, etc.). (2) In terms of spot prices, the combination of exploration and adjustment costs can significantly prolong the period of time that a spot price stays at the bottom, which explains why spot prices of many commodities can be quite low for a long time. (3) In terms of futures prices, the model can explain some stylized facts observed in futures markets, such as backwardation, contango, Samuelson effect, and higher volatility conditional on backwardation.

Journal of Economic Literature Classification: C61, C68, Q30, Q31.

Keywords: Exhaustible resources, Exploration, Production adjustment costs.

1 Introduction

Exhaustible (or non-renewable) resources, such as fossil fuels (coal, oil, natural gas) and metal ores, essentially cannot renew themselves in a meaningful human time frame. In contrast, resources such as timber and wind are renewable. For general background of natural resources and environmental economics, see, e.g., Pearce and Turner (1990), Maler and Vincent (2005, Volumes 1, 2, 3), Perman, Ma, Common, Maddison, and McGilvray (2012), Stavins (2012), Tietenberg and Lewis (2014).

In a seminal paper Hotelling (1931) studies the equilibrium prices of exhaustible resources, where a deterministic model is used to establish a fundamental principle, known as Hotelling's rule, that the net price (i.e., the flow price at which the resource, once extracted, can be sold on the market minus the marginal cost of extracting) of an exhaustible resource grows at the risk-free rate in a competitive equilibrium. If the marginal cost of extracting is constant, then Hotelling's rule predicts that the spot price should grow exponentially, which we shall refer to as Hotelling's rule for the spot price. Although this seminal paper provides an important understanding of the economics of exhaustible resources, there are two major issues for both spot prices and futures prices.

First, most empirical studies do not support the theoretical predictions from Hotelling's rule for spot prices. For example, Gaudet (2007) carries out a comprehensive empirical study using the data of 10 exhaustible resources over the last century and finds that (p. 1037) "it is very hard to detect any trend in the actual price levels of those resources and certainly not the kind of positive trend that is suggested by the above statement of Hotelling's rule." For surveys on the subjects, see, e.g., Devarajan and Fisher (1981), Krautkraemer (1998) and references therein. To give an illustration, the left panel of Figure 1 shows the historic prices of crude oil, one of the most important exhaustible resources, during the last 30 years. In addition to dramatic changes, the recent persistence of low oil price around \$50 per barrel has drawn a lot of attention around the world, as the low price level not only adds fuel to the debate about the political and economic policies related to the consumption of exhaustible resources, but also has big impacts on people's daily life; for example, the low price may encourage the use of fuel products, which could in turn accelerate the greenhouse gas levels.

Secondly, the deterministic model in Hotelling (1931) cannot effectively explain some stylized facts observed in futures markets of exhaustible recourses, such as (a) backwardation and contango

(Litzenberger and Rabinowitz (1995)); (b) Samuelson effect, i.e., the term structure of volatility decreases as maturity increases, c.f. Samuelson (1965); (c) and the volatility of futures prices being higher conditional on backwardation (Routledge, Seppi, and Spatt (2000)). See the right panel of Figure 1 for an illustration of crude oil futures prices across different maturities.

To address the first issue, many deterministic models are proposed as extensions of the Hotelling model, which include, e.g., a more realistic and complex form of extraction costs (Solow and Wan (1976), Levhari and Liviatan (1977)), exploration activities (Pindyck (1978b)), durability of resources (Levhari and Pindyck (1981)), market imperfections (Salant (1976), Pindyck (1978a)), set-up costs (Hartwick, Kemp, and Long (1986), Holland (2003)), and backstop technology (Heal (1976), Tahvonen (1997)). Tailored for oil production, Anderson et al. (2018) propose a new model by focusing on the physical pressure constraint to the existing oil wells in the production process. They show that this pressure constraint will lead to a low elasticity of production in existing wells to oil prices but a high elasticity of exploration (drilling) activities of new wells to oil prices. In terms of equilibrium oil prices, their model can generate a U-shaped price profile.

1.1 Our Contribution

This paper focuses on stochastic models for exhaustible resources, in order to study both spot and futures prices simultaneously. We develop a general equilibrium model to study both spot and futures markets of exhaustible resources in a comprehensive setting, including exploration, stochastic demand, stochastic reserve, and, in particular, production adjustment costs.

To the best of our knowledge, this is the first time these factors are studied simultaneously in the field of exhaustible resources. In contrast to the most existing literature, we highlight the importance of production adjustment cost, which is a fundamental friction in the process of resource extraction. The production adjustments here can be quite general, e.g., sudden or gradual changes of the instantaneous production level. Adjustment costs are important at several levels. First, at the unit or plant level, direct costs are usually incurred from adjusting production rate. For example, stopping the injection of steam into oil sand reservoirs would result in a long and expensive re-start. Second, at the firm level, production adjustment costs include set-up costs (such as costs of tunneling, drilling wells, building pipelines), capacity expansion investments, hiring and firing costs, etc. Third, at the industry level, entry and exit costs can also be viewed as production

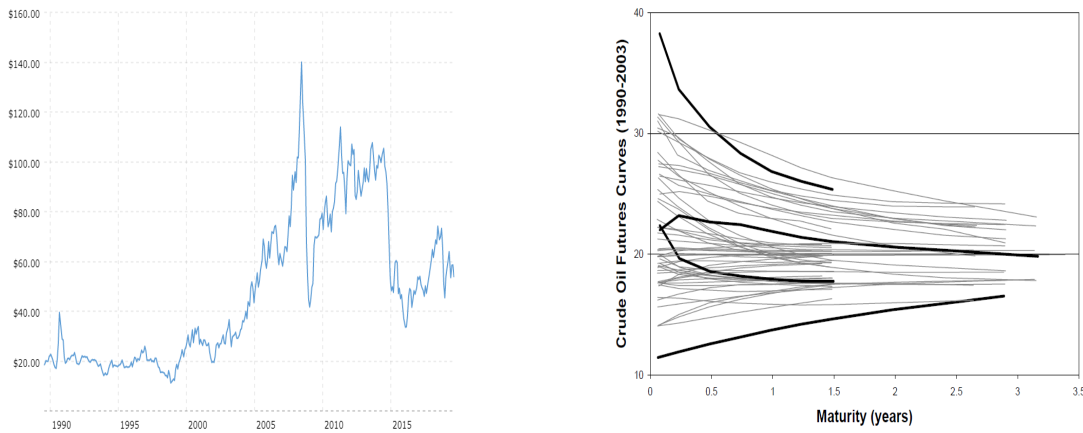

Figure 1. Historic data of spot prices and futures curves for the crude oil. *Left*: Oil prices for the last 30 years (Source: www.macrotrends.net). Historically the oil prices can have complicated dynamics, rather than going up exponentially. *Right*: Crude oil futures prices across different maturities (Source: Casassus et al. (2018)). Futures curves can be both in backwardation (going downward) and contango (going upward).

adjustment costs.

More precisely, the contribution of this paper is fourfold:

1. Theoretically, we incorporate production adjustment costs into both the Hotelling model and the model in Pindyck (1980), and show that in the presence of production adjustment costs, there exists a unique extraction path of an exhaustible resource in the socially optimal way, which can be reproduced by a competitive market equilibrium; see Theorems 1 and 2.

2. Consistent with the finding in Anderson et al. (2018) for oil, our model can generate low elasticity of production and high elasticity of exploration with respect to prices for general exhaustible resources (e.g., coal, natural gas, etc.); see Section 4.2.

3. In terms of spot prices and production policy, we find that the combination of exploration and production adjustments can significantly prolong the period of time that spot prices stay at the bottom, which, in turn, explains why prices of many exhaustible commodities can be quite low for a long time; see Section 4.3.

4. In terms of futures prices, we find that adjustment costs combined with uncertainty (e.g., stochastic demand and stochastic reserve) can generate interesting stylized facts such as (a) backwardation and contango, (b) Samuelson effect, and (c) higher volatility conditional on

Table 1. A Comparison of Related Literature

<table>
<tr><td></td><td colspan="6">Deterministic Model</td></tr>
<tr><td></td><td colspan="2">Price Profile</td><td colspan="2">Elasticity of Production</td><td colspan="2">Elasticity of Exploration</td></tr>
<tr><td>Hotelling (1931)</td><td colspan="2">Exponential Growth</td><td colspan="2">High</td><td colspan="2">N.A.</td></tr>
<tr><td>Anderson et al. (2018)</td><td colspan="2">U-shape</td><td colspan="2">Low</td><td colspan="2">High</td></tr>
<tr><td>This Paper</td><td colspan="2">U-shape</td><td colspan="2">Low</td><td colspan="2">High</td></tr>
<tr><td></td><td colspan="6">Stochastic Model with Demand Uncertainty</td></tr>
<tr><td></td><td>Reserve Uncertainty</td><td>Exploration Activity</td><td>Adjustment Costs</td><td>Volatility of Futures Prices</td><td>Elasticity of Production and Exploration</td><td>Persistence of Low Prices</td></tr>
<tr><td>Pindyck (1980)</td><td>✓</td><td>✓</td><td></td><td></td><td></td><td></td></tr>
<tr><td>Carlson, Khokher, and Titman (2007)</td><td></td><td></td><td>✓</td><td>✓</td><td></td><td></td></tr>
<tr><td>This Paper</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td></td><td colspan="6">Applications of Singular Control</td></tr>
<tr><td>Constantinides (1986)</td><td colspan="6">Transaction Costs</td></tr>
<tr><td>Abel and Eberly (1996)</td><td colspan="6">Reversible/irreversible Investments</td></tr>
<tr><td>Dai, Liu, Yang, and Zhong (2015)</td><td colspan="6">Capital Gain Taxes</td></tr>
<tr><td>Bolton, Chen, and Wang (2011)</td><td colspan="6">Optimal Corporate Finance Decisions</td></tr>
<tr><td>This Paper</td><td colspan="6">Exhaustible Resources</td></tr>
</table>

backwardation; see Section 4.4.

1.2 Literature Review

In terms of deterministic models, the model in Anderson et al. (2018) is related to our model because two models can generate the same stylized fact in terms of the elasticity of production and of exploration with respect to price shocks. More precisely, a low elasticity of production and a high elasticity of drilling to price shocks arise due to the presence of production constraint. The main difference is that their model emphasizes the role of physical pressure constraint in oil wells in the production process, while our model highlights the production adjustment costs in a general way. As a result, their model is carefully designed to fit the oil production in wells, but our model can be used in any kind of exhaustible resource such as coal, natural gas, etc.

Several stochastic models for exhaustible resources are closely related to this paper, as indicated in the upper panel of Table 1. Using a discrete model of stochastic demand Weinstein and Zeckhauser (1975) show that Hotelling's rule still holds for the expected price if the marginal extraction cost is constant but fails for a more general form of extraction cost, and that the optimal extraction path will deviate from the socially optimal one if producers are not risk neutral. Pindyck (1980) extends their result to a continuous-time model with stochastic demand, stochastic reserve, and exploration, and finds that exploration can generate a U-shaped price profile, indicating that the price can stay at the bottom for a certain time period. This paper extends the above two papers by considering stochastic demand, stochastic reserve, exploration, and production adjustment costs in a unified framework, in addition to studying futures prices and the volatility of futures prices.

There are very few studies focusing on exhaustible resources with production adjustment costs; the only paper that we know of is Carlson et al. (2007), where an equilibrium model is proposed with production adjustment costs depending on the historical average production. They show that adjustment costs naturally generate an inaction region in which production is perfectly inelastic, so that prices will have stickiness with respect to the changes in the stochastic demand. Consequently, the paper also demonstrates that including adjustment costs can generate a U-shaped price profile. In addition, they can explain many stylized facts about the futures prices (e.g., backwardation and contango) and the volatility of futures prices (e.g., Samuelson effect and higher volatility conditional on backwardation).

The present paper complements Carlson et al. (2007) in several ways. (a) We have exploration activities and stochastic reserve, as in Pindyck (1980). (b) Different from the adjustment costs depending on the historical average production, the adjustment costs here arise from instantaneous upward and downward adjustments. The motivation for this type of adjustment costs is twofold:

First, a large class of adjustment costs can naturally fit into this framework better than models based on the use of historic average production. For example, costs arising from sudden changes of regulation are more suitably modeled by instantaneous adjustment costs. Indeed, the Obama-era regulation, called the New Source Performance Standards, sets a limit of 1,400 pounds of CO_2 per megawatt hour of electricity produced, which is quite lower than the limit in most existing plants; as a result, producers must take actions and incur the correspondingly sudden costs to reduce the emissions of CO_2. We can parsimoniously model reversible investment, free disposal, and costly exit by setting the parameter of downward adjustment to be negative, zero and positive, respectively; in addition, an initial lump upward adjustment cost leads to the set-up cost in a parsimonious way.

Secondly, we gain analytical tractability because of the availability of the well-developed tools from singular control, which avoids the path-dependency caused by the use of historical average production in Carlson et al. (2007). In particular, a semi-closed form solution is available for the basic model presented in Section 2.

In terms of methodologies, singular control has been widely used in economics, such as transaction costs (e.g., Magill and Constantinides (1976), Constantinides (1986), Davis and Norman (1990), Shreve and Soner (1994), Liu and Loewenstein (2002), and Dai and Yi (2009)), reversible/irreversible investments (e.g., Abel and Eberly (1996), Guo and Pham (2005) and Merhi and Zervos (2007)), capital gains taxes (e.g., Tahar, Soner, and Touzi (2010), Dai et al. (2015), Cai, Chen, and Dai (2018)), and optimal decisions in corporate finance (e.g., Bolton et al. (2011)). The lower panel in Table 1 gives a comparison of various applications of singular control. To our knowledge, this is the first time that singular control is applied to the field of exhaustible resources.

Other stochastic models in the literature are also loosely connected with our paper. Arrow and Chang (1982) consider a model where new discovery follows a Poisson process with controllable intensity, and demonstrate that the price may follow a cyclic pattern. In addition to adding new reserves, Quyen (1991) also considers another function of exploration as learning the uncertainty of total reserve size. For both spot and futures prices, inventory models (e.g., Pindyck (2001) and Knittel and Pindyck (2016)), production models (e.g., Kogan, Livdan, and Yaron (2009); Casassus et al. (2018)) and macro-finance models (e.g., Hitzemann (2016); Ready (2018)) are proposed, however, under the assumption that resources are unlimited.

The rest of the paper is organized as follows. In Section 2 we begin with a basic model by adding production adjustment costs to the Hotelling model in a deterministic setting. The basic model is extended to include stochastic demand, stochastic reserve, and exploration activities in Section 3. We conduct an extensive numerical analysis in Section 4; in particular, all the stylized facts stated above can be easily generated by our model. Section 5 concludes. Some technique results and details of numerical method are documented in the appendix. All proofs are given in the online supplement.

2 Adding Adjustment Costs to the Hotelling Model

We first incorporate adjustment costs into the Hotelling model; the simple deterministic setting does provide a good start point to understand how production adjustment costs affect the classic theory of exhaustible resources. Extensions with stochastic demand, stochastic reserve and exploration activities will be studied in the next section.

2.1 A Social Planner's Problem

The basic settings are the same as those in Hotelling (1931): In a continuous time economy with infinite horizon, there is a resource that is depletable and nondurable. The quantity of total reserves of this exhaustible resource is known and finite. Extracted resources cannot be stored and must be consumed immediately. The demand side is characterized by the following inverse demand function with a general form

$$p_t = P(Q_t), \tag{1}$$

where $P(\cdot)$ is a univariate decreasing function, t is the time index, and $p_t \geqslant 0$ and $Q_t \geqslant 0$ are the price and the aggregate production rate at time t, respectively.

What is new here is that we introduce upward and downward production adjustment costs. More precisely, let I_t and D_t be the cumulative upward and downward adjustments of production rate up to time t, respectively, such that the aggregate production rate Q is governed by[1]

$$dQ_t = dI_t - dD_t, \quad Q_{0^-} = q. \tag{2}$$

Here, our production adjustments can be quite general, e.g., sudden or gradual changes of the instantaneous production level. In particular, (a) a sudden increase of production level may occur if new equipments are installed, which can be modeled by a jump as $I_t - I_{t^-} > 0$ at time t; (b) a gradual increase of production level due to the improvement of productivity may be modeled as $dI_t = i\,dt$ for some constant rate $i > 0$.

[1] Q_{t^-} represents the production rate just before time t. The same notation is used throughout this paper.

In terms of the aggregate production rate Q, the reserve level follows

$$dR_t = -Q_t dt, \quad R_0 = r. \tag{3}$$

There will be proportional adjustment costs $\eta_+ dI_t$ and $\eta_- dD_t$ at time t. The social planner's objective function is

$$S(\{I_t, D_t\}) := \int_0^\infty e^{-\beta t}\Big([U(Q_t) - C\, Q_t]dt - \eta_+ dI_t - \eta_- dD_t\Big),$$

where $U(\cdot)$ is a social utility function with the form $U(Q) := \int_0^Q P(q)dq$, $C \geqslant 0$ is the marginal extraction cost, and β is the discount rate.

The objective function in Hotelling (1931) is a special case of $S(\{I_t, D_t\})$ with $\eta_+ = \eta_- = 0$. Note that the integration in the objective function can include any jumps at time 0.

Assumption 1. The parameters of production adjustment cost satisfy $\eta_+ \geqslant 0$ and $\eta_+ + \eta_- \geqslant 0$.

The intuition behind this assumption is as follows. An upward adjustment is always costly, while a downward adjustment is likely beneficial; more precisely, $\eta_- > 0$ can be viewed as the case of costly dismantling and cleaning-up;[2] $\eta_- = 0$ could be thought of as the case of free disposals; and $\eta_- < 0$ could be interpreted as the case of partial refund, for example, selling equipments. The condition $\eta_- + \eta_+ \geqslant 0$ is a natural requirement such that a producer cannot benefit by simply increasing and decreasing production simultaneously.

In summary, for a given initial state $(r, q) \in [0, \infty)^2$, the socially optimal production problem can be described as the following singular control problem:

$$V(r, q) := \sup_{\{I_t, D_t\} \in \mathcal{A}(r,q)} S(\{I_t, D_t\}), \tag{4}$$

subject to the dynamics (2) and (3). Here $\mathcal{A}(r, q)$ is the set of all (r, q)-*admissible* production strategies such that: (a) I_t and D_t are nonnegative, nondecreasing, and are right continuous with

[2] In the same spirit of Dixit (1989) and Leahy (1993), the case that $\eta_+ > 0$ and $\eta_- > 0$ might also be viewed as entry and exit costs. More precisely, Dixit (1989) and Leahy (1993) model entry and exit as real options with exercise prices being the fixed entry and exit costs, and thereby characterizing entry and exit thresholds as two points. Here we can model entry and exit as options with exercise prices being the proportional upward and downward adjustment costs, respectively. However, the two thresholds are characterized by two optimal boundaries instead of two points, because we have an additional dimension due to the total resource being finite, see Figure 2 for an illustration.

left limit; (b) $Q_t \geqslant 0$; and (c) $R_t \geqslant 0$.

2.2 Optimal Production Policy

In this subsection we present the optimal production policy as shown in Figure 2, whose rigorous statement can be found in Theorem 3 in Appendix A.1.1.

Before describing the optimal policy, we present an assumption.

Assumption 2. The inverse demand function satisfies $\int_0^q P(x)dx < \infty$ for $q < \infty$, and $P(0) > C + \beta\eta_+$.

Note that Assumption 2 is necessary for the social planner's problem. The integrable condition for the demand function guarantees that the social surplus is well-defined. The other condition $P(0) > C + \beta\eta_+$ states that the "choke" price must be greater than the marginal extraction cost plus the cost from initial upward adjustments of production; if this condition dose not hold, it is never optimal, for example, to exploit the oil in the deep ocean with a huge cost.

Under Assumptions 1 and 2, there exist two boundaries that split the state space into three regions: upward adjustment region $\mathcal{I}$, no-adjustment region $\mathcal{N}$, and downward adjustment region $\mathcal{D}$. In addition, these two boundaries can be characterized by two smooth and monotonically increasing functions $Q_I(\cdot\,; \eta_+, \eta_-)$ and $Q_D(\cdot\,; \eta_-)$ such that the following simple structure holds:

$$\begin{cases} \mathcal{I} = \{(r,q) \in [0,\infty)^2 \,|\, r > 0,\, 0 \leqslant q \leqslant Q_I(r)\}, \\ \mathcal{N} = \{(r,q) \in [0,\infty)^2 \,|\, r > 0,\, Q_I(r) < q < Q_D(r)\}, \\ \mathcal{D} = \{(r,q) \in [0,\infty)^2 \,|\, r > 0,\, Q_D(r) \leqslant q < \infty\}. \end{cases} \tag{5}$$

More precisely, there are three scenarios for the optimal adjustment decisionpolicy consists of three cases (c.f. the left panel of Figure 2):

(a) Suppose the initial state is at point $A \in \mathcal{I}$. First, an upward adjustment of production rate is needed so that the post-adjustment state hits the optimal boundary of upward adjustment. Afterward, there are no adjustments in production rate and the current extraction rate remains unchanged (i.e., the reserve level is decreasing at constant rate) until the optimal boundary of

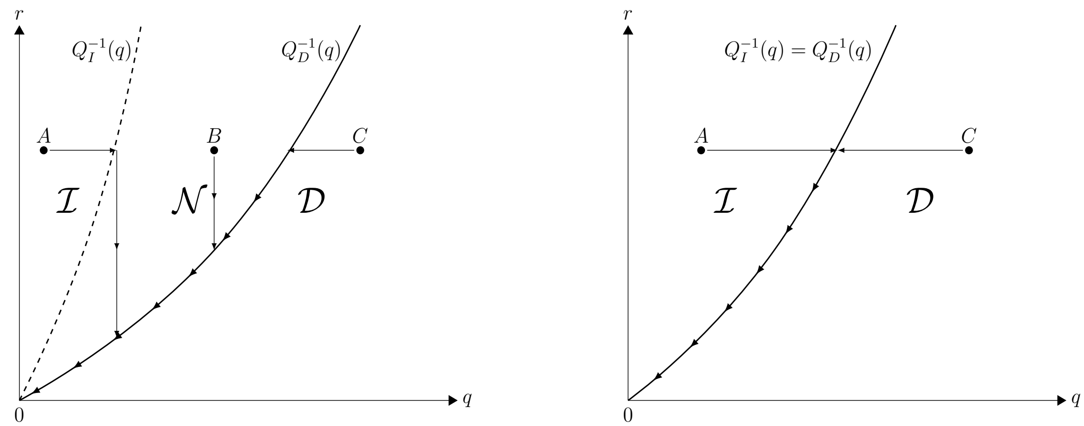

Figure 2. The optimal adjustment policy for two cases: $\eta_+ > 0$ and $\eta_- < C/\beta$; and $\eta_+ + \eta_- = 0$. *Left Panel*: For the case $\eta_+ > 0$ and $\eta_- < C/\beta$, the whole region (first quadrant) is divided by the two optimal adjustment boundaries (characterized by the two increasing functions Q_I^{-1} (dashed line) and Q_D^{-1} (solid line) respectively, where notation f^{-1} is the inverse of a function f) into three regions: upward-adjustment $\mathcal{I}$, no-adjustment $\mathcal{N}$, and downward-adjustment $\mathcal{D}$ from left to right in turn. The optimal adjustment policy of production rate consists of (a) a possible initial jump to the boundaries of no-adjustment regions; (b) constant production rate until the first time at which the state touches the optimal downward adjustment boundary; and (c) continuously decreasing production rate along the optimal downward adjustment boundary. *Right Panel*: For the case $\eta_+ + \eta_- = 0$, no-adjustment region does not exist as in Hotelling (1931) with the special case $\eta_\pm = 0$. Two optimal boundaries of upward and downward adjustment coincide, i.e., $Q_I \equiv Q_D$. Therefore, the optimal strategy consists of (a) a possible initial jump to the optimal rate; and (b) continuously decreasing production along the optimal boundary.

downward adjustment is touched. At last, the state simply moves along the optimal boundary of downward adjustment.

(b) Suppose the initial state is at point $B \in \mathcal{N}$. Then there are no adjustments in production rate and the current extraction rate remains unchanged until the optimal boundary of downward adjustment is touched. Afterward, the state simply moves along the optimal boundary of downward adjustment.

(c) Suppose the initial state is at point $C \in \mathcal{D}$. Then a downward adjustment of production rate is required so that the post-adjustment state hits the optimal boundary of downward adjustment. Afterward, the state simply moves along the optimal boundary of downward adjustment.

Note that, different from the case of no adjustment costs in Hotelling (1931), the optimal production policy here is characterized by three regions. Furthermore, the upward and downward adjustments are not symmetric, due to the fact that the total resource is finite. In Appendix A.1.1, we also study asymptotic behavior of the optimal boundaries, as the reserve level goes to infinity, and give some comparative statics. In particular, if $\eta_+ + \eta_- = 0$, i.e., investment being totally refundable which covers the special case Hotelling (1931), then the no-adjustment region does *not* exist and thus continuously adjusting is always optimal; see the right panel of Figure 2.

2.3 Competitive Equilibrium

Having described the socially optimal extraction path, it is natural to ask whether this path can be supported by competitive markets. Thanks to the assumption of constant proportional adjustment costs, we can prove that our economy is convex, and get the equilibrium price via the second welfare theorem;[3] alternatively, we can also prove the existence of competitive equilibrium price directly, armed with the semi-closed form solution (see online supplement §II).

Theorem 1 (Competitive Equilibrium with Adjustment Costs)**.** *Suppose that Assumptions 1 and 2 hold, that initially there is no production and the total quantity of reserves is $r > 0$, and that there are finite number of identical price-taking producers. Let $\{I_t^*, D_t^*\}_{t\geqslant 0}$ be the optimal production policy of the social planner's problem (4) stated in Section 2.2, and let $\{R_t^*, Q_t^*\}_{t\geq 0}$ be the solution of the system (2) and (3) under the optimal policy $\{I_t^*, D_t^*\}_{t\geqslant 0}$. Then there exists a competitive equilibrium which can be characterized by the reserve level $\{R_t^*\}_{t\geqslant 0}$ and production rate $\{Q_t^*\}_{t\geqslant 0}$ together with the price $\{p_t^*\}_{t\geqslant 0}$ satisfying $p_t^* = P(Q_t^*)$. More precisely,*

$$p_t^* = \begin{cases} P(Q_I(r)) & \text{for} \quad 0 \leqslant t \leqslant t^*, \\ [P(Q_I(r)) - C + \beta\eta_-]e^{\beta(t-t^*)} + (C - \beta\eta_-) & \text{for} \quad t > t^*, \end{cases}$$

where

$$t^* := \frac{r - Q_D^{-1}(Q_I(r))}{Q_I(r)}, \tag{6}$$

$Q_I(\cdot)$ *and* $Q_D(\cdot)$ *are functions given in Section 2.2, and* Q_D^{-1} *is the inverse function of* Q_D*.*

[3] Hartwick et al. (1986) show that in general an economy with set-up costs does not have a competitive equilibrium, due to the non-convexity of the economy. Here, by using a linear structure of adjustment costs, our model still retains the convexity of the economy. A simple way to understand the convexity is that we can view our model as an entry and exit model for firms with constant-to-scale production technology.

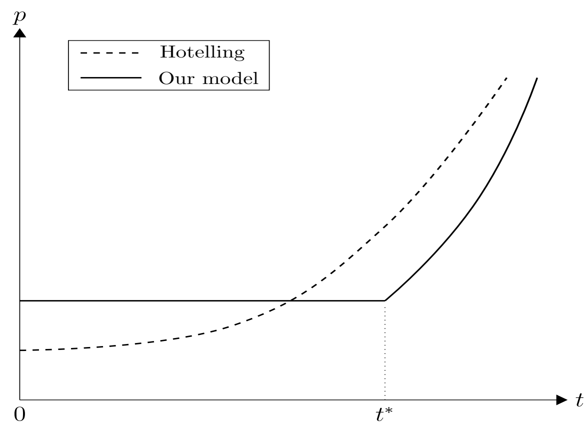

Figure 3. Competitive equilibrium prices with and without production adjustment costs. The dashed line, which is exponentially increasing at interest rate (no extraction cost), represents the classic Hotelling's price without adjustment costs. The solid line represents the competitive equilibrium price with costly downward adjustments (i.e., $\eta_- > 0$) in the basic model. There are two stages: (a) constant price in the period $[0, t^*]$, where t^* depends on both initial reserve level and adjustment cost parameters $\eta_\pm$; and (b) exponentially increase period, in which the growth rate depends on the parameter of downward adjustment cost, i.e., η_-.

Moreover, the price after t^* satisfies

$$dp_t^*/dt = \beta(p_t^* - C + \beta\eta_-), \qquad t > t^*. \tag{7}$$

In particular, if $\eta_+ + \eta_- = 0$, then the duration of constant price is zero, i.e., $t^* = 0$, and thereby the price dynamics (7) is satisfied for all $t > 0$.

Note that the growth rate of the equilibrium price in (7) depends on the sign of η_-. If $\eta_- = 0$, the growth rate coincides with that of no production adjustment costs. If $\eta_- > 0$ (< 0), then the growth rate is bigger (smaller) than that without production adjustment costs; see Figure 3 for an illustration of an equilibrium price p_t^* with $\eta_- > 0$. From the comparative statics results (A.12) in Appendix A.1.1, we can see that an increase of upward and/or downward adjustment costs will make the initial equilibrium price before t^* more expensive and the duration of this constant price longer.

We have seen that in the deterministic model only one no-adjustment period arises. However, we shall demonstrate that by adding uncertainty, there could be many no-adjustment periods in general.

3 Adding Adjustment Costs to Pindyck (1980)

In this section, we extend our basic model to a general one which incorporates demand uncertainty, reserve uncertainty, exploration activity, and adjustment costs. In other words, we will incorporate adjustment costs into the model in Pindyck (1980). To the best of our knowledge, we are the first to consider these four factors simultaneously in the economics of exhaustible resources. As we shall see, our general model not only easily generates many interesting phenomena studied in the literature, but also naturally gives rise to persistence of low prices with a U-shaped profile, which can explain why many commodities of exhaustible resources may stay low for a long time.

3.1 The Setting

To introduce the first three factors, i.e., demand uncertainty, reserve uncertainty and exploration, we follow the settings in Pindyck (1980). More precisely: (i) For stochastic demand side, we follow Pindyck (1980) by modifying the demand relation (1) as $p_t = X_t P(Q_t)$, where X_t is a diffusion process with

$$dX_t = \mu_x(X_t)dt + \sigma_x(X_t)dB_t^x, \quad X_0 = x. \tag{8}$$

Here $\mu_x(\cdot)$ and $\sigma_x(\cdot)$ are deterministic functions, and B^x is a standard Brownian motion in a filtrated probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$. (ii) For exploration activity, we follow Pindyck (1978b) and Pindyck (1980) by assuming that the cumulative discoveries at time t, Y_t, satisfy

$$dY_t = f(Y_t, W_t)dt, \quad Y_0 = y, \tag{9}$$

where $W_t \geqslant 0$ is the exploratory effort (a control variable) at time t, and $f(\cdot,\cdot)$ is the discovery rate with the property $\partial f/\partial Y < 0 < \partial f/\partial W$. That is, the more cumulative discoveries, the less new discoveries; and the more exploratory efforts, the more new discoveries. (iii) For stochastic reserve, following Pindyck (1980), we assume

$$dR_t = dY_t - Q_t dt + \sigma_r dB_t^r, \quad R_0 = r, \tag{10}$$

where Q_t is the aggregate production rate, σ_r is a positive constant, and B^r is another standard Brownian motion in the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$. Since there is no direct strong evidence showing the correlation between B^r and B^x, following Pindyck (1980), we assume that they are independent. In addition, we assume that the probability measure $\mathbb{P}$ is risk neutral; or equivalently, we explicitly assume that producers are all risk neutral.[4]

So far, the above settings are exactly those used in Pindyck (1980). Now, we add production adjustment costs in the same way as in the basic model. That is, there is a pair of production adjustment strategies $\{I_t, D_t\}_{t\geqslant 0}$ controlling the production rate $\{Q_t\}_{t\geqslant 0}$ via the dynamic equation (2).

3.2 A Social Planner's Problem

A social measure of use of an exhaustible resource is the discounted social surplus minus extraction costs, exploration costs and adjustment costs, which can be written as

$$\tilde{S}(\{I_t, D_t, W_t\}) = \int_0^\infty e^{-\beta t}\Big([\tilde{U}(Q_t, X_t) - C(R_t)Q_t - \tilde{C}(W_t)]dt - \eta_+ dI_t - \eta_- dD_t\Big),$$

where $\tilde{U}(q,x) = x\int_0^q P(z)dz$, $C(\cdot)$ is the marginal extraction cost depending on the current reserve level, $\tilde{C}(\cdot)$ is the exploration cost, and $\eta_\pm$ are parameters of adjustment costs satisfying Assumption 1 as in the basic model. Therefore, for any initial state $(r,q,x,y) \in [0,\infty)^4$, the social planner's problem is expressed as

$$\tilde{V}(r,q,x,y) := \sup_{\{I_t, D_t, W_t\}\in\tilde{\mathcal{A}}(r,q,x,y)} \mathbb{E}\Big[\tilde{S}(\{I_t, D_t, W_t\})\,\Big|\,(R_0, Q_{0-}, X_0, Y_0) = (r,q,x,y)\Big], \tag{11}$$

subject to the dynamics (2), (8), (9) and (10). Here $\tilde{\mathcal{A}}(r,q,x,y)$ is the set of all (r,q,x,y)-*admissible* production strategies defined as follows. A production strategy $\{I_t, D_t, W_t\}_{t\geqslant 0}$ is said (r,q,x,y)-admissible provided that: (a) I_t and D_t are adapted, nonnegative, nondecreasing, and right-continuous-left-limit processes; (b) production rate Q_t must be non-negative; (c) W_t is adapted and non-negative process; and (d) the reserve level must be non-negative.

[4]This assumption is important to the result stated in Theorem 2 below that the social optimal consumption path can be reproduced by a competitive equilibrium, since Pindyck (1980) shows that this result does not hold when producers are risk averse, in the same setting without the adjustment costs.

3.3 Optimal Production Policy

To solve the social planner's problem, we need some assumptions below.

Assumption 3. The marginal extraction cost function $C(\cdot)$ is decreasing and convex with respect to reserve level, i.e., $C' < 0 \leqslant C''$.

Assumption 3, which is assumed in Pindyck (1980), gives producers an incentive to do exploration activities; otherwise, Pindyck (1978b) shows that producers will postpone exploration until the current proved reserves are depleted in the absence of adjustment costs.[5]

Assumption 4. The exploration cost function $\tilde{C}(\cdot)$ is increasing and convex with respect to exploration effort W, i.e., $\tilde{C}' > 0$ and $\tilde{C}'' \geqslant 0$.

This is a quite standard assumption, as in Pindyck (1980), which helps to make the exploration effort rate be finite.

Assumption 5. The marginal discovery cost function $\tilde{C}'/(\partial f/\partial W)$ is increasing with respect to exploration effort W, given the level of cumulative discoveries.

Assumption 5 is used also in Pindyck (1980), which helps to uniquely determine the optimal exploration effort. Intuitively, it states that the growth rate of exploration cost is higher than that of new discovery, given the level of cumulative discoveries.

Assumption 6. The discovery rate function $f(\cdot,\cdot)$ is concave in both two arguments.

Assumption 6 is new, which can be interpreted as that new discoveries will become harder and harder when the proved discoveries increase. Moreover, this assumption, combined with the convexity of extraction cost and exploration cost, helps to guarantee the convexity of our economy and the concavity of the value function $\tilde{V}$ defined in (11).

Based on the above assumptions, solving the social optimization problem (11) is equivalent to solving a corresponding Hamilton-Jacobi-Bellman (HJB) equation (A.13) given in Appendix A.1.2. Moreover, we can rigorously characterize the optimal production policy. However, for the

[5] In practice, the extraction cost could be very complicated due to other factors such as development of technology, see Levhari and Liviatan (1977) for some discussion on a general form of extraction cost.

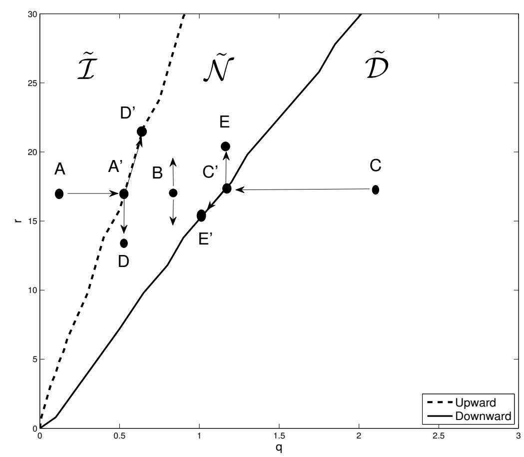

Figure 4. Numerical illustration of optimal boundaries. As in the deterministic case, the whole region is divided by the two optimal boundaries into three regions: upward-adjustment $\tilde{\mathcal{I}}$, no-adjustment $\tilde{\mathcal{N}}$, and downward-adjustment regions $\tilde{\mathcal{D}}$ from left to right in turn. The optimal boundaries are given in Q-R plane (by fixing demand level $X = 40$ and cumulative discoveries $Y = 1$). Parameters are summarized in Table 2.

purpose of both exposition and comparison, we shall use a numerical result to illustrate the optimal production policy below and summarize these results rigorously in Theorem 4 in Appendix A.1.2.

To be more precise, Figure 4 shows that the basic profile of optimal adjustment policy in the Q-R plane (by fixing demand level and cumulative discoveries) is retained as in Figure 2 for the deterministic model. There also exist two monotonically increasing boundaries, which divide the whole state space into three regions: upward adjustment region $\tilde{\mathcal{I}}$, downward adjustment region $\tilde{\mathcal{D}}$, and no-adjustment region $\tilde{\mathcal{N}}$. In Appendix A.3, we further display the profile of the optimal boundaries from different viewpoints, such as in Q-X plane and Q-Y plane by fixing other corresponding coordinates. In addition, some sensitive analysis is studied there.

Compared to the deterministic case, however, the optimal adjustment strategies are more complicated (c.f. Figure 4):

(a) Initially, the optimal adjustment policy is the same with that in the deterministic model. That is, an instantaneous adjustment may be required if and only if the state is outside of no-adjustment region $\tilde{\mathcal{N}}$; and if so, the post-adjustment state must be on the corresponding optimal boundary. For example, if the initial state is at point $B \in \tilde{\mathcal{N}}$, no adjustments are needed; if the initial state is at point $A \in \tilde{\mathcal{I}}$ ($C \in \tilde{\mathcal{D}}$), the post-adjustment state is on the optimal boundary of upward (downward) adjustment, i.e., $A \to A'$ ($C \to C'$).

(b) After the initial time, the adjustment policy differs significantly from that in the deterministic model because of uncertainty and exploration. At every instant of time, there are three scenarios: (i) if the state is in the no-adjustment region, then simply retain the current production rate, despite the reserve level may decrease or increase; see point $B \in \tilde{\mathcal{N}}$ in Figure 4; (ii) if the state is on the optimal boundary of upward adjustment (see point A'), then there are two subcases: The first one is that if the reserve level decreases, the current production rate is retained and the state moves to the inside of no-adjustment region (i.e., $A' \to D$); the second one is that if the reserve level increases, possibly due to new discoveries from exploration and/or some random perturbations, production increases along the optimal boundary of upward adjustment according to optimal upward adjustment policy (i.e., $A' \to D'$); and (iii) if the state is on the optimal boundary of downward adjustment (see point C'), similar to item (ii), two subcases are possible, e.g., $C' \to E$ or $C' \to E'$.

It is worth to point out that different from only one period of no-adjustment in the deterministic case, there are many periods of no-adjustment in the current stochastic model. Because random perturbations make the reserve level up and down *non-monotonically*, in contrast to monotonic decrease in the base model, the state could hit the optimal boundaries and then move back into the no-adjustment region many times. Consequently, there exist many no-adjustment periods.

3.4 Competitive Equilibrium

Analogy to Theorem 1, we can apply the second welfare theorem to obtain the following result, whose proof is given in online supplement §IV, thanks to the convexity of the economy.

Theorem 2 (Competitive Equilibrium with Uncertainty, Exploration, and Adjustment Costs)**.** *Suppose that Assumptions 1-6 hold. Let* $\{I_t^*, D_t^*, W_t^*\}_{t \geqslant 0}$ *be the optimal production policy of the social planner's problem* (11)*, and let* $\{R_t^*, Q_t^*, X_t^*, Y_t^*\}_{t \geqslant 0}$ *be the solution of the system* (2)*,* (8)*,* (9) *and* (10) *under the optimal strategy* $\{I_t^*, D_t^*, W_t^*\}_{t \geqslant 0}$*. Then, there exists a competitive equilibrium characterized by* $\{R_t^*, Q_t^*, X_t^*, Y_t^*\}_{t \geqslant 0}$ *together with the price process* $\{p_t^*\}_{t \geqslant 0}$ *satisfying* $p_t^* = X_t^* \, P(Q_t^*)$*.*

Theorem 2 helps us to conveniently calculate prices in a competitive equilibrium by directly solving a social planner's optimization problem. In addition, this can be done equivalently by solving the corresponding HJB equation (A.13) in Appendix A.1.2.

4 Numerical Analysis

Theoretically, we can use existing numerical methods to solve the social optimization problem (11) in a quite straightforward way. However, the current problem is four-dimensional and does not permit dimension reduction,[6] which makes this problem quite challenging. Here, we use penalty method (e.g., Dai and Zhong (2010)) to solve the HJB equation (A.13) associated with the value function $\tilde{V}$; the technical details of the numerical algorithm are given in Appendix B. In particular, we shall use our model to generate all the stylized facts presented earlier, see Figures 7, 8, and 9.

To be concrete, we fix the demand function with a constant elastic form which satisfies Assumption 2, i.e., $P(Q) = \gamma Q^{\gamma-1}$ with $\gamma \in (0, 1)$. For the demand factor, we consider temporary demand shocks since the mean-reverting property is commonly recognized in real markets, i.e.,

$$\frac{dX_t}{X_t} = \kappa(\mu_T - \ln X_t)dt + \sigma_T dB_t^x,$$

where $\kappa > 0$, $\mu_T \in \mathbb{R}$, and $\sigma_T > 0$ are all constants. For the exploration, we take $\tilde{C}(w) = p_1 w$ and $f(y, w) = (p_3 - p_4 y)w^{p_2}$ with $p_i > 0, i = 1, \ldots, 4$. The extraction cost is assumed to be the form of $C(r) = C_0/r$ with $C_0 > 0$. It is clear that Assumptions 3-6 are satisfied.

4.1 Parameters

We first take the parameters used in Carlson et al. (2007); see part one in Table 2. Second, we use a backwarded futures curve that is used in Carlson et al. (2007), denoted by F_T^b with maturity T (the initial state is fixed at $(R_0, X_0, Z_0) = (18, 45, 1.9)$), as a benchmark to calibrate the parameters in Pindyck (1980).[7] More precisely, the parameters in the second part of Table 2 are obtained by

[6]Sometimes, due to the homotheticity of the value function, there will be a dimension reduction, e.g., Dai et al. (2015) for a tax problem.

[7]The original initial state used in Carlson et al. (2007) is $(R_0, X_0, Z_0) = (18, 40, 1.8)$ as in Footnote 24 (p. 1678). To generate backwardation, we set $X_0 = 45$ a little higher than the long-run average 40, and pick the initial historic average production level $Z_0 = 1.9$ such that the initial state is in the no-adjustment region of their model.

Table 2. Base Parameter Values for Numerical Analysis. This table contains three sets of parameters used in this paper. Part one copies the parameters used originally in Carlson et al. (2007). Part two collects the parameters used in Pindyck (1980), whose values here are obtained by calibrating the futures curves to the corresponding benchmark generated by Carlson et al. (2007). Finally, part three gives the key parameters of adjustment cost and initial production rate, which are new in our model, as our baseline case.

(I) The original parameters in Carlson et al. (2007)

Parameter Name	Symbol	Value
Risk-free interest rate	β	0.05
CKT adjustment cost	δ	0.50
CKT weight of historic average	ϕ	1.00
Long-run average of temporary demand	μ_T	3.69
Rate of mean-reversion of temporary demand	κ	1.00
Volatility of temporary demand	σ_T	0.15
Elasticity of demand	γ	0.50
Extraction cost	C_0	5.00
Initial reserve level	R_0	18
Initial demand level	X_0	45
Initial average production rate	Z_0	1.9

(II) Parameters in Pindyck (1980) calibrated here to Carlson et al. (2007)

Parameter Name	Symbol	Value
Exploration parameter	p_1	1.00
Exploration parameter	p_2	0.50
Exploration parameter	p_3	50.0
Exploration parameter	p_4	25.0
Volatility of reserve	σ_r	0.05
Initial discovery level	Y_0	0

(III) New parameters in our model (the baseline case)

Parameter Name	Symbol	Value
Upward adjustment cost	η_+	0.10
Downward adjustment cost	η_-	0.10
Initial production rate	Q_{0-}	1.9

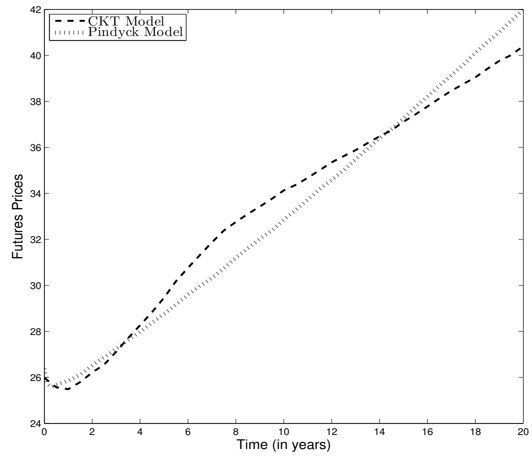

Figure 5. Parameter Calibration. Using backwarded futures curve generated by Carlson et al. (2007) as a benchmark (dashed line), we calibrate parameters in the model from Pindyck (1980) by matching a futures curve (dotted line) to the benchmark as closely as possible. All parameters are reported in Table 2.

minimizing the following relative error, with the initial state is fixed at $Y_0 = 0$:

$$||(F_T^m - F_T^b) \cdot \omega_T||/||F_T^b \cdot \omega_T||,$$

where the correponding futures curve generated from the model in Pindyck (1980) is denoted by F_T^m, and the weight w_T is defined as $\omega_T := (T - t_0, T - t_1, \cdots, T - t_N)/\sum_{i=0}^{N}(T - t_i)$ with $t_i = iT/N$, $i = 0, 1, \cdots, N$. The corresponding two futures curves are plotted in Figure 5. Finally, for the 3 new parameters in our model, we select $\eta_\pm = 0.1$ and initial production rate $Q_{0^-} = 1.9$ as our base case in the third part of Table 2, and will use a set of other values for robustness check.

4.2 A Low Elasticity of Production and A High Elasticity of Exploration

Anderson et al. (2018) show that production from existing oil wells is *insensitive* to demand shocks, while exploration (i.e. drilling activity) of new wells is *sensitive* to demand shocks. Their model is tailored for oil well production and focuses on oil well pressure constraint. Although our model is different from theirs, their observed stylized facts of the low elasticity of production and the high elasticity of drilling with respect to prices (demand shocks) can also be recovered by our model based on adjustment costs. In fact, because the drilling activity in their model can be

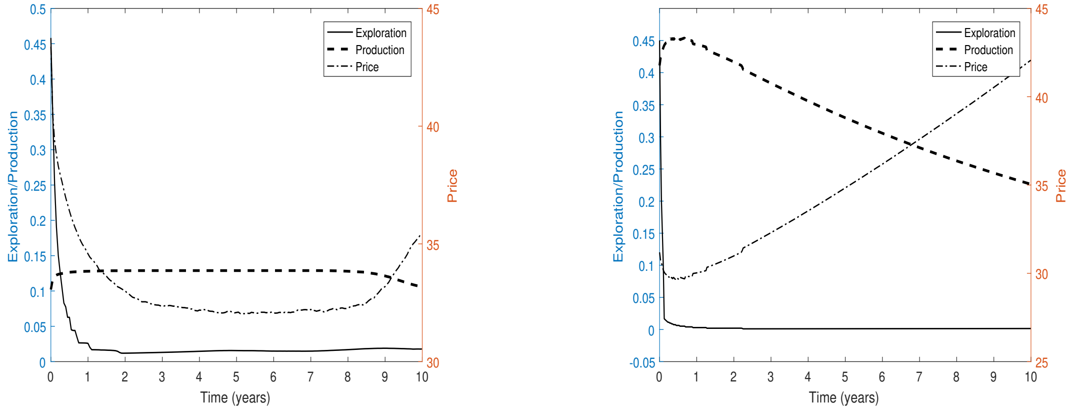

Figure 6. Elasticity of Production/Exploration. Our model (left panel) shows that production (dashed line) is not sensitive with respect to demand shocks but exploration (solid line) is sensitive to demand shocks, mainly due to adjustment costs related to production. This pattern is similar to that in Figure 7 (panel A) in Anderson et al. (2018). In contrast, it is difficult for the model in Pindyck (1980) (right panel) to generate low elasticity of production to price shocks, in the absence of adjustment costs. The initial state is fixed at $R_0 = 5, Q_0 = 0, X_0 = 45, Y_0 = 0$, and other parameters are taken from Table 2.

viewed as exploration activity in our model, the high elasticity of exploration comes from no direct constraint on exploration activity, and, in contrast, the low production elasticity comes naturally from adjustment costs. As an illustration, Figure 6 displays a similar pattern as shown in Figure 7 (panel A) in Anderson et al. (2018). Note that without adjustment cost it is difficult for a model, such as Pindyck (1980), to produce a similar pattern.

4.3 Spot Prices: U-shaped Profile, Persistence of Low Prices

4.3.1 U-shaped Profile. The exponentially growth profile of price predicted by the basic Hotelling model is not consistent with the observed data (e.g., Gaudet (2007)). In contrast, a U-shaped price profile is supported by many empirical studies (e.g., Slade (1982)).

4.3.2 Persistence of Low Prices. Moreover, spot prices of many commodities, such as crude oil, can be quite low for a long time. In this subsection, we show that our model including stochastic demand, stochastic reserve, exploration, and particularly adjustment costs, not only can generate a U-shaped price profile but also can lead to persistence of low spot prices, as shown in Figure 7.

It is not surprising to see that our model can generate the dynamics of U-shaped spot prices, as it is known that either adjustment costs (e.g., Carlson et al. (2007) with adjustment costs based

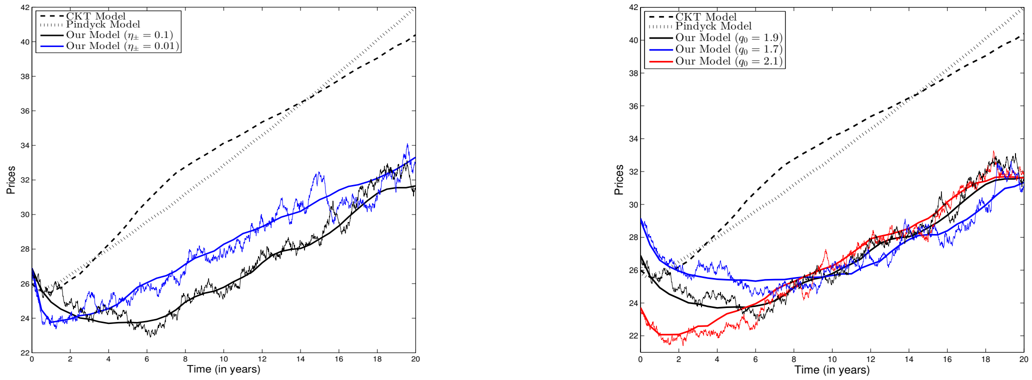

Figure 7. Persistence of Low Prices in our model. The expected spot prices under 3 models, Carlson et al. (2007) model, Pindyck (1980) model, and our model. The thin solid lines indicate sample paths under our model. This figure shows that although either adjustment costs with demand uncertainty (dashed line for the Carlson et al. (2007) model with initial state $(R_0, X_0, Z_0) = (18, 45, 1.9)$) or exploration alone (dotted line for the Pindyck (1980) model with initial state $(R_0, X_0, Y_0) = (18, 45, 0)$) may generate a U-shaped price profile, the combination of adjustment costs and exploration in our model (solid line for our model) can significantly prolong the period of time that the price stays at the bottom. *Left*: The case with different adjustment costs $\eta_{\pm}$ ($\eta_{\pm} = 0.1$, the black line, $\eta_{\pm} = 0.01$, the blue line), but with the baseline initial production rate Q_{0-}. *Right*: The case with different initial production rate Q_{0-} ($Q_{0-} = 1.9$, the black solid line, $Q_{0-} = 1.7$, the blue solid line, $Q_{0-} = 2.1$, the red solid line), but with the baseline adjustment costs $\eta_{\pm} = 0.1$. All other parameters are summarized in Table 2.

on historical averages) with stochastic demand or exploration (Pindyck (1980)) alone can generate a U-shaped spot price. What is more interesting is that in our setting the instantaneous upward and downward adjustment costs lead to a no-adjustment region, which means that the production levels tend to sticky; this stickiness, in combination with the possibility of new discovery of reserves due to exploration, leads to a much stronger persistence of a spot price staying at bottom, which implies that the spot price can be quite low for a long time starting from the beginning.

Our general model is also robust in terms of generating a U-shaped price dynamics, even if under some initial conditions that the U-shape disappears in either the Carlson et al. (2007) model or the Pindyck (1980)) model; see Figure 8 for illustration.

4.4 Futures Prices

We denote by $F(t;T)$ the futures price at date t for a unit of a commodity to be delivered at date $T > t$. Under the risk neutral measure, futures price is the conditional expectation of the

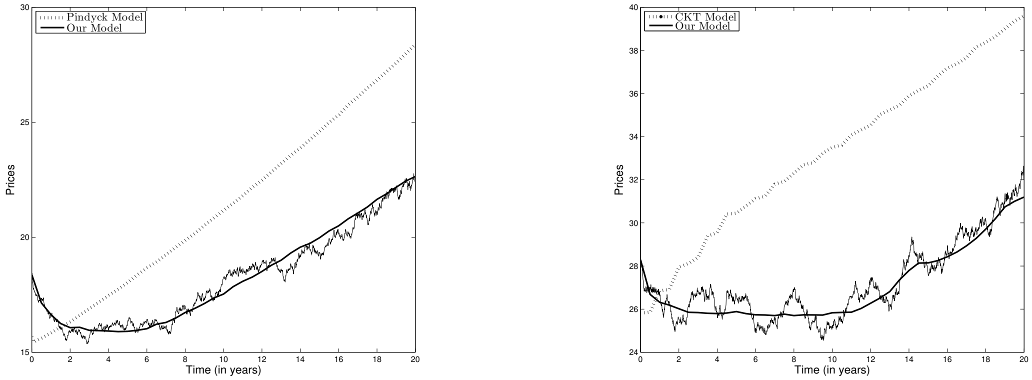

Figure 8. Robustness for U-shaped price dynamic in our model. This figure shows that for some initial conditions our model can still have the U-shape price dynamics, although the U-shape may disappear in either the Carlson et al. (2007) model or the Pindyck (1980)) model. *Left*: Disappearance of U-shape for Pindyck model. An initial high reserve level postpones the exploration in Pindyck model, hence a U-shaped price cannot appear in his model (dotted line). In contrast, our model can still give rise to a U-shaped price (solid line) due to the existence of no-adjustment region and a temporary high demand level. The initial states are fixed at $(R_0, X_0, Y_0) = (40, 45, 0)$, and $(R_0, Q_{0-}, X_0, Y_0) = (40, 3.0, 45, 0)$ for Pindyck model and our model, respectively. *Right*: Disappearance of U-shape for the CKT model. As demand level becomes steady, a U-shaped price cannot be generated by CKT model (dotted line), while our model can still produce a U-shaped price (solid line) due to exploration. The initial states are fixed at $(R_0, X_0, Z_0) = (18, 40, 1.8)$ and $(R_0, Q_{0-}, X_0, Y_0) = (18, 1.8, 40, 0)$ for the CKT model and our model, respectively. The thin solid line plots a sample path, and all other parameters are summarized in Table 2.

future spot prices, i.e., $F(t;T) := \mathbb{E}_t[p_T]$.[8] The standard term structure of volatility is defined as $\text{TV}(T-t) := \sqrt{\text{Var}_t(\log(p_T))/(T-t)}$. Using cross-sectional averages of simulated future spot prices, we obtain estimates of futures prices.

4.4.1 Backwardation and Contango. Our model can generate backwardation and contango naturally, as prices are just affected by small and temporary demand shocks in the no-adjustment region. More precisely, in that region, if the temporary demand is relatively high, the current price is also high. Then, as temporary shocks disappear, the futures prices return to the long-run average consequently. Hence, backwardation occurs in this scenario. Contango is just the opposite case. See the left panel of Figure 9 for illustration.

[8]Since interest rate is constant in our model, there is no difference between forward prices and futures prices.

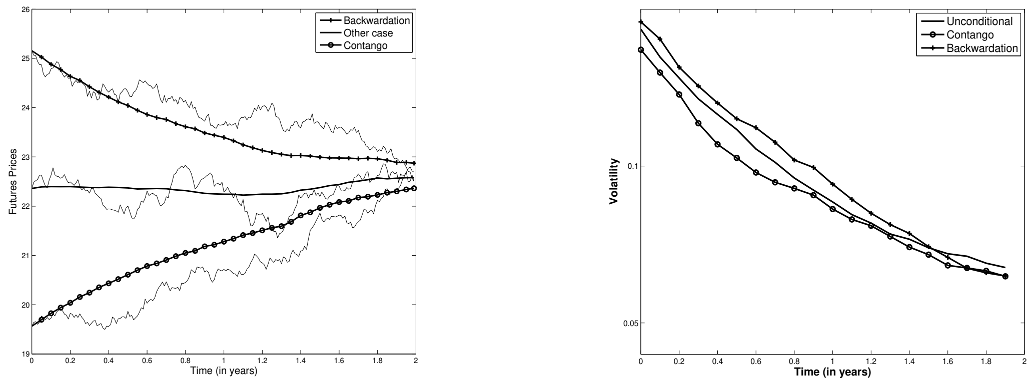

Figure 9. Term structure of futures prices and term structure of volatilities of futures in our model. *Left*: Our model can generate backwardation, contango, other patterns for the term structure of futures. *Right*: Our model can also generate the Samuelson effect (the term structure of volatility decreases as maturity increases) and higher volatility conditional on backwardation. The thin solid line plots a sample path. All parameters are summarized in Table 2 and the number of simulation runs is 2000.

4.4.2 Volatility. Empirically the term structure of futures price volatility typically declines as time to maturity increases, which is called "Samuelson Effect." In our adjustment cost model, this can be interpreted as a result of a mean-reverting process. More precisely, in the no adjustment region, price just reflects the behavior of demand side; particularly, when the demand is mean-reverting, so is price. Thus, the volatility of a futures contract will decrease across time, as the long run average is more certain than today's futures price due to the mean reverting, leading to the Samuelson effect.

The second stylized fact related to the volatility of futures prices, as reported in Litzenberger and Rabinowitz (1995), is that the volatility conditional on backwardation is usually higher than that conditional on contango (Routledge et al. (2000) and Kogan et al. (2009)). In our model, this phenomenon can be explained by the *asymmetry* of the optimal upward and downward adjustments. More precisely, when the optimal upward adjustment boundary is reached, the scarcity of the resource pushes the state into the no-adjustment region. In contrast, when the downward adjustment boundary is reached, the scarcity of the resource pushes the state downward along this boundary, see Figure 2 for illustration (the deterministic case).[9] This asymmetry makes, roughly speaking, the production adjustments less sensitive to shocks on the upward adjustment boundary

[9] A similar asymmetric adjustment property is also noted in Bloom (2009) but in a quite different context to study the impact of uncertainty (second-moment) shocks.

than on the downward adjustment boundary. Meanwhile, backwardation (contango) happens more likely when the state is on the upward (downward) adjustment boundary because the other adjustment boundary is remote. Other things being equal, production adjustments tend to mitigate demand shocks and in turn to reduce price volatility. Therefore, the fact that futures prices being less sensitive to shocks on upward adjustment boundary (backwardation) than on downward one (contango) results in a higher volatility conditional on backwardation. This is illustrated in the right panel of Figure 9.

5 Conclusion

We propose an equilibrium model of exhaustible resources in a comprehensive setting, including exploration, stochastic demand, stochastic reserve, and production adjustment costs. The adjustment costs include costs related to both sudden and gradual changes of production. In terms of spot prices, we find that the combination of exploration and adjustment costs can significantly prolong the period of time that a spot price stays at the bottom, which can explain why spot prices of many commodities may be quite low for a long time. In terms of futures prices, the model can explain some stylized facts observed in futures markets, such as backwardation, contango, Samuelson effect, and higher volatility conditional on backwardation. Despite that the current model is relatively stylized, it suggests that adjustment costs may play an important role to understand the complex dynamics of exhaustible resource markets.

Several possible extensions are left for future studies. This paper only treats competitive markets. The case of oligopoly may be an interesting topic to study later. Furthermore, although we have exploration in our model, more significant substitute technology (e.g., new energy vs. crude oil) might be another critical factor to understand exhaustible resource markets.

Appendix

A Some Additional Results

In this appendix, we give some additional results.

A.1 Social Optimality

We first give a characterization of the social planner's problem by its corresponding Hamilton-Jacobi-Bellman (HJB) equation. Then, a rigorous statement of the solution is presented.

A.1.1 The Deterministic Model. For the basic model, the value function V given in (4) is expected to solve

$$\max\left\{-q\frac{\partial V}{\partial q} - \beta V + U(q) - C\,q,\ \frac{\partial V}{\partial q} - \eta_+,\ -\frac{\partial V}{\partial q} - \eta_-\right\} = 0, \quad \text{for } r > 0,\ q \geqslant 0, \tag{A.1}$$

with the boundary condition $V(0,q) = -\eta_- q$, which comes from the fact that production must be stopped when a resource is used up. However, other conditions are unknown. In the literature of control theory this is known as the state-constrained problem.

In terms of the HJB equation (A.1), we can define upward, downward, no-adjustment regions, and two optimal boundaries as follows.

$$\mathcal{I} \quad := \quad \left\{(r,q) \in [0,\infty)^2 \;\middle|\; \frac{\partial V}{\partial q} = \eta_+\right\}, \tag{A.2}$$

$$\mathcal{D} \quad := \quad \left\{(r,q) \in [0,\infty)^2 \;\middle|\; -\frac{\partial V}{\partial q} = \eta_-\right\}, \tag{A.3}$$

$$\mathcal{N} \quad := \quad \left\{(r,q) \in [0,\infty)^2 \;\middle|\; -\eta_- < \frac{\partial V}{\partial q} < \eta_+\right\}, \tag{A.4}$$

$$\Gamma_I \quad := \quad \bar{\mathcal{I}} \cap \bar{\mathcal{N}}, \quad \Gamma_I \;:=\; \bar{\mathcal{D}} \cap \bar{\mathcal{N}}, \tag{A.5}$$

where the notation $\bar{A}$ represents the closure of the set A.

From the above definition, the optimal rule for an upward or a downward adjustment is to make the marginal benefit of an adjustment equal to its marginal cost. In the no-adjustment region $\mathcal{N}$, we have

$$-q\frac{\partial V(r,q)}{\partial r} - \beta V(r,q) + U(q) - C\,q = 0. \tag{A.6}$$

Equation (A.6) essentially states an average principle in contrast to a standard marginal principle in the absence of production adjustment costs. To see this, it is convenient to rewrite equation (A.6) as

$$\frac{\partial V(r,q)}{\partial r} + \beta\frac{V(r,q)}{q} = \frac{U(q)}{q} - C.$$

If we interpret $\beta V/q$ as average capital rent and $U(q)/q$ as average price, then (A.6) states that scarcity rent plus average capital rent must equal average net price. At any time, among these three policies (i.e., upward, downward, or no adjustments), the optimal one is to make the value function attain its maximum.

Before making a rigorous statement of solution to the social optimization problem (4), we introduce, for later use, q_I and q_D:

$$q_I := P^{-1}(C + \beta\eta_+), \quad q_D := \begin{cases} P^{-1}(C - \beta\eta_-) & \text{if} \quad \eta_- < C/\beta, \\ \infty & \text{if} \quad \eta_- \geqslant C/\beta, \end{cases} \tag{A.7}$$

where P^{-1} is the inverse function of P. Moreover, we define $R_D : [0, q_D) \mapsto [0, \infty)$ as the following integration:

$$R_D(q) := -\frac{1}{\beta}\int_0^q \frac{xU''(x)}{U'(x) + \beta\eta_- - C}dx, \tag{A.8}$$

and define $R_I : [0, q_I) \mapsto [0, \infty)$ implicitly as the unique solution of the following equation

$$e^{-\beta\frac{R_I(q)-R_D(q)}{q}}\Big(1 + \beta\frac{R_I(q) - R_D(q)}{q}\Big) = 1 - \frac{\beta(\eta_- + \eta_+)}{U'(q) - C + \beta\eta_-}. \tag{A.9}$$

The following theorem characterizes the optimal extraction strategy as shown in Figure 2, whose proof is given in §I in the online supplement.

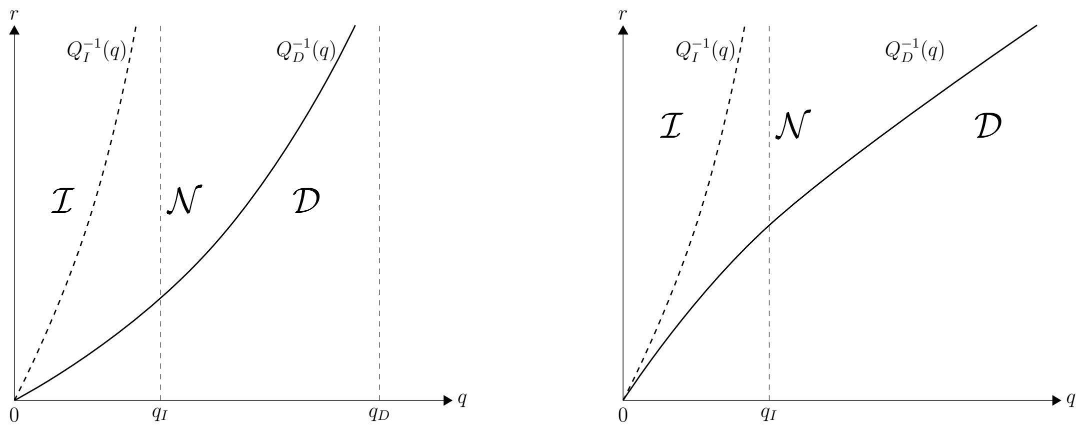

Figure 10. Asymptotic Behavior of Optimal Boundaries in the Basic Model. *Left*: When $\eta_- < C/\beta$, there are two finite limits for the optimal boundaries as reserve level goes to infinity. *Right*: When $\eta_- \geqslant C/\beta$, there is only one finite limit for the optimal upward boundary, the other limit goes to infinity as reserve level goes to infinity.

Theorem 3 (Social Optimality with Adjustment Costs)**.** *Let Assumptions 1 and 2 hold. There exits a unique socially optimal extraction strategy* $\{I_t^*, D_t^*\}_{t\geqslant 0}$ *given by*

$$I_t^* = [Q_0^* - Q_{0-}]^+ \mathbf{1}_{\{t\geqslant 0\}}, \quad D_t^* = [Q_{0-} - Q_0^*]^+ + [Q_0^* - Q_D(R_t^*)]^+, \tag{A.10}$$

where constant $Q_0^* = (Q_{0-} \vee Q_I(R_0)) \wedge Q_D(R_0)$, $\mathbf{1}_{\{\cdot\}}$ *is an indicator function, and* R_t^* *is the reserve level at time* t.[10] $Q_I(\cdot\,; \eta_+, \eta_-) : [0, \infty) \mapsto [0, q_I)$ *and* $Q_D(\cdot\,; \eta_-) : [0, \infty) \mapsto [0, q_D)$ *are respectively the inverse functions of* R_I *and* R_D *given by* (A.9) *and* (A.8)*, and satisfy, by suppressing parameters* $\eta_\pm$*, that* $Q_I(0) = Q_D(0) = 0$*, and that*

$$\lim_{r\to\infty} Q_I(r) = q_I, \quad \lim_{r\to\infty} Q_D(r) = q_D, \tag{A.11}$$

$$\frac{\partial Q_I}{\partial \eta_-} < 0, \quad \frac{\partial Q_D}{\partial \eta_-} > 0, \quad \frac{\partial Q_I}{\partial \eta_+} < 0, \quad \frac{\partial Q_D}{\partial \eta_+} = 0, \tag{A.12}$$

with q_I *and* q_D *given in* (A.7)*. In addition, the characterization* (5) *holds.*

Particularly, if $\eta_+ + \eta_- = 0$*, then two optimal boundaries coincide, i.e.,* $Q_I \equiv Q_D$*, and the no-adjustment region dose not exist, i.e.,* $\mathcal{N} = \{\emptyset\}$*.*

The optimal adjustment policy given in (A.10) is just a mathematical presentation of the optimal strategy stated in Section 2.2. Note that the optimal strategy $\{I_t^*\}_{t\geqslant 0}$ denotes the cumulative

[10] Here, $z^+ = \max\{z, 0\}$. $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$.

upward adjustments and has only two scenarios in our deterministic model: (a) there is an initial jump from zero to an optimal level say I_0, then $I_t^* = I_0$ for all $t \geqslant 0$; (b) there is no initial upward adjustment, so $I_t^* = 0$ forever. In contrast, downward adjustment strategy $\{Q_t^*\}_{t\geqslant 0}$ is more complicated, which includes adjustments at future time rather than only at initial time. As mentioned before, for the current deterministic case, the optimal policy consists of only three scenarios depending uniquely on the initial state (R_0, Q_{0^-}). If $(R_0, Q_{0^-}) = A \in \mathcal{I}$ in the left panel of Figure 2, that is, initially production rate is too low, then an upward adjustment is required and no further upward adjustment is needed. So, $I_0^* > 0$ and $I_t^* = I_0^*$ forever. Keep producing at the current rate until the critical time t^* such that the state is on the optimal downward adjustment boundary. During the period $[0, t^*]$, the optimal downward adjustment policy in (A.10) is that $D_t^* = 0$. After t^*, D_t^* increases as both production rate and reserve level decrease to zero. Points $B \in \mathcal{N}$ and $C \in \mathcal{D}$ in the left panel of Figure 2 represent the other two scenarios and can be explained similarly.

(A.11) presents the asymptotic behavior of the optimal adjustment policy as total reserves go to infinity. Even with infinite reserves, i.e., scarcity disappears, there still exists a no-adjustment region due to adjustment costs. (A.7) shows that the lower bound of this region is always bigger than zero and that its upper bound is finite if and only if downward adjustments are indeed beneficial, see Figure 10. Otherwise, there are no incentives to reduce production. (A.12) collects some comparative statics results, which indicate that the no-adjustment region becomes wider as adjustment costs increase and is consistent with the intuition. Specifically, $\partial Q_I/\partial\eta_- < 0$ ($\partial Q_D/\partial\eta_- > 0$) implies that, for a fixed reserve level, an increase of downward adjustment cost leads to a decrease (increase) of the optimal boundary of upward (downward) adjustment. Consequently, the no-adjustment region becomes wider as the downward adjustment cost increases. Similarly, $\partial Q_I/\partial\eta_+ < 0$ also indicates that an increase of upward adjustment cost will decrease the upward boundary. However, $\partial Q_D/\partial\eta_+ = 0$ states that the downward adjustment boundary is irrelevant to the upward adjustment cost parameter η_+. It may be somewhat surprising at the first glance. To understand this, recall the optimal adjustment policy that eventually results in a continuously downward adjusting part because of the scarcity of an exhaustible resource. In online supplement §I.6, we show that, during this stage, the current problem is equivalent to the classic Hotelling's problem with a modified marginal extraction cost $C - \beta\eta_-$. The optimal boundary can be directly derived from the Hotelling's solution which is independent of the upward adjustment parameter η_+. Finally, from (A.12), it is also easy to see that $(Q_I^{-1} - Q_D^{-1})/q$, i.e., the duration of no adjustment, is increasing with respect to adjustment costs. Therefore, in words, an increase of adjustment cost, *ceteris paribus*, will make the initial production more conservative and the duration of constant production longer.

A.1.2 The Stochastic Model. On the other hand, for the extended model, the value function $\tilde{V}$ given in (11) is also expected to solve

$$\max\Big\{\tilde{\mathcal{L}}\tilde{V} + \tilde{U}(q,x) - C(r)q + \max_{w\geqslant 0}\Big(f(w,y)\Big[\frac{\partial \tilde{V}}{\partial r} + \frac{\partial \tilde{V}}{\partial y}\Big] - \tilde{C}(w)\Big), \\ \frac{\partial \tilde{V}}{\partial q} - \eta_+,\ -\frac{\partial \tilde{V}}{\partial q} - \eta_-\Big\} = 0, \quad \text{for } (r,q,x,y) \in \Omega := [0,\infty)^4, \tag{A.13}$$

where

$$\tilde{\mathcal{L}}\tilde{V} := \frac{1}{2}\sigma_r^2\frac{\partial^2 \tilde{V}}{\partial r^2} + \frac{1}{2}\sigma_x^2(x)\frac{\partial^2 \tilde{V}}{\partial x^2} - q\frac{\partial \tilde{V}}{\partial r} + \mu_x(x)\frac{\partial \tilde{V}}{\partial x} - \beta\tilde{V}.$$

As in the basic model, boundary conditions are implicitly determined by the equation.

Similarly, we can define upward, downward and no-adjustment regions as follows.

$$\tilde{\mathcal{I}} := \Big\{(r,q,x,y)\,\Big|\,\frac{\partial \tilde{V}(r,q,x,y)}{\partial q} > \eta_+\Big\}, \tag{A.14}$$

$$\tilde{\mathcal{D}} := \Big\{(r,q,x,y)\,\Big|\,\frac{\partial \tilde{V}(r,q,x,y)}{\partial q} < -\eta_-\Big\}, \tag{A.15}$$

$$\tilde{\mathcal{N}} := \Big\{(r,q,x,y)\,\Big|\,-\eta_- < \frac{\partial \tilde{V}(r,q,x,y)}{\partial q} < \eta_+\Big\}. \tag{A.16}$$

For later use, we also introduce two critical optimal boundaries of upward and downward adjustment:

$$\tilde{\Gamma}_I := \overline{\tilde{\mathcal{I}}} \cap \overline{\tilde{\mathcal{N}}}, \qquad \tilde{\Gamma}_D := \overline{\tilde{\mathcal{D}}} \cap \overline{\tilde{\mathcal{N}}}. \tag{A.17}$$

For the optimal adjustment rule, we have a similar relation with equation (A.6). In addition, the optimal exploration effort must satisfy the following first order condition

$$\frac{\partial\tilde{V}(R_t,Q_t,X_t,Y_t)}{\partial r}+\frac{\partial\tilde{V}(R_t,Q_t,X_t,Y_t)}{\partial y}=\frac{\tilde{C}'(W_t)}{\partial f(Y_t,W_t)/\partial w}, \tag{A.18}$$

which implies, similar to Pindyck (1978b), that the marginal value of exploration effort must equal its marginal cost.

Now we have the following result for the social planner's problem (11), whose proof is presented in online supplement §III.

Theorem 4 (Social Optimality with Uncertainty, Exploration, and Adjustment Costs)**.** *Suppose that Assumptions 1-6 hold, and that there is a unique classic solution* φ *to the HJB equation* (A.13) *with two smooth optimal boundaries* $\tilde{\Gamma}_I$ *and* $\tilde{\Gamma}_D$ *defined by* (A.17)*. Furthermore, assume that for any admissible strategy, the transversality condition holds, i.e.,* $\lim_{t\to\infty} e^{-\beta t}\varphi(R_t,Q_t,X_t,Y_t)=0$*. Then the solution to the HJB equation* (A.13) *equals the value function defined in* (11)*, i.e.,* $\varphi=\tilde{V}$*, and there is a unique optimal policy* $\{I_t^*,D_t^*,W_t^*\}_{t\geqslant 0}$ *given by*

$$\begin{aligned} I_t^* &= \int_0^t \mathbf{1}_{\{(R_u^*,Q_u^*,X_u^*,Y_u^*)\in\tilde{\Gamma}_I\}} dI_u^*, && \text{(A.19)}\\ D_t^* &= \int_0^t \mathbf{1}_{\{(R_u^*,Q_u^*,X_u^*,Y_u^*)\in\tilde{\Gamma}_D\}} dD_u^*, && \text{(A.20)}\\ W_t^* &= F\Big(\frac{\partial\tilde{V}(R_t^*,Q_t^*,X_t^*,Y_t^*)}{\partial r}+\frac{\partial\tilde{V}(R_t^*,Q_t^*,X_t^*,Y_t^*)}{\partial y}\Big), && \text{(A.21)}\end{aligned}$$

where $\mathbf{1}_{\{\cdot\}}$ *is an indicator function,* $F(\cdot)$ *is the inverse function of marginal cost of discovery* $\tilde{C}'/(\partial f/\partial w)$*, and* $(R_t^*,Q_t^*,X_t^*,Y_t^*)$ *is the state at time* t*.*

In light of the optimal adjustment strategies, we can see that except for a possible initial jump to the optimal boundary, upward (downward) production adjustments are required if and only if the state touches the optimal upward (downward) adjustment boundary. Due to the introduction of Brownian motions and constant proportional adjustment costs, the optimal adjustment magnitude is set in a marginal way so as to minimize the adjustment costs. Mathematically, these processes given in (A.19) and (A.20) are called local time. In particular, in contrast to the deterministic case, continuously adjustment is suboptimal in current stochastic case. The optimal exploration effort (A.21) comes from the first order condition (A.18). Thanks to the Assumption 5, the optimal exploration effort rate (A.21) is uniquely determined by this first order condition.

A.2 Hotelling's Rule for Shadow Price

As the Hotelling's rule is a fundamental result in the classic theory of exhaustible resources, it is natural to ask whether it still holds in the presence of production adjustment costs or not. Although the Hotelling's rule may not hold for the spot prices, due to adjustment cost and stochastic features, we shall show that the Hotelling's rule still holds for the (properly defined) shadow prices, if we interpret it properly.

A.2.1 Deterministic Model. Recall that the optimal adjustment policy described in Section 2.2 consists of no-adjustment and continuously adjusting parts. We begin with the no-adjustment region. We define the shadow price of an exhaustible resource by $\lambda_t := \frac{\partial V}{\partial r}(R_t^*, Q_t^*)$, where R_t^* and Q_t^* are under optimal adjustment policy. Noting that Q_t^* is constant in $\mathcal{N}$, differentiating equation (A.6) with respect to r, and using dynamics equation (3), we then have

$$\frac{d\lambda_t}{dt} = \frac{dV_r(R_t^*, Q_t^*)}{dt} = V_{rr}(R_t^*, Q_t^*)\frac{dR_t}{dt} = \beta V_r(R_t^*, Q_t^*) = \beta\lambda_t, \tag{A.22}$$

where V_r and V_{rr} are partial derivatives. This implies that the Hotelling's rule still holds in the no adjustment region for the so defined shadow price.

Next, in the continuously adjusting stage, the state is required to stay at the optimal boundary of downward adjustment all the time. In online supplement §I.6, we have shown that, in this continuously adjusting stage, the current problem is equivalent to the classic Hotelling's problem with the modified marginal extraction cost $C - \beta\eta_-$. Hence, the Hotelling's rule must also hold in this stage. In fact, by differentiating the equation (A.6) with respect to r, the equation $q\partial^2 V/\partial r^2 + \beta\partial V/\partial r = 0$ holds on the optimal downward adjustment boundary due to the high contact condition discussed in online supplement §I.6. Thus, by using this high contact condition again, i.e., $\partial^2 V/(\partial r \partial q) = 0$, the above calculation in (A.22) works as well; in this two-dimensional problem, the high contact condition implies that the value function is twice continuously differentiable on downward adjustment boundary. Therefore, the Hotelling's rule also holds in the presence of production adjustment costs for the shadow price. Thus, we have the following:

Proposition 1 (Hotelling's Rule for Shadow Price with Adjustment Costs)**.** *Assume that* $\{R_t^*, Q_t^*\}_{t\geqslant 0}$ *are reserve level and production rate under the optimal strategy* $\{I_t^*, D_t^*\}_{t\geqslant 0}$*. Defining the shadow price of the resource as* $\lambda_t := \frac{\partial V}{\partial R}(R_t^*, Q_t^*)$*, where* V *is the value function define by* (4)*, then the shadow price grows at the interest rate, i.e.,* $d\lambda_t/dt = \beta\lambda_t$*.*

A.2.2 Stochastic Model. Now let us turn to the general case. Since we have introduced randomness and exploration activity simultaneously, the regularity of both the value function and the optimal boundaries are not clear so far, and its analysis may need advanced mathematical techniques and is beyond the scope of this paper. However, under the assumption that both the value function and the optimal boundaries are sufficiently smooth, we can discuss some implications of the general model.

From Theorem 4, we know that the optimal adjustment strategies are continuous except for a possible initial jump, which is similar to the deterministic model. However, continuously adjusting along downward adjustment region is no longer optimal. Instead, the adjustment is made in a marginal way by reflecting the post-state into the no-adjustment region whenever the state touches one of the optimal boundaries. Overall, the state will stay in the region consisting of no-adjustment region and optimal upward and downward adjustment boundaries. In addition, marginal upward adjustment is possible in this stochastic setting. Therefore, we split the analysis into two cases: no adjustment, and marginal adjustment.

First of all, let us consider the case of no adjustment. Using the similar argument for the basic model, we have

$$\mathbb{E}d\tilde{V}_r(R_t^*, Q_t^*, X_t^*, Y_t^*) = \Big[\beta\tilde{V}_r(R_t^*, Q_t^*, X_t^*, Y_t^*) + C'(R_t^*)Q_t^*\Big]dt, \tag{A.23}$$

where $R_t^*, Q_t^*, X_t^*, Y_t^*$ are under the optimal strategy. Therefore, if $C' = 0$, then the stochastic version of the Hotelling's rule for the shadow price still holds. Similarly, we can derive that

$$\mathbb{E}d\tilde{V}_y(R_t^*, Q_t^*, X_t^*, Y_t^*) = \Big[\beta\tilde{V}_y(R_t^*, Q_t^*, X_t^*, Y_t^*) - \tilde{C}'(W_t^*)\frac{f_y}{f_w}(Y_t^*, W_t^*)\Big]dt, \tag{A.24}$$

where we have used the first order condition (A.18). Consequently,

$$\mathbb{E}d\Big(\tilde{V}_r+\tilde{V}_y\Big)=\Big[\beta\Big(\tilde{V}_r+\tilde{V}_y\Big)+C'(R_t^*)Q_t^*-\tilde{C}'(W_t^*)\frac{f_y}{f_w}(Y_t^*,W_t^*)\Big]dt. \tag{A.25}$$

Now let us consider the case of marginal adjustments. As in the base model, the regularity of both value function and optimal boundaries indicates that the high contact conditions hold on both two optimal boundaries. Armed with these conditions, a straightforward calculation shows that the above results (A.23)-(A.25) hold as well. Therefore, we have

Proposition 2 (Extended Hotelling's Rule for Shadow Price). *Let $\{R_t^*,Q_t^*,X_t^*,Y_t^*\}_{t\geqslant 0}$ be the solution of the system (2),(8),(9) and (10) under the optimal strategy $\{I_t^*,D_t^*,W_t^*\}_{t\geqslant 0}$ given in Theorem 4. Define the shadow prices of the resource and of the exploration opportunity by $\lambda_t^r=\frac{\partial\tilde{V}}{\partial R}(R_t^*,Q_t^*,X_t^*,Y_t^*)$ and $\lambda_t^y=\frac{\partial\tilde{V}}{\partial Y}(R_t^*,Q_t^*,X_t^*,Y_t^*)$ respectively. If both the marginal extraction cost and exploration cost are constant, then these two shadow prices both grow at interest rate, i.e., $d\lambda_t^i/dt=\beta\lambda_t^i$ for $i=\{r,y\}$.*

These results are similar to that studied in Pindyck (1978b, 1980) with no adjustment costs. The intuition is as follows. As we discussed in the basic model, adjustment costs have no impacts on the scarcity of the resource, so does in the current case. Different from the basic model, exploration opportunity is another exhaustible resource. Thus, the Hotelling's rule for shadow price may still hold, if the marginal "extraction" cost is constant.

A.3 Further Analysis of Optimal Boundaries for the Stochastic Model

Since that the optimal extraction strategies heavily depend on the optimal boundaries of upward and downward adjustments, in this subsection, we will conduct an extensive numeric analysis about these optimal boundaries.

A.3.1 Basic Profile. Note that the optimal boundaries are indeed hyperplanes in the current four-dimensional case. In addition to the basic profile illustrated in Section 3.3 in Q-R plane by fixing demand level and cumulative discoveries, we further provide some descriptions from other angles in Figures 11.

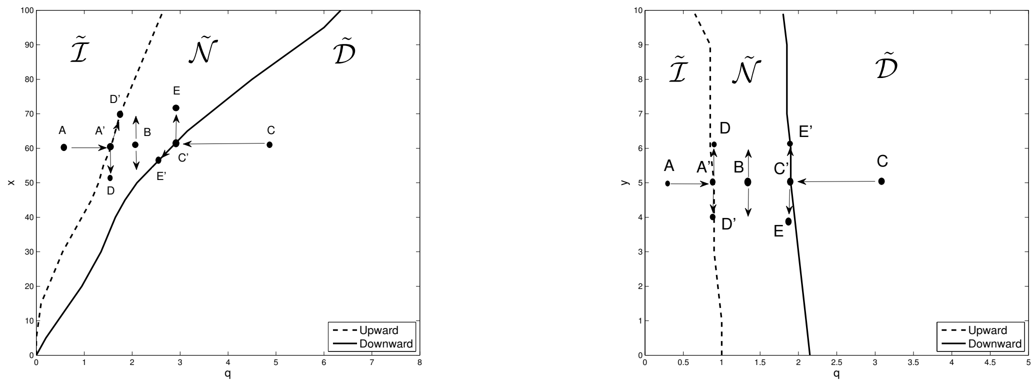

Figure 11. Numerical analysis of optimal boundaries in Q-X plane and in Q-Y plane. *Left*: The whole region is divided by the two optimal boundaries in Q-X plane (by fixing $R = 20$, $Y = 1$) into three regions: upward-adjustment $\tilde{\mathcal{I}}$, no-adjustment $\tilde{\mathcal{N}}$, and downward-adjustment $\tilde{\mathcal{D}}$ regions from left to right in turn. Parameters are summarized in Table 2. Due to the mean-reversion of demand, the no-adjustment region is relatively narrow in the long-run average demand level ($X_{\text{mean}} = 40$). Moreover, the optimal adjustment policy consists of (a) a possible initial adjustment which is same with that in the base case; (b) three adjustment strategies according to the state being in the no-adjustment region, on the upward boundary, and on the downward boundary, respectively. *Right*: The optimal boundaries in Q-Y plane (by fixing $X = 40$, $R = 40$). The whole region is still divided by the two optimal boundaries into three regions. However, two optimal boundaries are decreasing with respect to cumulative discoveries instead of increasing.

Left panel of Figure 11 describes the optimal boundaries in Q-X plane by fixing reserve level and cumulative discoveries ($R = 20$, $Y = 1$). Due to the mean-reverting assumption, the no-adjustment region is relatively narrow at the long-run average of demand (about 40 in the base case). For a temporary demand shock, the high level of demand will go back to its long-run average in the future, so it is not optimal to increase the production according to a transitorily high demand.

Moreover, from the angle of Q-X plane, the optimal adjustment policy is very similar to that in Q-R plane in Section 3.3. That is, the initial adjustment consists of a jump such that the post-adjustment state is on the optimal boundary if and only if the initial state is outside of the no-adjustment region, see $\tilde{\mathcal{I}} \ni A \to A'$ and $\tilde{\mathcal{D}} \ni C \to C'$. After that, at any instant of time, there are three scenarios: (i) if the state is in the no-adjustment region, then simply retain the current production rate despite the demand level may decrease or increase, see point $B \in \tilde{\mathcal{N}}$; (ii) if the state is on the optimal boundary of upward adjustment (see point A'), then there are two subcases. The first one is that if demand level decreases, the current production rate is retained therefore no adjustments are needed and the state moves to the inside of no-adjustment region (i.e., $A' \to D$); the other is that if demand level increases, production increases along the optimal boundary of upward adjustment due to optimal upward adjustment policy (i.e., $A' \to D'$); and (iii) if the state is on the optimal boundary of downward adjustment (see point C'), similar to item (ii), two subcases are possible. The first one is that if demand level increases, the current production rate is retained therefore no adjustments are needed (i.e., $C' \to E$); the other is that if demand level decreases, production decreases along the optimal boundary of downward adjustment due to optimal downward adjustment policy (i.e., $C' \to E'$).

In contrast, the right panel of Figure 11 illustrates the impact of exploration on the optimal adjustment boundary by fixing both reserve and demand levels ($R = 40$, $X = 40$). Except for the same property that the space is still divided into three regions by two optimal boundaries, the main difference is that two optimal boundaries are decreasing with respect to cumulative discoveries instead of increasing both in Q-R plane and Q-X plane. The interpretation is as follows. When cumulative discoveries are small, it is easy to find new discovery from exploration activity. Therefore, producers should take these potential new reserves into account for their extraction plan. As a result, the more potential reserves due to current low cumulative discoveries encourage producers to extract more. In contrast, less potential reserves due to current high cumulative discoveries make producers extract more conservative.

Finally, regarding the optimal adjustment policy projected into Q-Y plane, the strategy is the same as in Q-X plane. That is, except for an initial jump (if needed) to one of the two optimal boundaries, afterward adjustments happen if and only if the state is on one of the two optimal boundaries and, more importantly, the magnitude of adjustment is set in a marginal way along the optimal boundary. Therefore, in short, the optimal adjustment policy is to make the state in the no-adjustment region *plus* two optimal boundaries all the time.

A.3.2 Sensitive Analysis. The sensitive result to demand volatility is presented in the left panel of Figure 12. An increase of volatility from 15% to 20% will push the optimal boundary of upward and downward to left and right respectively but in a non-significant way. Consequently, the no-adjustment region becomes a little wider. The right panel reports the sensitive result to the interest rate. An increase of interest rate from 5% to 8% will push the two optimal boundaries to right simultaneously. As a result, the no-adjustment region shifts to right. This is consistent with

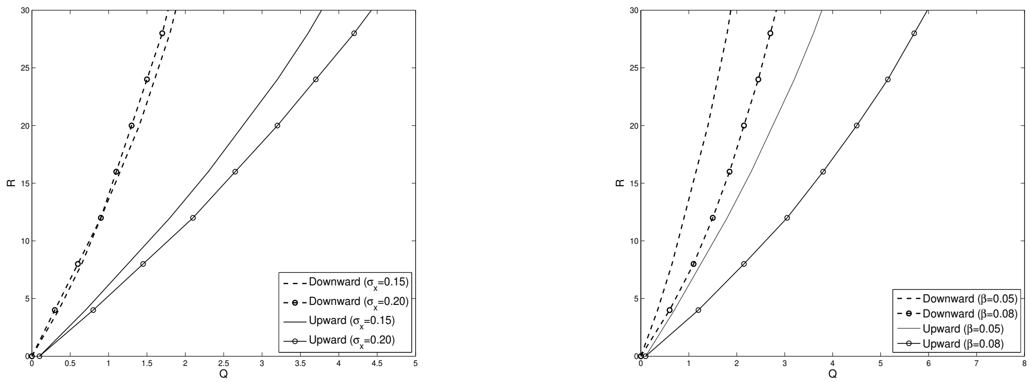

Figure 12. Sensitive analysis of optimal boundaries. The optimal boundaries are much more sensitive with respect to interest rate than to demand volatility. Parameters are summarized in Table 2. *Left*: the sensitivity of optimal boundaries to the demand volatility ($\sigma_{x,\text{low}} = 15\%$ and $\sigma_{x,\text{high}} = 20\%$). *Right*: the sensitivity of optimal boundaries to the interest rate ($\beta_{\text{low}} = 5\%$ and $\beta_{\text{high}} = 8\%$).

Hotelling's model. It is clear if we consider a two-period model. In the last period, all reserves will be produced. If interest rate increases, the value of holding reserves for another period falls. Therefore, more people choose to produce the resource in the first period.

B Numerical Method

In this appendix, we use penalty method to numerically solve social planner's problem, i.e., the HJB equation (A.13). Since the current problem is four-dimensional, we need some simplifications and reasonable boundary conditions.

B.1 Approximation

First of all, under the optimal exploratory effort W_t^*, we have

$$\max\{f^*\tilde{V}_y + \mathscr{L}\tilde{V} + \tilde{U}(x,q) - C(r)q - \tilde{C}(w^*),\ \tilde{V}_q - \eta_+,\ -\tilde{V}_q - \eta_-\} = 0, \tag{B.1}$$

where $f^* = f(y, w^*)$ and

$$\mathscr{L}\tilde{V} = \frac{1}{2}\sigma_r^2\tilde{V}_{rr} + \frac{1}{2}\sigma_x^2 x^2\tilde{V}_{xx} + (f^* - q)\tilde{V}_r + \kappa(\mu_T - \ln x)x\tilde{V}_x - \beta\tilde{V}.$$

Since $f^* > 0$, the equation (B.1) can be viewed as a backward parabolic equation where y can be regarded as a time variable. Note that $f^* \searrow 0$ as $y \to y_{max}$, therefore, a reasonable terminal condition is $\tilde{V}(r, q, x, y_{max}) = u(r, q, x)$, where u is the solution to the problem with stochastic demand and reserve only:

$$\max\{\tilde{\mathscr{L}}u + \tilde{U}(x, q) - C(r)q,\ u_q - \eta_+,\ -u_q - \eta_-\} = 0, \tag{B.2}$$

where

$$\tilde{\mathscr{L}}u = \frac{1}{2}\sigma_r^2 u_{rr} + \frac{1}{2}\sigma_x^2 x^2 u_{xx} - q u_r + \kappa(\mu_T - \ln x) x u_x - \beta u.$$

Next, note that both problems (B.1) and (B.2) are variational inequalities with derivatives constraints. To solve this type variational inequality, we use the following approximation problem

$$f^* V_y^a + \mathscr{L} V^a + \tilde{U}(x, q) - C(r)q - \tilde{C}(w^*) + PN \times (V_q^a - \eta_+)^+ + PN \times (-V_q^a - \eta_-)^+ = 0,$$

for (B.1), where $PN > 0$ is a sufficiently large constant and

$$\tilde{\mathscr{L}}u^a + \tilde{U}(x, q) - C(r)q + PN \times (V_q^a - \eta_+)^+ + PN \times (-V_q^a - \eta_-)^+ = 0,$$

for (B.2), respectively.

Now let us turn to the boundary conditions.

B.2 Boundary Conditions

First, for the problem (B.2), for a sufficiently large $q = q_{max}$, decreasing the production rate is optimal. This intuition suggests a boundary condition for large q, i.e., $u_q^a(r, q_{max}, x) = -\eta_-$. However, in contrast, the boundary condition for $q = 0$ is determined by the equation itself instead of being given a priori because it may be optimal to stop production. So, on this boundary, we use the equation.

For x direction, since no direct information can be used, we set these two boundary conditions as second derivatives being zero, i.e., $u_{xx}^a(r, q, x_{max/min}) = 0$.

Finally, for r direction, when reserve level is sufficiently large, the marginal value of resource is small. Thus, we have $u^a_{rr}(r_{max}, q, x) = 0$. When reserves are exhausted, production must be stopped, therefore $u^a(0, q, x) = -\eta_- q$.

Now for the problem (B.1), similarly we have the following boundaries conditions:

$$V^a_q(r, q_{max}, x, y) = -\eta_-, \quad V^a_{xx}(r, q, x_{max/min}, y) = 0, \quad V^a_{rr}(r_{max}, q, x, y) = 0,$$

and the equation satisfies on the boundary $q = 0$. However, the boundary condition on $r = 0$ is different. Here, since exploration is perfectly anticipated, production may be maintained as a certain level, which is implicitly determined by the equation. Hence, on the boundary $r = 0$, we still use the equation.

B.3 Convergence

With the above boundary conditions, one can show that the solution of the above approximation problem converges to the solution of the original problem; see e.g., Dai and Zhong (2010).

B.4 Algorithm to Implement the Approximation

Having been specified the approximate problems and corresponding boundary conditions, now let us give a detailed algorithm to implement the approximation. Here, we use the finite difference method.

More precisely, for the approximate problem (B.2), denote $dx = (x_{max} - x_{min})/nx$, $dr = r_{max}/nr$, $dq = q_{max}/nq$, $x_i = x_{min} + (i-1)dx$ for $i = 1, \cdots, nx$, $r_j = (j-1)dr$ for $j = 1, \cdots, nr$, $q_k = (k-1)dq$ for $k = 1, \cdots, nq$, $u_{i,j,k} = u(x_i, r_j, q_k)$, and

$$\begin{aligned}
D_i^+ u &= (u_{i+1,j,k} - u_{i,j,k})/dx, & D_i^- u &= (u_{i,j,k} - u_{i-1,j,k})/dx,\\
D_j^+ u &= (u_{i,j+1,k} - u_{i,j,k})/dr, & D_j^- u &= (u_{i,j,k} - u_{i,j-1,k})/dr,\\
D_k^+ u &= (u_{i,j,k+1} - u_{i,j,k})/dq, & D_k^- u &= (u_{i,j,k} - u_{i,j,k-1})/dq,\\
D_{ii} u &= (u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k})/dx^2, & D_{jj} u &= (u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k})/dr^2.
\end{aligned}$$

Then, the algorithm can be summarized as follow.

Step 1. Set error tolerance $\epsilon \ll 1$ and an initial guess $u^{(0)}$, e.g., $u^{(0)} = 0$.

Step 2. Solve for $u^{(n)}$, $n = 1, 2, \cdots$, the solution of the following equation:

$$
\begin{aligned}
0 \;=\; & \frac{1}{2}\sigma_r^2 D_{jj}u + \frac{1}{2}\sigma_x^2 x_i^2 D_{ii}u - q_k D_j^- u - \beta u_{i,j,k} + \tilde{U}(x_i, q_k) - C(r_j)q_k \\
& + [\kappa(\mu_t - \ln x_i)]^+ D_i^+ u - [\kappa(\mu_t - \ln x_i)]^- D_i^- u \\
& + PN \times \mathbf{1}_{\{D_k^+ u^{(n-1)} - \eta_+ > 0\}} \Big(D_k^+ u - \eta_+ \Big) + PN \times \mathbf{1}_{\{-D_k^- u^{(n-1)} + \eta_- > 0\}} \Big(- D_k^- u + \eta_- \Big),
\end{aligned}
$$

coupled with the boundary conditions prescribed in the previous subsection. Here, we use the implicit and upwind scheme to guarantee the stability of the algorithm.

Step 3. Calculate the relative error $\mathrm{Err} = ||u^{(n)} - u^{(n-1)}||/||u^{(n-1)}||$. If $\mathrm{Err} \leqslant \epsilon$, then stop and set $u = u^{(n)}$; otherwise, return to step 2.

Similarly, for the problem (B.1), we denote $dy = y_{max}/ny$ and $y_t = (t-1)dy$ for $t = 1, \cdots, ny$. Then, the algorithm can be summarized as follow.

Step 1. Set error tolerance $\epsilon \ll 1$ and the terminal value $V_{ny,i,j,k} = u_{i,j,k}$. We solve problem (B.1) backward one layer by one layer.

Step 2. Given $V_{t,i,j,k}$ for $t = 2, \cdots, ny-1$, $i = 1, \cdots, nx$, $j = 1, \cdots, nr$, and $k = 1, \cdots, nq$, calculate $V_{t-1,i,j,k}$ as follows.

(a) Set an initial guess $v^{(0)}$, e.g., $v^{(0)} = V_{t,i,j,k}$.

(b) Solve for $v^{(n)}$, $n = 1, 2, \cdots$, the solution of the following equation:

$$
\begin{aligned}
0 \;=\; & f(y_t, w_{t-1,i,j,k}^{(n)}) \frac{V_{t,i,j,k} - v_{i,j,k}}{dy} \\
& + \frac{1}{2}\sigma_r^2 D_{jj}v + \frac{1}{2}\sigma_x^2 x_i^2 D_{ii}v - \beta v_{i,j,k} + \tilde{U}(x_i, q_k) - C(r_j)q_k - \tilde{C}(w_{t-1,i,j,k}^{(n)}) \\
& + [f(y_t, w_{t-1,i,j,k}^{(n)}) - q_k]^+ D_j^+ v - [f(y_t, w_{t-1,i,j,k}^{(n)}) - q_k]^- D_j^- v \\
& + [\kappa(\mu_t - \ln x_i)]^+ D_i^+ v - [\kappa(\mu_t - \ln x_i)]^- D_i^- v \\
& + PN \times \mathbf{1}_{\{D_k^+ v^{(n-1)} - \eta_+ > 0\}} \Big(D_k^+ v - \eta_+ \Big) + PN \times \mathbf{1}_{\{-D_k^- v^{(n-1)} + \eta_- > 0\}} \Big(- D_k^- v + \eta_- \Big),
\end{aligned}
$$

coupled with the boundary conditions prescribed in the previous subsection. Here

$w_{t-1,i,j,k}^{(n)}$ is obtained from the first order condition (A.18)

$$\frac{V_{t,i,j,k} - v_{i,j,k}^{(n)}}{dy} + D_j v^{(n)} = \frac{D\,\tilde{C}(w^{(n)})}{D_w f(y_t, w^{(n)})}.$$

Note that we again use the implicit and upwind scheme to guarantee the stability of the algorithm.

(c) Calculate the relative error $\text{Err} = ||v^{(n)} - v^{(n-1)}||/||v^{(n-1)}||$. If $\text{Err} \leqslant \epsilon$, then stop and set $V_{t-1,i,j,k} = v^{(n)}$; otherwise, return to step (b).

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Online Supplement

Exhaustible Resources with Adjustment Costs: Spot and Futures Prices

I The Proof of Theorem 3

In what follows, to prove Theorem 3, we first show that the HJB equation (A.1) admits a unique classic solution. Then, based on this solution, we construct a candidate optimal adjustment strategy. Finally, by a standard verification argument, we show that this solution is indeed the value function defined by (4) and that the constructed strategy is the optimal strategy.

First, we show the uniqueness of the solution to the HJB equation (A.1). For the current state constrained problem, if we restrict the solution to satisfy continuity and some growth conditions, then the assertion follows from a standard argument by using viscosity approach, see, e.g., Crandall, Ishii, and Lions (1992).

I.1 Continuity and Growth Conditions of the Value Function

Lemma I.1. *The value function is concave in* $(0,\infty)^2$*. Consequently, it is continuous in* $(0,\infty)^2$*.*

Proof. For a given initial state $(r^i, q^i) \in (0,\infty)^2$ and its corresponding admissible strategy $\{I_t^i, D_t^i\}_{t\geqslant 0}$ for $i = 1, 2$, define the following initial state and strategy

$$(r^\omega, q^\omega) := \omega(r^1, q^1) + (1-\omega)(r^2, q^2),\ \{I_t^\omega, D_t^\omega\} := \omega\{I_t^1, D_t^1\} + (1-\omega)\{I_t^2, D_t^2\} \ \text{ with } \omega \in (0,1).$$

Then from the definition, it is easy to see $\{I_t^\omega, D_t^\omega\}_{t\geqslant 0}$ is an (r^ω, q^ω)-admissible strategy. In addition, $(R_t^\omega, Q_t^\omega) = \omega(R_t^1, Q_t^1) + (1-\omega)(R_t^2, Q_t^2)$ for $t \geqslant 0$. Therefore, due to the concavity of U, we have

$$\begin{aligned} V(r^\omega, q^\omega) \ &\geqslant \ \int_0^\infty e^{-\beta t}\Big([U(Q_t^\omega) - C\,Q_t^\omega]dt - \eta_+ dI_t^\omega - \eta_- dD_t^\omega\Big) \\ &\geqslant \ \omega \int_0^\infty e^{-\beta t}\Big([U(Q_t^1) - C\,Q_t^1]dt - \eta_+ dI_t^1 - \eta_- dD_t^1\Big) \\ &\quad + (1-\omega)\int_0^\infty e^{-\beta t}\Big([U(Q_t^2) - C\,Q_t^2]dt - \eta_+ dI_t^2 - \eta_- dD_t^2\Big). \end{aligned}$$

By taking all admissible strategies over $\mathcal{A}(r^i, q^i)$ for $i = 1, 2$, we have $V(r^\omega, q^\omega) \geqslant \omega V(r^1, q^1) + (1-\omega)V(r^2, q^2)$. This completes the proof. □

Next, we show the value function is continuous in $[0,\infty)^2$.

Lemma I.2. *Let* $K = \max\{\eta_+, |\eta_-|\}$, *then for any* $(r, q^i) \in [0,\infty)^2$ *for* $i = 1, 2$,

$$|V(r,q^1) - V(r,q^2)| \leqslant K|q^1 - q^2|.$$

Proof. Without loss of generality, assuming $q^1 \geqslant q^2$, the definition implies that

$$V(r,q^1) - V(r,q^2) \geqslant -\eta_+(q^1 - q^2) \geqslant -K|q^1 - q^2|.$$

Similarly, $V(r,q^1) - V(r,q^2) \leqslant K|q^1 - q^2|$. Thus the lemma follows. □

Lemma I.3. *The value function is bounded by*

$$-\eta_- q \leqslant V(r,q) \leqslant -\eta_- q + A\,U(r) + B\,q\Big(1 - e^{-\beta\frac{r}{q}}\Big) \quad \forall \quad (r,q) \in [0,\infty)^2,$$

where $A = \max\{1, 1/\beta\}$ *and* $B = \max\{\eta_-, 0\}$. *Consequently, combined with Lemmas I.1 and I.2, the value function is continuous in* $[0,\infty)^2$.

Proof. For the first part, for any $(r,q) \in [0,\infty)^2$, by taking the strategy of shutting down the production which is suboptimal, the definition implies that $V(r,q) \geqslant -\eta_- q$.

On the other hand, define $\varphi(r,q) := -\eta_- q + A\,U(r) + B\,q\Big(1 - e^{-\beta\frac{r}{q}}\Big)$. Then,

$$\varphi_q = -\eta_- + B\left(1 - \left[1 + \beta\frac{r}{q}\right]e^{-\beta\frac{r}{q}}\right), \quad \varphi_r = A\,U'(r) + \beta B e^{-\beta\frac{r}{q}}.$$

By using the fact that $(1+t)e^{-t} \in [0,1]$ and $B \in [0, \eta_- + \eta_+]$, we have $\varphi_q \in [-\eta_-, \eta_+]$. On the other hand,

$$\begin{aligned}
& -q\varphi_r - \beta\varphi + U(q) - C\,q \\
= \; & -AqU'(r) - \beta Bqe^{-\beta\frac{r}{q}} - \beta\left(-\eta_- q + A\,U(r) + B\,q\Big[1 - e^{-\beta\frac{r}{q}}\Big]\right) + U(q) - C\,q \\
= \; & -AqU'(r) - \beta A\,U(r) + U(q) - \beta[B - \eta_-]q - C\,q \\
\leqslant \; & -qU'(r) - U(r) + U(q) \leqslant -rU'(r) < 0.
\end{aligned}$$

Therefore, $\max\{-q\varphi_r - \beta\varphi + U(q) - C\,q,\ \varphi_q - \eta_+,\ -\varphi_q - \eta_-\} \leqslant 0$. Then, from the standard argument of verification (see subsection I.5 below), we have $V \leqslant \varphi$. □

The bound conditions in Lemma I.3 give us the following growth conditions of the value function:

$$\lim_{r\to\infty} \frac{V(r,q)+\eta_- q}{r} = 0 \quad \text{and} \quad \lim_{q\to\infty} \frac{V(r,q)+\eta_- q}{q} = 0. \tag{I.1}$$

I.2 Uniqueness of the Solution

Note that without the growth condition (I.1), there is an infinite number of classical solutions to the HJB equation (A.1). In fact, one can check that, for sufficiently large constants C_1 and C_2, the function $\phi(r,q) = -\eta_- q + C_1 U(r) + C_2 r$ is a classic solution to the HJB equation (A.1). The detailed proof of uniqueness requires the notion of viscosity solution and is quite standard with the help of continuity and growth conditions. Therefore, we omit it for simplicity and refer interested readers to Crandall et al. (1992) for details.

I.3 Construction of a Solution to the HJB Equation (A.1)

At any time t, the optimal production adjustment policy depends only on the state (R_t, Q_{t^-}). First, in the no-adjustment region $\mathcal{N}$, we have (A.6), which admits a general solution $V(r,q) = B(q)e^{-\beta r/q} + (U(q) - Cq)/\beta$, where B is an unknown function of q. In the upward adjustment region $\mathcal{I}$, we have $\partial V/\partial q = \eta_+$, which also has a general solution $V(r,q) = \eta_+ q + A_I(r)$, where A_I is an unknown function of r. Similarly, in the downward adjustment region $\mathcal{D}$, the general solution is of the form $V(r,q) = -\eta_- q + A_D(r)$, where A_D is an unknown function of r. Since production must be stopped at the exhaustion time, the boundary condition $V(0,q) = -\eta_- q$ holds, which in turn implies that $A_D(0) = 0$.

Now according to the optimal adjustment policy and equations (A.2), (A.3), and (A.6), we have five unknown functions $\{A_I(r), A_D(r), B(q), Q_I(r), Q_D(r)\}$ to be determined. To solve them, we must impose five relations. First, we know the value function V is continuous. So, by continuity, the following conditions must hold

$$\begin{aligned}
\Big(\eta_+ q + A_I(r)\Big)\Big|_{q=Q_I(r)} &= \left(B(q)e^{-\beta r/q} + \frac{U(q)-Cq}{\beta}\right)\Big|_{q=Q_I(r)}, &\text{(I.2)}\\
\Big(-\eta_- q + A_D(r)\Big)\Big|_{q=Q_D(r)} &= \left(B(q)e^{-\beta r/q} + \frac{U(q)-Cq}{\beta}\right)\Big|_{q=Q_D(r)}. &\text{(I.3)}
\end{aligned}$$

Next, from the optimal rule of adjustment, we also have

$$\begin{aligned}
\eta_+ &= \frac{d}{dq}\left(B(q)e^{-\beta r/q} + \frac{U(q)-Cq}{\beta}\right)\Big|_{q=Q_I(r)}, &\text{(I.4)}\\
-\eta_- &= \frac{d}{dq}\left(B(q)e^{-\beta r/q} + \frac{U(q)-Cq}{\beta}\right)\Big|_{q=Q_D(r)}. &\text{(I.5)}
\end{aligned}$$

Finally, from the subsection I.6, another key condition comes from a *high contact condition* on the optimal boundary of downward adjustment, i.e.,

$$0 = \frac{d^2}{dq^2}\left(B(q)e^{-\beta r/q} + \frac{U(q)-Cq}{\beta}\right)\Big|_{q=Q_D(r)}. \tag{I.6}$$

Here, for simplicity, we only present the construction of these functions as follows. One can easily check that the five unknown functions $\{A_I(r), A_D(r), B(q), Q_I(r), Q_D(r)\}$ satisfy the relations (I.2)–(I.6).

Suppose that $\eta_+ \geqslant 0$ and $\eta_- + \eta_+ \geqslant 0$ and that q_I and q_D are defined by (A.7). First of all, we construct Q_I and Q_D as inverse functions of R_I and R_D given in (A.9) and (A.8), respectively. That is, $R_D : [0, q_D) \mapsto [0, \infty)$ is defined explicitly as the following integration:

$$R_D(q) := -\frac{1}{\beta}\int_0^q \frac{xU''(x)}{U'(x) + \beta\eta_- - C}dx.$$

And $R_I : [0, q_I) \mapsto [0, \infty)$ is defined implicitly as the unique solution to the following equation

$$e^{-\beta\frac{R_I(q)-R_D(q)}{q}}\left(1 + \beta\frac{R_I(q)-R_D(q)}{q}\right) = 1 - \frac{\beta(\eta_- + \eta_+)}{U'(q) - C + \beta\eta_-}.$$

It is clear that R_D is increasing from the definition. Now we claim that R_I is well defined and also increasing. Indeed, denote

$$\begin{aligned} f(t) &:= e^{-t}(1+t), && \text{for} \quad t \geqslant 0, \\ g(q) &:= 1 - \frac{\beta(\eta_- + \eta_+)}{U'(q) - C + \beta\eta_-}, && \text{for} \quad 0 \leqslant q \leqslant q_I. \end{aligned}$$

It is clear that $f(0) = 1$, $\lim_{t\to\infty} f(t) = 0$, and $f'(t) < 0$ for $t \in (0, \infty)$.

Case 1. If $\eta_- + \eta_+ = 0$, then $g(q) \equiv 1$. Therefore $R_I(q) \equiv R_D(q)$ for $0 \leqslant q \leqslant q_I = q_D$.

Case 2. If $\eta_+ + \eta_- > 0$, then it is clear that

$$g(q_I) = 0, \quad \lim_{q\to 0^+} g(q) = \frac{U'(0) - C - \beta\eta_+}{U'(0) - C + \beta\eta_-} \leqslant 1, \quad g'(q) < 0 \quad \text{in} \quad (0, q_I).$$

Therefore, there exists a unique increasing mapping $T : [0, q_I) \mapsto [0, \infty)$ such that $f(T(q)) = g(q)$, and

$$R_I(q) = \frac{qT(q)}{\beta} + R_D(q), \quad \text{for} \quad q \in [0, q_I],$$

which is strictly increasing and no less than R_D. Thus, as $q \to 0^+$, $T(q) \geqslant 0$, we have $R_I(0) = R_D(0) = 0$ and

$$R_I'(0) - R_D'(0) = \lim_{q\to 0^+} \frac{R_I(q) - R_D(q)}{q} = \lim_{q\to 0^+} \frac{T(q)}{\beta} \geqslant 0.$$

In particular, if $U'(0) = P(0) = \infty$ then $g(0^+) = 1$, which in turn implies $R_I'(0) = R_D'(0)$.

A further calculation shows

$$\begin{aligned} R_I' &= R_D' + \frac{-(\eta_- + \eta_+)qU''(q)}{(U'(q) - C + \beta\eta_-)(U'(q) - C - \beta\eta_+)} \Big(1 + \frac{q}{\beta(R_I - R_D)}\Big) + \frac{R_I - R_D}{q} \\ &= \frac{-qU''(q)}{\beta(U' - C - \beta\eta_+)} + \frac{-(\eta_- + \eta_+)qU''(q)}{(U'(q) - C + \beta\eta_-)(U'(q) - C - \beta\eta_+)} \frac{q}{\beta(R_I - R_D)} + \frac{R_I - R_D}{q}. \end{aligned}$$

Therefore, $R_I' \geqslant -qU''/\beta(U' - C - \beta\eta_+) \geqslant R_D'$. Also, it is easy to see

$$\frac{\partial R_D}{\partial \eta_-} < 0, \quad \frac{\partial R_I}{\partial \eta_-} > 0, \quad \frac{\partial R_D}{\partial \eta_+} = 0, \quad \frac{\partial R_I}{\partial \eta_+} > 0,$$

From the above argument, obviously, Q_I and Q_D satisfy the conditions stated in Theorem 3.

Next, based on the functions R_I, R_D, Q_I and Q_D, we can define functions B, A_I and A_D as follows.

$$\begin{cases} B(q) := -e^{\beta \frac{R_D(q)}{q}} \frac{q}{\beta}(U'(q) - C + \beta\eta_-), & \text{for} \quad 0 \leqslant q < q_D, \\ A_I(r) := e^{-\beta \frac{r}{Q_I(r)}} B(Q_I(r)) + \frac{1}{\beta}\Big(U(Q_I(r)) - (C + \beta\eta_+)Q_I(r)\Big), & \text{for} \quad 0 \leqslant r < \infty, \\ A_D(r) := \frac{1}{\beta}\Big(U(Q_D(r)) - Q_D(r)U'(Q_D(r))\Big), & \text{for} \quad 0 \leqslant r < \infty. \end{cases}$$

With the above preparation, we conjecture that the following function is a solution to the HJB equation (A.1).

$$\varphi(r,q) := \begin{cases} \eta_+ q + A_I(r) & \text{for} \quad r \geqslant 0,\ 0 \leqslant q \leqslant Q_I(r), \\ e^{-\beta\frac{r}{q}} B(q) + \frac{1}{\beta}\Big(U(q) - C\,q\Big) & \text{for} \quad r \geqslant 0,\ Q_I(r) < q < Q_D(r), \\ -\eta_- q + A_D(r) & \text{for} \quad r \geqslant 0,\ Q_D(r) \leqslant q < \infty. \end{cases}$$

In addition, from the construction, it is clear that the following regularity results hold:

$$\varphi \in C^{1,1}((0,\infty)^2) \cap C^{2,2}((0,\infty)^2 \backslash \Gamma_I) \cap C([0,\infty)^2),$$
$$R_i \in C^{\infty}([0,q_i)), \quad Q_i \in C^{\infty}([0,\infty)) \quad \text{for} \quad i = \{I, D\}.$$

I.4 Variational Property of the Candidate Solution

Now we need to check that the function φ is indeed a solution to HJB equation (A.1). First, from the construction and the property of R_I and R_D, it is easy to see that the boundary condition holds, that is, $\varphi(0,q) = -\eta_- q$. Next we have to verify the variational property of the HJB equation, i.e., $\min\{q\varphi_r + \beta\varphi - U(q) + C\,q,\ -\varphi_q + \eta_+,\ \varphi_q + \eta_-\} = 0$, where we use subscript to represent partial derivative for simplicity.

Case 1. If $(r,q) \in \mathcal{D}$, then $\varphi(r,q) = -\eta_- q + A_D(r)$. Thus

$$\varphi_q + \eta_- = 0, \quad -\varphi_q + \eta_+ = \eta_- + \eta_+ > 0.$$

For a fixed r, define $F(q) := q\varphi_r + \beta\varphi - U(q) + C\,q = qA'_D + \beta(-\eta_- q + A_D) - U(q) - C\,q$. Then

$$F'(q) \quad = \quad A'_D(r) - \beta\eta_- - U'(q) + C = U'(Q_D(r)) - U'(q) \geqslant 0$$

since $q \geqslant Q_D(r)$. Also note $F(Q_D(r)) = 0$, thus $F(q) \geqslant 0$ for $q \geqslant Q_D(r)$.

Case 2. If $(r, q) \in \mathcal{N}$, then $\varphi(r, q) = B(q)e^{-\beta\frac{r}{q}} + \frac{U(q)-C\,q}{\beta}$. Thus $q\varphi_r + \beta\varphi - U(q) + C\,q = 0$. Note that B satisfies the ODE $B'(q) + \beta R_D(q)B(q)/q^2 = B(q)/q$. A tedious calculation shows

$$\begin{aligned}
\varphi_{qq} &= e^{-\beta\frac{r}{q}}\Big(B''(q) + \frac{2\beta}{q^2}rB'(q) - \frac{2\beta}{q^3}rB(q) + \frac{\beta^2}{q^4}r^2B(q)\Big) + \frac{U''(q)}{\beta} \\
&= e^{-\beta\frac{r}{q}}\Big(\beta^2\frac{R_D^2}{q^2}B(q) - \beta^2\frac{2R_Dr}{q^4}B(q) + \beta^2\frac{r^2}{q^4}B(q) - \beta\frac{R'_D}{q^2}B(q)\Big) + \frac{U''(q)}{\beta} \\
&= e^{-\beta\frac{r}{q}}\frac{\beta^2(r - R_D(q))^2}{q^4}B(q) + \frac{U''(q)}{\beta}\Big(1 - e^{-\beta\frac{r-R_D(q)}{q}}\Big) \\
&\leqslant 0,
\end{aligned}$$

since $R_D(q) \leqslant r$ and $B(q) < 0$. On the other hand, we have $\varphi_q(r, Q_I(r)) = \eta_+$ and $\varphi_q(r, Q_D(r)) = -\eta_-$. Therefore, $-\eta_- \leqslant \varphi_q \leqslant \eta_+$.

Case 3. If $(r, q) \in \mathcal{I}$, then $\varphi(r, q) = \eta_+ q + A_I(r)$. Thus

$$\varphi_q + \eta_- = \eta_+ + \eta_- > 0, \quad -\varphi_q + \eta_+ = 0.$$

Similar to case 1, define $F(q) := q\varphi_r + \beta\varphi - U(q) + C\,q$, for a fixed r. Then $F'(q) = A'_I(r) + \beta\eta_+ + C - U'(q)$. Note that φ_r is continuous across the free boundary Γ_I, then

$$\begin{aligned}
A'_I(r) &= \varphi_r(r, Q_I(r)^-) = \varphi_r(r, Q_I(r)^+) \\
&= -e^{-\beta\frac{r}{Q_I(r)}}\beta\frac{B(Q_I(r))}{Q_I(r)} \\
&= \frac{[U'(Q_I(r)) - C - \beta\eta_+]Q_I(r)}{Q_I(r) + \beta[r - R_D(Q_I(r))]} \\
&\leqslant U'(Q_I(r)) - C - \beta\eta_+.
\end{aligned}$$

So, $F'(q) \leqslant U'(Q_I(r)) - U'(q) \leqslant 0$ follows from the concavity of U. On the other hand, note that $F(Q_I(r)) = 0$. Therefore, $F(q) \geqslant 0$ for $q \leqslant Q_I(r)$.

I.5 Construction of Optimal Adjustment Policy and Verification Argument

First, based on previous functions Q_I and Q_D, define the adjustment strategy by (A.10). It is easy to see that the strategy is admissible. Now let us show that this strategy is indeed the optimal one and the solution φ coincides with the value function V. This follows by a standard verification argument due to the regularity of the solution to HJB equation (A.1).

For an arbitrary initial state $(r, q) \in [0, \infty)^2$, and an arbitrary (r, q)-admissible strategy $\{I_t, D_t\}_{t \geqslant 0}$, denote (R_t, Q_t) the corresponding processes under the strategy $\{I_t, D_t\}$ with initial value $(R_0, Q_{0^-}) = (r, q)$. Since φ satisfies the HJB equation (A.1), we have

$$\begin{aligned}
\varphi(r,q) &= e^{-\beta t}\varphi(R_t,Q_t) - \Big\{ \int_0^t e^{-\beta s}[\varphi_r(R_s,Q_s)dR_s - \beta\varphi(R_s,Q_s)]ds \\
&\quad + \int_0^t e^{-\beta s}\varphi_q(R_s,Q_s)(dQ_s^c + \sum_{0\leqslant s\leqslant t} e^{-\beta s}\Big(\varphi(R_s,Q_s) - \varphi(R_s,Q_{s^-})\Big)\Big\} \\
&= e^{-\beta t}\varphi(R_t,Q_t) - \Big\{ \int_0^t e^{-\beta s}[-Q_s\varphi_r(R_s,Q_s) - \beta\varphi(R_s,Q_s)]ds \\
&\quad + \int_0^t e^{-\beta s}\Big(\varphi_q(R_s,Q_s)dI_s^c + \varphi_q(R_s,Q_s)dD_s^c\Big) \\
&\quad + \sum_{0\leqslant s\leqslant t} e^{-\beta s}\Big(\varphi(R_s,Q_s) - \varphi(R_s,Q_{s^-})\Big)\Big\},
\end{aligned}$$

for all $t \geqslant 0$, where Q_t^c, I_s^c and D_s^c are the continuous parts of each counterpart.

Note that, for the last term,

$$\varphi(R_s,Q_s) - \varphi(R_s,Q_{s^-}) = \varphi(R_s,Q_{s^-}+\Delta Q_s) - \varphi(R_s,Q_{s^-}) = \Delta Q_s \int_0^1 \varphi_q(R_s,Q_{s^-}+\lambda\Delta Q_s)d\lambda,$$

where ΔQ_s is the jump at time s, and is equal to $\Delta I_s - \Delta D_s$. Therefore,

$$\begin{aligned}
\varphi(R_s,Q_s) - \varphi(R_s,Q_{s^-}) &= (\Delta I_s - \Delta D_s)\int_0^1 \varphi_q(R_s,Q_{s^-}+\lambda\Delta Q_s)d\lambda \\
&\leqslant \Delta I_s \int_0^1 \eta_+ d\lambda + \Delta D_s \int_0^1 d\lambda = \eta_+\Delta I_s + \eta_-\Delta D_s,
\end{aligned}$$

where we have used the fact that $\varphi_q \in [-\eta_-, \eta_+]$ from the HJB equation.

Letting $t \to \infty$ and using transversality condition, $\lim_{t\to\infty} e^{-\beta t}\varphi(R_t, Q_t) \geqslant 0$, we have

$$\begin{aligned}\varphi(r,q) &\geqslant \lim_{t\to\infty} e^{-\beta t}\varphi(R_t,Q_t) + \int_0^t e^{-\beta s}\Big([U(Q_s) - C\,Q_s]ds - \eta_+ dI_s - \eta_- dD_s\Big)\\ &\geqslant \int_0^\infty e^{-\beta t}\Big([U(Q_t) - C\,Q_t]dt - \eta_+ dI_t - \eta_- dD_t\Big).\end{aligned}$$

Thus $\varphi(r,q) \geqslant V(r,q)$ follows from the arbitrariness of the strategy.

On the other hand, using the same calculation as before but with the strategy $\{I_t^*, D_t^*\}$ given by (A.10), we have equalities instead of inequalities, i.e.,

$$\varphi(r,q) = \int_0^\infty e^{-\beta t}\Big([U(Q_t^*) - C\,Q_t^*]dt - \eta_+ dI_t^* - \eta_- dD_t^*\Big).$$

Thus, $\varphi(r,q) \leqslant V(r,q)$ follows from the definition of the value function. Therefore, $V(r,q) = \varphi(r,q)$ for all $(r,q) \in [0,\infty)^2$, and the extraction policy defined by (A.10) is the socially optimal extraction path. Since the value function is the unique solution to the HJB equation (A.1), the uniqueness of the optimal adjustment strategy follows from the unique structure of this solution.

To complete the proof of Theorem 3, let us come back to the high contact condition used in (I.6).

I.6 The High Contact Condition

The high contact condition has been widely used in the literature of transaction costs models or singular stochastic control problems in mathematical literature. Roughly speaking, the high contact condition in those literature comes from the property of diffusion processes; see Dumas (1991) for more details. But differently, we are here dealing with a deterministic problem. More interestingly, this high contact condition is required only on the optimal boundary of downward but not upward adjustment. Both this high contact condition and its asymmetric property seem quite surprising at first. In this appendix we give a short discussion about this critical condition.

To see this, we need to come back to the original optimization problem for the production adjustment costs. Since there will be a continuous adjustment part, without loss of generality, we can assume initially the state is on the optimal boundary of downward adjustment. Then

$(I_t, D_t) = (0, q - Q_t)$ for all $t \geqslant 0$. Using integration by parts, we have

$$\begin{aligned} V(r,q) &= \sup_{\{I_t,D_t\}\in\mathcal{A}(r,q)} \int_0^\infty e^{-\beta t}\Big([U(Q_t) - C\,Q_t]dt - \eta_+ dI_t - \eta_- dD_t\Big) \\ &= \sup_{\{Q_t\geqslant 0\}} \int_0^\infty e^{-\beta t}\Big([U(Q_t) - C\,Q_t]dt + \eta_- dQ_t\Big) \\ &= \sup_{\{Q_t\geqslant 0\}} \int_0^\infty e^{-\beta t}\Big(U(Q_t) - [C - \beta\eta_-]Q_t\Big)dt - \eta_- q. \end{aligned}$$

This implies that once the state is on the optimal boundary of downward adjustment, the optimization problem with marginal downward adjustment cost η_- is equivalent to the problem first reducing the production rate to zero and then doing a problem with no adjustment costs but with a modified marginal extraction cost $C - \beta\eta_-$. Therefore, the first order condition for the problem without adjustment costs implies the high contact condition. However, due to the non-renewability of the resource, this equivalence dose not hold on the optimal boundary of upward adjustment, which implies that the high contact condition fails on this boundary. Roughly speaking, the optimal boundary of upward adjustment plays a role of an optimal capacity for a producer.

Furthermore, from this equivalent statement, we can directly obtain the optimal boundary of downward adjustment. In fact, from dynamic programming, the optimal control problem

$$V^H(r) = \sup_{\{Q_t\geqslant 0\}} \int_0^\infty e^{-\beta t}\Big(U(Q_t) - [C - \beta\eta_-]Q_t\Big)dt$$

satisfies the HJB equation

$$\max_{q\geqslant 0}\left\{U(q) - (C - \beta\eta_-)q - q\frac{dV^H(r)}{dr}\right\} - \beta V^H = 0. \tag{I.7}$$

Then the first order condition is

$$\frac{dV^H(r)}{dr} = U'(q) - (C - \beta\eta_-).$$

Thus, under optimal production rate Q_t^*, by differentiating the above equation with respect to t, we have

$$U''(Q_t^*)\frac{dQ_t^*}{dt} = \frac{dR_t^*}{dt}\frac{d^2V^H}{dr^2}\Big|_{r=R_t^*} = -Q_t^*\frac{d^2V^H}{dr^2}\Big|_{r=R_t^*} = \beta\frac{dV^H}{dr}\Big|_{r=R_t^*} = \beta\Big(U'(Q_t^*) - (C - \beta\eta_-)\Big),$$

where the third equality follows from differentiating the equation (I.7) and using the first order condition. On the other hand, from the dynamics of reserve, i.e., $dR_t^* = -Q_t^* dt$ and the fact $\lim_{t\to\infty} R_t^* = 0$ and the expression of dQ_t^*, we have

$$R_t^* = \int_t^\infty Q_s^* ds = -\frac{1}{\beta}\int_0^{Q_t^*} \frac{xU''(x)}{U'(x) - (C - \beta\eta_-)}dx,$$

which is exactly the definition of function R_D. This, in turn, explains why the upward adjustment parameter η_+ has no impacts on the optimal boundary of downward adjustment, i.e., $\partial R_D/\partial\eta_+ = 0$.

As a remark, we conjecture that when uncertainties (e.g., demand shocks) in diffusion forms are taken into consideration, the high contact condition may hold on both two optimal boundaries.

II The Proof of Theorem 1

II.1 An Application of Welfare Theorems

The assertion follows from the application of the second welfare theorem. The following proof is largely inspired by Stockey and Lucas (1989) (Ch. 15 and 16) and Lucas and Prescott (1971).

First of all, let's define the commodity space

$$L := \left\{ \{\ell_t\}_{t\geqslant 0} \;\middle|\; \begin{array}{l} \ell_t \text{ is non-negative and right-continuous-left-limit process, with} \\ \ell_{0-} \geqslant 0 \text{ is given and } ||\ell|| := \sup_{t\geqslant 0} e^{-\beta t}|\ell_t| < \infty \end{array} \right\}.$$

Then, for consumers, the commodity points set $\mathbf{X} := L^4 = L\times L\times L\times L$. For firms, the production set is defined as

$$\mathbf{Y} := \left\{ \{Q_t, C_t, I_t, D_t\}_{t\geqslant 0} \in \mathbf{X} \;\middle|\; \begin{array}{l} (I_t, D_t) \in \mathcal{A}(R_0, Q_{0-}) \text{ with } (R_0, Q_{0-}) \in [0,\infty)^2, \\ (R_t, Q_t) \text{ satisfy (3), (2), and } C_t \geqslant C\, Q_t \end{array} \right\}.$$

The interpretation of a point (Q, C, I, D) in the commodity space is that (Q_t, C_t, I_t, D_t) denote respectively the level of production, cost of production, cumulative upward adjustment cost, and cumulative downward adjustment cost at instant time of t.

Therefore, our economy consists of a single consumer whose preference, given by

$$u(\{Q_t, C_t, I_t, D_t\}) = \int_0^\infty e^{-\beta t}\Big([U(Q_t) - C_t]dt - \eta_+ dI_t - \eta_- dD_t\Big),$$

is defined on $\mathbf{X}$, and a single firm, whose production set is $\mathbf{Y}$.

Consequently, an allocation (Q, C, I, D) is *Pareto optimal* in this economy if

(P) $u(\{Q_t, C_t, I_t, D_t\}) \geqslant u(\{Q'_t, C'_t, I'_t, D'_t\})$ for all $(Q', C', I', D') \in \mathbf{Y}$.

And, a *competitive equilibrium* is defined as an allocation (Q, C, I, D) together with a price process $p = \{p_t\}_{t\geqslant 0} \in L$ such that

(**E**1) $(Q, C, I, D) \in \mathbf{Y}$,

(**E**2) $p_t = P(Q_t)$ for all $t \geqslant 0$,

(**E**3) $\int_0^\infty e^{-\beta t}\Big([p_t\, Q_t - C_t]dt - \eta_+ dI_t - \eta_- dD_t\Big) \geqslant \int_0^\infty e^{-\beta t}\Big([p_t\, Q'_t - C'_t]dt - \eta_+ dI'_t - \eta_- dD'_t\Big)$ for all $(Q', C', I', D') \in \mathbf{Y}$.

Now we are ready to show the following result.

Lemma II.1 (The First Welfare Theorem)**.** *Suppose (Q, C, I, D, p) is a competitive equilibrium, then (Q, C, I, D) is Pareto optimal.*

Proof. For any $(Q', C', I', D') \in \mathbf{Y}$, the concavity of U implies that

$$\begin{aligned}
u(\{Q'_t, C'_t, I'_t, D'_t\}) &= \int_0^\infty e^{-\beta t}\Big([U(Q'_t) - C'_t]dt - \eta_+ dI'_t - \eta_- dD'_t\Big) \\
&\leqslant \int_0^\infty e^{-\beta t}\Big([U(Q_t) + U'(Q_t)(Q'_t - Q_t) - C'_t]dt - \eta_+ dI'_t - \eta_- dD'_t\Big) \\
&= u(\{Q_t, C_t, I_t, D_t\}) + \int_0^\infty e^{-\beta t}\Big([p_t(Q'_t - Q_t) - (C'_t - C_t)]dt \\
&\quad -\eta_+ d(I'_t - I_t) - \eta_- d(D'_t - D_t)\Big) \\
&\leqslant u(\{Q_t, C_t, I_t, D_t\}),
\end{aligned}$$

where the last inequality comes from (**E**3). Thus the lemma follows. □

Next, we want to show the reverse one, that is

Lemma II.2 (The Second Welfare Theorem)**.** *Suppose* (Q, C, I, D) *is Pareto optimal, then* (Q, C, I, D) *together with prices* p *defined in* $(\mathbf{E}2)$ *is a competitive equilibrium.*

Proof. Since $(\mathbf{E}1)$ and $(\mathbf{E}2)$ are satisfied automatically, we only need to show that $(\mathbf{E}3)$ holds.

For this, first, we show that the production set $\mathbf{Y}$ is convex. For any initial state $(r, q) \in [0, \infty)^2$ and any two points $(Q^i, C^i, I^i, D^i) \in \mathbf{Y}$ for $i = 1, 2$, we define

$$(Q^\omega, C^\omega, I^\omega, D^\omega) = \omega(Q^1, C^1, I^1, D^1) + (1 - \omega)(Q^2, C^2, I^2, D^2).$$

From the properties of $\{I^1_t, D^1_t\}_{t\geqslant 0}$ and $\{I^2_t, D^2_t\}_{t\geqslant 0}$, $\{I^\omega_t, D^\omega_t\}_{t\geqslant 0}$ are non-decreasing, non-negative, and right-continuous-left-limit processes, which implies that condition (a) in the definition of $\mathcal{A}(r, q)$ holds.

In addition, from the proof of Lemma I.1, we have

$$Q^\omega_t = \omega Q^1_t + (1 - \omega)Q^2_t, \quad R^\omega_t = \omega R^1_t + (1 - \omega)R^2_t \quad \forall\, t \geqslant 0.$$

Since $R^i_t \geqslant 0$ and $Q^i_t \geqslant 0$ for $i = 1, 2$, so are R^ω_t and Q^ω_t. Thus, conditions (b) and (c) in the definition of $\mathcal{A}(r, q)$ hold as well. Therefore, $\{I^\omega_t, D^\omega_t\}_{t\geqslant 0} \in \mathcal{A}(r, q)$. Moreover,

$$C^\omega_t = \omega C^1_t + (1 - \omega)C^2_t \geqslant \omega C\, Q^1_t + (1 - \omega)C\, Q^2_t = C\, Q^\omega_t \quad \text{for } t \geqslant 0.$$

Hence, $(Q^\omega, C^\omega, I^\omega, D^\omega) \in \mathbf{Y}$ by the definition and $\mathbf{Y}$ is convex.

Now, for any $(Q', C', I', D') \in \mathbf{Y}$ and $\omega \in [0, 1]$, define

$$\begin{aligned}
&(Q^\omega, C^\omega, I^\omega, D^\omega) = \omega(Q', C', I', D') + (1 - \omega)(Q, C, I, D) \in \mathbf{Y},\\
&f(\omega) = \int_0^\infty e^{-\beta t}\Big([U(Q^\omega_t) - C^\omega_t]dt - \eta_+ dI^\omega_t - \eta_- dD^\omega_t\Big).
\end{aligned}$$

Since (Q, C, I, D) is Pareto optimal, thus

$$0 \geqslant f'(0) = \int_0^\infty e^{-\beta t}\Big([U'(Q_t)(Q'_t - Q_t) - (C'_t - C_t)]dt - \eta_+ d(I'_t - I_t) - \eta_- d(D'_t - D_t)\Big).$$

Then, by the definition of p in (**E**2), (**E**3) follows immediately from the above inequality. This completes the proof. □

Therefore, the proof of Theorem 1 follows immediately from the above lemma.

II.2 A Direct Proof

In this deterministic model, alternatively, we can give a direct proof as follow. To this end, suppose there is a representative producer in the economy, then we need to show that the depletion path $\{q_t\}_{t\geqslant 0} = \{Q_t^*\}_{t\geqslant 0}$, and the price given by $\{p_t\}_{t\geqslant 0} = \{P(Q_t^*)\}_{t\geqslant 0}$ form a Walransion equilibrium, where $\{Q_t^*\}_{t\geqslant 0}$ is the socially optimal production rate under the optimal adjustment policy $\{I_t^*, D_t^*\}_{t\geqslant 0}$.

First, note that the price is defined as $p_t = P(Q_t^*) = U'(Q_t^*)$, which means the price equals the marginal benefit of consumption in every period, so consumers are optimizing. Then, we only need to focus on producers' problem and market clearing condition.

Next, given the price p_t exogenously, let us consider the optimal production of the representative producer. Mathematically, we need to solve the following optimization problem:

$$\sup_{\{I,D\}\in\mathcal{A}(r,0)} \int_0^\infty e^{-\beta t}\Big([p_t - C]q_t dt - \eta_+ dI_t - \eta_- dD_t\Big), \tag{II.1}$$

subject to (3) and (2). Note that, by using integration by parts, problem (II.1) is equivalent to the following problem

$$\sup_{\{I,D\}\in\mathcal{A}(r,0)} \int_0^\infty e^{-\beta t}\Big([p_t - C + \beta\eta_-]q_t dt - (\eta_+ + \eta_-)dI_t\Big), \tag{II.2}$$

subject to (3), and (2).

Problem (II.1) or (II.2) is difficult to solve since it depends not only on current reserve level and production rate but also on time explicitly. For later use, we denote its optimal value as $v(r, 0, 0)$. However, the next lemma implies that the optimal production strategies are restricted in a class in which increase of production rate can take place only at the initial time. To give some intuitions, recall Hotelling's result with no extraction costs, that is, the price will grow at interest rate. Roughly speaking, the logic behind this simple rule is that if the price grows slower than interest rate, producers will extract the resource today instead of tomorrow, because they can earn the interest rate. On the contrary, if the price grows faster than interest rate, they will postpone the production for the future. Now in our case with production adjustment costs, the price will on average grow slower than interest rate. So, producers will produce as fast as they can. But, the production adjustment costs prevent them from extracting all resources at the very beginning.

Lemma II.3. *Suppose* $\{I_t, D_t\}_{t\geqslant}$ *is an optimal strategy, then* $dI_0 > 0$ *and* $dI_t = 0$ *for all* $t > 0$.

Proof. First we show production must start initially. To see this, suppose $\{I_t, D_t\}_{t\geqslant 0}$ is an optimal strategy. Further assume that the optimal starting time is $T_1 \geqslant 0$. From Lemma II.6 below we know that no production at all is not optimal, which implies that $T_1 < \infty$. Since $\eta_+ + \eta_- \geqslant 0$, simultaneous upward and downward adjustments are not optimal, which in turn implies that $dD_{T_1} = 0$ and $dI_{T_1} > 0$. By denoting $q_1 := dI_{T_1}$, then $v(r, 0, 0) = L_1(T_1)$, where

$$L_1(T_1) := \int_{T_1}^{\infty} e^{-\beta t}\Big([p_t - C + \beta\eta_-]q_t dt - (\eta_+ + \eta_-)dI_t\Big).$$

Therefore,

$$\frac{dL_1}{dT_1} = -e^{-\beta T_1}(p_{T_1} - C - \beta\eta_+)q_1 < 0,$$

since $p_t \geqslant \beta\eta_+ + C$ for all $t \geqslant 0$ and $T_1 < \infty$. Thus, $T_1 = 0$.

Next we show $dI_t = 0$ for any $t > 0$. If not, then there exists

$$T_2 = \inf\{t > 0 \mid dI_t > 0\},$$

with convention that $\inf\{\emptyset\} = \infty$.

Thus, $dI_{T_2} > 0$, and $v(r, 0, 0) = L_2(T_2)$, where

$$\begin{aligned} L_2(T_2) &:= \int_0^{\infty} e^{-\beta t}\Big([p_t - C + \beta\eta_-]q_t dt - (\eta_+ + \eta_-)dI_t\Big) \\ &= \int_0^{\infty} e^{-\beta t}[p_t - C + \beta\eta_-]q_t dt - \int_{T_2}^{\infty} e^{-\beta t}(\eta_+ + \eta_-)dI_t - (\eta_+ + \eta_-)dI_0. \end{aligned}$$

Consequently, the optimality of T_2 implies that

$$\frac{dL_2}{dT_2} = e^{-\beta T_2}(\eta_+ + \eta_-)dI_{T_2} = 0.$$

This in turn indicates that $T_2 = \infty$, since $dI_{T_2} > 0$.

This completes the proof. □

Then, using integration by parts, the optimization problem (II.1) or (II.2) is equivalent to

$$\sup_{\{\bar{q},\, 0 \leqslant q_t \leqslant \bar{q}\}} \int_0^\infty e^{-\beta t}[p_t - (c - \beta\eta_-)]q_t dt - (\eta_- + \eta_+)\bar{q},$$

subject to nonincreasing q_t and $\int_0^\infty q_t dt = r$.

Particularly, if $\eta_- + \eta_+ = 0$, i.e., the totally reversible investment case, then the above problem is equivalent to the original Hotelling problem with marginal extraction costs $C - \beta\eta_-$ or equivalently $C + \beta\eta_+$. Consequently, the socially optimal extraction path can be reproduced by a competitive market.

Note that $p_t - (C - \beta\eta_-)$ increases at interest rate β for $t > t^*$, where t^* is defined by (6). From that time on, the current problem is identical to the Hotelling's problem with constant marginal extraction cost $C - \beta\eta_-$. Thus, producers will be indifferent with production today or future. By contrast, in the constant price period $[0, t^*]$, producers will extract the resource as soon as possible due to the time value. Therefore, for a given initial production rate $\bar{q}$, which can be viewed as the capacity, a producer will use the maximum capacity to extract the resource until the critical time t^* or the exhausted time τ which comes first. Thus, the optimal production strategy admits the following properties.

Lemma II.4. *Suppose $(\bar{q}, \{q_t\}_{t \geqslant 0})$ is the optimal production strategy and τ is the exhaustion time. Then (a) if $\tau \leqslant t^*$, then $q_t = \bar{q}\mathbf{1}_{\{0 \leqslant t \leqslant \tau\}}$; if $\tau > t^*$, then $q_t = \bar{q}$ for $t \in [0, t^*]$.*

Proof. If the exhaustion time $\tau \leqslant t^*$, then

$$\int_0^\infty e^{-\beta t}(p_t - c + \beta\eta_-)q_t dt = (p_0 - c + \beta\eta_-)\int_0^\tau e^{-\beta t} q_t dt.$$

Due to the discount factor, it is easy to see that the optimal production rate is equal to $\bar{q}$. So, decreasing production is not optimal.

If $\tau > t^*$, then, owing to $e^{\beta t}(p_t - C + \beta\eta_-) = e^{-\beta t^*}(p_0 - C + \beta\eta_-)$ for $t \geqslant t^*$,

$$\begin{aligned}\int_0^\infty e^{-\beta t}(p_t - c + \beta\eta_-)q_t dt &= \int_0^{t^*} + \int_{t^*}^\infty e^{-\beta t}(p_t - c + \beta\eta_-)q_t dt \\ &= (p_0 - c + \beta\eta_-)\Big(\int_0^{t^*} e^{-\beta t} q_t dt + \int_{t^*}^\infty e^{-\beta t^*} q_t dt\Big) \\ &= (p_0 - c + \beta\eta_-)\Big(\int_0^{t^*} (e^{-\beta t} - e^{-\beta t^*}) q_t dt + e^{-\beta t^*} r\Big).\end{aligned}$$

Similar to the previous case, due to the discount factor, it is never optimal to decrease the production. Thus the lemma follows. □

The next lemma indicates that the exhaustion time $\tau > t^*$.

Lemma II.5. *The exhaustion time $\tau > t^*$, i.e., $\bar{q} < r/t^*$.*

Proof. If $\tau \leqslant t^*$, then from the above lemma, the optimization problem is equivalent to

$$\max_{\bar{q}\geqslant 0} \int_0^\tau e^{-\beta t}[p_0 - (C - \beta\eta_-)]\bar{q} dt - (\eta_- + \eta_+)\bar{q}.$$

subject to $r/\bar{q} \leqslant t^*$, or equivalently, $r/t^* - \bar{q} \leqslant 0$. Define the Lagrangian as follows

$$L(\lambda, \bar{q}) := \int_0^{\frac{r}{\bar{q}}} e^{-\beta t}[p_0 - (C - \beta\eta_-)]\bar{q} dt - (\eta_- + \eta_+)\bar{q} + \lambda\Big(\frac{r}{t^*} - \bar{q}\Big).$$

The Kuhn-Tucker first order and complementary slackness conditions for $\bar{q}$ are

$$\begin{aligned}&\frac{\partial L}{\partial \bar{q}} = \frac{1}{\beta}(p_0 - \beta\eta_+ - C) - \lambda - \frac{p_0 - (C - \beta\eta_-)}{\beta} e^{-\frac{\beta r}{\bar{q}}}\left(1 + \frac{\beta r}{\bar{q}}\right) = 0, \\ &\frac{r}{t^*} - \bar{q} \leqslant 0, \quad \lambda \geqslant 0, \quad \lambda\Big(\frac{r}{t^*} - \bar{q}\Big) = 0.\end{aligned}$$

Rearranging the above first equation gives

$$e^{-\frac{\beta r}{\bar{q}}}\left(1 + \frac{\beta r}{\bar{q}}\right) = 1 - \frac{\beta(\eta_- + \eta_+ + \lambda)}{p_0 - (C - \beta\eta_-)}.$$

If $\lambda > 0$ then $r/t^* - \bar{q} = 0$. On the other hand, recalling the definition of R_I and $p_0 = U'(q_0)$, and noting the function $e^{-t}(1+t)$ is strictly decreasing in $(0,\infty)$, we have

$$\frac{r}{\bar{q}} > \frac{R_I(q_0) - R_D(q_0)}{q_0} = t^*,$$

a contradiction. If $\lambda = 0$ then $r/t^* - \bar{q} < 0$. However, when $\lambda = 0$, the same argument implies $r/\bar{q} = t^*$, a contradiction. Thus, the lemma follows. □

Having described the basic profile of the optimal strategies, the following lemma shows that the optimal production clears the market.

Lemma II.6. *Given the price $\{p_t\}_{t\geqslant 0}$, $q_t = P^{-1}(p_t) = Q^*_t$ is an optimal production strategy, and the maximum profit is $e^{-\beta t^*}[p_0 - (C - \beta\eta_-)]r$.*

Proof. First, we show that the maximum profit is $e^{-\beta t^*}[p_0 - (C - \beta\eta_-)]r$. Since $\tau > t^*$ from the above lemma, the optimization problem (II.1) is equivalent to

$$\max_{\bar{q},\{q_t\}_{t\geqslant t^*}} \int_0^{t^*} e^{-\beta t}(p_0 - C + \beta\eta_-)\bar{q}dt + \int_{t^*}^{\infty} e^{-\beta t}(p_t - C + \beta\eta_-)q_t dt - (\eta_- + \eta_+)\bar{q},$$

subject to $0 \leqslant q_t \leqslant \bar{q}$ and nonincreasing q_t for $t \geqslant t^*$, and

$$\bar{q}t^* < r, \quad \int_{t^*}^{\infty} q_t dt + \bar{q}t^* = r. \tag{II.3}$$

A further calculation shows that

$$\begin{aligned}
& \max_{\bar{q},\{q_t\}_{t\geqslant t^*}} \int_0^{t^*} e^{-\beta t}(p_0 - C + \beta\eta_-)\bar{q}dt + \int_{t^*}^{\infty} e^{-\beta t}(p_t - C + \beta\eta_-)q_t dt - (\eta_- + \eta_+)\bar{q} \\
= \; & \max_{\bar{q}} (p_0 - c + \beta\eta_-)\Big(\int_0^{t^*} (e^{-\beta t} - e^{-\beta t^*})\bar{q}dt + e^{-\beta t^*} r\Big) - (\eta_- + \eta_+)\bar{q} \\
= \; & \max_{\bar{q}} \frac{p_0 - (C - \beta\eta_-)}{\beta}\Big(1 - \frac{\beta(\eta_- + \eta_+)}{p_0 - C + \beta\eta_-} - e^{-\beta t^*}(1 + \beta t^*)\Big)\bar{q} + e^{-\beta t^*}(p_0 - C + \beta\eta_-)r \\
= \; & e^{-\beta t^*}[p_0 - (C - \beta\eta_-)]r,
\end{aligned}$$

since, by the definition of t^*,

$$e^{-\beta t^*}(1 + \beta t^*) - 1 + \frac{\beta(\eta_- + \eta_+)}{p_0 - (C - \beta\eta_-)} = 0.$$

Thus, the maximum profit is $e^{-\beta t^*}[p_0 - (C - \beta\eta_-)]r$.

In addition, from the above arguments, we know that any strategy satisfying conditions in (II.3) is an optimal production strategy. In particular, $q_t = P^{-1}(p_t) = Q_t^*$ for $t \geqslant 0$ is an optimal one and clears the market. □

The Proof of Theorem 1. It follows immediately from Lemmas II.3-II.6. □

III The Proof of Theorem 4

In this appendix, we consider the extended model and prove Theorem 4.

First, by using Dynamic Programming Principle and the viscosity solution approach, one can show that the value function $\tilde{V}$ defined in (11) is a unique viscosity solution of the variational inequality (A.13) (see, e.g., Shreve and Soner (1994)). However, to define the proposed optimal strategy in Theorem 4, certain regularity of both value function and associated free boundaries is required. It is well-known that the regularity of free boundary problems is quite challenging especially in high dimensional cases. Therefore, we leave this as an open problem and proceed by explicitly assuming this regularity is satisfied.

Second, based on the concavity of the value function $\tilde{V}$ in (r, q) variables (see Lemma III.1 below), the whole region can be divided into the three regions defined in (A.14)–(A.16). We can use a similar verification argument as in the deterministic case to show that the optimal strategy $\{I_t^*, D_t^*, W_t^*\}_{t\geqslant 0}$ proposed in Theorem 4 is indeed optimal. Therefore, the social planner's problem admits a solution.

III.1 Concavity of the Value Function

Let us begin with a basic property of the value function.

Lemma III.1. *Assume that the discovery rate f is concave in both arguments. Then, the value function $\tilde{V}$ defined by (11) is concave in (r, q, y).*

Proof. The proof is similar to the Lemma I.1. First, let us fix $\{I_t^1, D_t^1, W_t^1\}_{t\geqslant 0} \in \tilde{\mathcal{A}}(r^1, q^1, x, y^1)$

and $\{I_t^2, D_t^2, W_t^2\}_{t\geqslant 0} \in \tilde{\mathcal{A}}(r^1, q^1, x, y^1)$. Then, for $\omega \in (0,1)$, we construct a new strategy

$$\{I_t^\omega, D_t^\omega, W_t^\omega\}_{t\geqslant 0} = \omega\{I_t^1, D_t^1, W_t^1\}_{t\geqslant 0} + (1-\omega)\{I_t^2, D_t^2, W_t^2\}_{t\geqslant 0}.$$

From the properties of $\{I_t^1, D_t^1, W_t^1\}_{t\geqslant 0}$ and $\{I_t^2, D_t^2, W_t^2\}_{t\geqslant 0}$, $\{I_t^\omega, D_t^\omega\}_{t\geqslant 0}$ are non-decreasing, non-negative, and right-continuous-left-limit processes, which implies condition (a) in the definition of $\tilde{\mathcal{A}}(r,q,x,y)$ holds. Similarly, the condition (c) holds, that is, $\tilde{W}^\omega$ is non-negative and adapted process.

Recall that, in the basic model, we have a very strong relationship that $(R_t^\omega, Q_t^\omega) = \omega(R_t^1, Q_t^1) + (1-\omega)(R_t^2, Q_t^2)$ due to the linearity of the dynamics. This in turn implies $R_t^\omega \geqslant 0$. Here, $Q_t^\omega = \omega Q_t^1 + (1-\omega)Q_t^2$ still holds due to the same reason. However, the nonlinearity of the new discovery rate $f(y,w)$ makes the dynamics nonlinear. Thus, the previous strong relation dose not hold. Indeed, from the dynamics (9), we have

$$d\Big(Y_t^\omega - \omega Y_t^1 - (1-\omega)Y_t^2\Big) = \Big(f(Y_t^\omega, W_t^\omega) - \omega f(Y_t^1, W_t^1) - (1-\omega) f(Y_t^2, W_t^2)\Big)dt.$$

Since $Y_0^\omega = Y_0^1 = Y_0^2$ and $f(y,w)$ is concave,

$$d(Y_0^\omega - \omega Y_0^1 - (1-\omega)Y_0^2) > 0.$$

This in turn implies that $Y_t^\omega - \omega Y_t^1 - (1-\omega)Y_t^2 > 0$ for a sufficiently small positive time t. Thanks to concavity of f, we claim that $Y_t^\omega - \omega Y_t^1 - (1-\omega)Y_t^2 \geqslant 0$ for all $t \geqslant 0$. To see this, define

$$T = \sup\{s > 0 \mid Y_t^\omega - \omega Y_t^1 - (1-\omega)Y_t^2 \geqslant 0 \quad \text{for all } t \in [0,s)\}.$$

That is, T is the largest time that the claim holds. Suppose $T < \infty$, then at T, we have

$$\begin{aligned}
& d\Big(Y_T^\omega - \omega Y_T^1 - (1-\omega)Y_T^2\Big) \\
= \; & \Big(f(Y_T^\omega, W_T^\omega) - \omega f(Y_T^1, W_T^1) - (1-\omega) f(Y_T^2, W_T^2)\Big)dt \\
= \; & \Big(f(\omega Y_T^1 + (1-\omega)Y_T^2, W_T^\omega) - \omega f(Y_T^1, W_T^1) - (1-\omega) f(Y_T^2, W_T^2)\Big)dt \\
\geqslant \; & 0.
\end{aligned}$$

Thus, the claim holds for a larger T', which contradicts the definition of T. This in turn implies $R_t^\omega \geqslant \omega R_t^1 + (1-\omega) R_t^2$ for all $t \geqslant 0$. Therefore, conditions (b) and (d) in the definition of $\tilde{\mathcal{A}}(r,q,x,y)$ hold. Thus, $\{I_t^\omega, D_t^\omega, W_t^\omega\}_{t\geqslant 0} \in \tilde{\mathcal{A}}(r,q,x,y)$.

Then, we have

$$
\begin{aligned}
& \tilde{V}(r^\omega, q^\omega, x, y^\omega) \\
\geqslant\ & \mathbb{E}\Big[\int_0^\infty e^{-\beta t}\Big((\tilde{U}(X_t, Q_t^\omega) - C(R_t^\omega)Q_t^\omega - \tilde{C}(W_t^\omega))dt - \eta_+ dI_t^\omega - \eta_- dD_t^\omega\Big)\Big] \\
\geqslant\ & \mathbb{E}\Big[\int_0^\infty e^{-\beta t}\Big((\tilde{U}(X_t, Q_t^\omega) - C(\omega R_t^1 + (1-\omega)R_t^2)Q_t^\omega - \tilde{C}(W_t^\omega))dt - \eta_+ dI_t^\omega - \eta_- dD_t^\omega\Big)\Big] \\
\geqslant\ & \omega\mathbb{E}\Big[\int_0^\infty e^{-\beta t}\Big((\tilde{U}(X_t, Q_t^1) - C(R_t^1)Q_t^1 - \tilde{C}(W_t^1))dt - \eta_+ dI_t^1 - \eta_- dD_t^1\Big)\Big] + \\
& (1-\omega)\mathbb{E}\Big[\int_0^\infty e^{-\beta t}\Big((\tilde{U}(X_t, Q_t^2) - C(R_t^2)Q_t^2 - \tilde{C}(W_t^2))dt - \eta_+ dI_t^2 - \eta_- dD_t^2\Big)\Big],
\end{aligned}
$$

where the second inequality comes from the decreasing property of C, and the third one is due to the convexity of C and $\tilde{C}$. Therefore, arbitrariness of the strategies implies that $\tilde{V}$ is concave in (r,q,y). □

III.2 Verification

Under the conditions stated in Theorem 4, the verification argument is similar to that in Davis and Norman (1990) and Dai et al. (2015). Here we only provide the main steps.

For an arbitrary initial state $(r,q,x,y) \in [0,\infty)^4$, and an arbitrary (r,q,x,y)-admissible strategy $\{I_t, D_t, W_t\}_{t\geqslant 0}$, denote (R_t, Q_t, X_t, Y_t) the corresponding processes under the strategy $\{I_t, D_t, W_t\}$ with initial value $(R_0, Q_{0^-}, X_0, Y_0) = (r,q,x,y)$. Since φ is a classic solution the HJB equation

(A.13) and the optimal boundaries $\tilde{\Gamma}_I$ and $\tilde{\Gamma}_D$ are smooth, we can directly use Ito's lemma. Thus,

$$\begin{aligned}
\varphi(r,q,x,y) &= e^{-\beta t}\varphi(R_t,Q_t,X_t,Y_t) - \Big\{ \int_0^t e^{-\beta s}\Big[-\beta\varphi(R_s,Q_s,X_s,Y_s)ds \\
&\quad +\varphi_r(R_s,Q_s,X_s,Y_s)dR_s + \frac{1}{2}\varphi_{rr}(R_s,Q_s,X_s,Y_s)(dR_s)^2 \\
&\quad +\varphi_x(R_s,Q_s,X_s,Y_s)dX_s + \frac{1}{2}\varphi_{xx}(R_s,Q_s,X_s,Y_s)(dX_s)^2 \\
&\quad +\varphi_y(R_s,Q_s,X_s,Y_s)dY_s\Big] + \int_0^t e^{-\beta s}\varphi_q(R_s,Q_s,X_s,Y_s)dQ_s^c \\
&\quad + \sum_{0\leqslant s\leqslant t} e^{-\beta s}\Big(\varphi(R_s,Q_s,X_s,Y_s) - \varphi(R_s,Q_{s^-},X_s,Y_s)\Big)\Big\} \\
&= e^{-\beta t}\varphi(R_t,Q_t,X_t,Y_t) - \Big\{ \int_0^t e^{-\beta s}\Big[-\beta\varphi(R_s,Q_s,X_s,Y_s) \\
&\quad [f(Y_s,W_s)-Q_s]\varphi_r(R_s,Q_s,X_s,Y_s) + \frac{1}{2}\sigma_r^2\varphi_{rr}(R_s,Q_s,X_s,Y_s) \\
&\quad \mu(X_s)\varphi_x(R_s,Q_s,X_s,Y_s) + \frac{1}{2}\sigma_x^2(X_s)\varphi_{xx}(R_s,Q_s,X_s,Y_s) \\
&\quad +f(Y_s,W_s)\varphi_y(R_s,Q_s,X_s,Y_s)\Big]ds \\
&\quad + \int_0^t e^{-\beta s}\Big(\varphi_q(R_s,Q_s,X_s,Y_s)dI_s^c + \varphi_q(R_s,Q_s,X_s,Y_s)dD_s^c\Big) \\
&\quad + \sum_{0\leqslant s\leqslant t} e^{-\beta s}\Big(\varphi(R_s,Q_s,X_s,Y_s) - \varphi(R_s,Q_{s^-},X_s,Y_s)\Big) \\
&\quad + \int_0^t \sigma_r\varphi_r(R_s,Q_s,X_s,Y_s)dB_s^r + \int_0^t \sigma_x(X_s)\varphi_x(R_s,Q_s,X_s,Y_s)dB_s^x\Big\}.
\end{aligned}$$

for all $t \geqslant 0$, where Q_t^c, I_t^c and D_t^c are the continuous parts of each counterpart.

Note that, by taking expectation, the last two terms regarding stochastic integral are zero since they are martingales. Moreover, for the each term in the summation,

$$\begin{aligned}
\varphi(R_s,Q_s,X_s,Y_s) - \varphi(R_s,Q_{s^-},X_s,Y_s) &= \varphi(R_s,Q_{s^-}+\Delta Q_s,X_s,Y_s) - \varphi(R_s,Q_{s^-},X_s,Y_s) \\
&= \Delta Q_s \int_0^1 \varphi_q(R_s,Q_{s^-}+\lambda\Delta Q_s,X_s,Y_s)d\lambda,
\end{aligned}$$

where ΔQ_s is the jump at time s, and is equal to $\Delta I_s - \Delta D_s$. Therefore,

$$\begin{aligned}
\varphi(R_s,Q_s,X_s,Y_s) - \varphi(R_s,Q_{s^-},X_s,Y_s) &= (\Delta I_s - \Delta D_s)\int_0^1 \varphi_q(R_s,Q_{s^-}+\lambda\Delta Q_s,X_s,Y_s)d\lambda \\
&\leqslant \Delta I_s \int_0^1 \eta_+ d\lambda + \Delta D_s \int_0^1 d\lambda = \eta_+\Delta I_s + \eta_-\Delta D_s,
\end{aligned}$$

where we have used the fact that $\varphi_q \in [-\eta_-, \eta_+]$ from the HJB equation.

Then, by using the HJB equation again,

$$\begin{aligned}\varphi(r,q,x,y) &\geqslant \mathbb{E}e^{-\beta t}\varphi(R_t,Q_t,X_t,Y_t)\\ &\quad +\mathbb{E}\int_0^t e^{-\beta s}\Big([\tilde{U}(X_s,Q_s)-C(R_s)Q_s-\tilde{C}(W_s)]ds-\eta_+dI_s-\eta_-dD_s\Big).\end{aligned}$$

By sending $t\to\infty$, and using the following transversality condition

$$\lim_{t\to\infty} e^{-\beta t}\varphi(R_t,Q_t,X_t,Y_t)=0,$$

we have

$$\varphi(r,q,x,y)\geqslant \mathbb{E}\int_0^\infty e^{-\beta t}\Big([\tilde{U}(X_s,Q_s)-C(R_s)Q_s-\tilde{C}(W_s)]dt-\eta_+dI_t-\eta_-dD_t\Big).$$

Thus $\varphi(r,q,x,y)\geqslant \tilde{V}(r,q,x,y)$ follows from the arbitrariness of the strategy.

On the other hand, using the same calculation as before but with the strategy $\{I_t^*, D_t^*, W_t^*\}$ given by (A.19), (A.20), and (A.21), we have equalities instead of inequalities, i.e.,

$$\varphi(r,q,x,y)=\mathbb{E}\int_0^\infty e^{-\beta t}\Big([\tilde{U}(X_t^*,Q_t^*)-C(R_t^*)Q_t^*-\tilde{C}(W_t^*)]dt-\eta_+dI_t^*-\eta_-dD_t^*\Big).$$

Thus, $\varphi(r,q,x,y)\leqslant \tilde{V}(r,q,x,y)$ follows from the definition of the value function. Therefore, $\tilde{V}(r,q,x,y)=\varphi(r,q,x,y)$ for all $(r,q,x,y)\in[0,\infty)^4$, and the extraction policy defined by (A.19), (A.20), and (A.21) is the socially optimal extraction path. Since the value function is the unique solution to the HJB equation (A.13), the uniqueness of the optimal adjustment strategy follows from the unique structure of this solution. Therefore, Theorem 4 follows.

IV The Proof of Theorem 2

The proof is similar to the deterministic case, which is an application of the second welfare theorem.

In the presence of uncertainty, we begin with a formal statement of a probability space (Ω, $\mathcal{F}$, $\{\mathcal{F}\}_{t\geqslant 0}$, $\mathbb{P}$), where the filtration $\{\mathcal{F}\}_{t\geqslant 0}$ is generated by the two Brownian motions $\{B_t^x, B_t^r\}_{t\geqslant 0}$.

Then, the commodity space is defined by

$$L := \left\{ \{\ell_t\}_{t\geqslant 0} \;\middle|\; \begin{array}{l} \ell_t \text{ is non-negative, right-continuous-left-limit and adapted process, with} \\ \ell_{0^-} \geqslant 0 \text{ is given and } ||\ell|| := \sup_{t\geqslant 0} e^{-\beta t}|\ell_t| < \infty \ a.s. \end{array} \right\}.$$

For consumers, the commodity points set $\mathbf{X} := L^5$. For firms, the production set is defined as

$$\mathbf{Y} := \left\{ \{Q_t, C_t, \tilde{C}_t, I_t, D_t\}_{t\geqslant 0} \in \mathbf{X} \;\middle|\; \begin{array}{l} (R_0, Q_{0^-}, X_0, Y_0) \in [0,\infty)^4, \\ (I_t, D_t, W_t) \in \tilde{\mathcal{A}}(R_0, Q_{0^-}, X_0, Y_0), \\ (R_t, Q_t, X_t, Y_t) \text{ satisfy (10), (2), (8), and (9)}, \\ C_t \geqslant C(R_t, Q_t), \quad \tilde{C}_t \geqslant \tilde{C}(W_t) \end{array} \right\}.$$

The interpretation of a point $(Q, C, \tilde{C}, I, D)$ in the commodity space is that $(Q_t, C_t, \tilde{C}_t, I_t, D_t)$ represent respectively the level of production, cost of production, cost of exploration, cumulative upward adjustment cost, and cumulative downward adjustment cost at instant time of t.

Therefore, our economy consists of a single consumer whose preference, given by

$$u(\{Q_t, C_t, \tilde{C}_t, I_t, D_t\}) = \mathbb{E}\int_0^\infty e^{-\beta t}\Big([\tilde{U}(Q_t, X_t) - C_t - \tilde{C}_t]dt - \eta_+ dI_t - \eta_- dD_t\Big),$$

is defined on $\mathbf{X}$, and a single firm, whose production set is $\mathbf{Y}$.

Consequently, an allocation (Q, C, I, D) is *Pareto optimal* in this economy if

($\mathbf{P}$') $u(\{Q_t, C_t, \tilde{C}_t, I_t, D_t\}) \geqslant u(\{Q'_t, C'_t, \tilde{C}'_t, I'_t, D'_t\})$ for all $(Q', C', \tilde{C}'_t, I', D') \in \mathbf{Y}$.

And, a *competitive equilibrium* is defined as an allocation $(Q, C, \tilde{C}, I, D)$ together with a price process $p = \{p_t\}_{t\geqslant 0} \in L$ such that

($\mathbf{E}$1') $(Q, C, \tilde{C}, I, D) \in \mathbf{Y}$,

($\mathbf{E}$2') $p_t = X_t P(Q_t)$ for all $t \geqslant 0$,

($\mathbf{E}$3') $\mathbb{E}\int_0^\infty e^{-\beta t}\Big([p_t Q_t - C_t - \tilde{C}_t]dt - \eta_+ dI_t - \eta_- dD_t\Big) \geqslant \mathbb{E}\int_0^\infty e^{-\beta t}\Big([p_t Q'_t - C'_t - \tilde{C}'_t]dt - \eta_+ dI'_t - \eta_- dD'_t\Big)$ for all $(Q', C', \tilde{C}', I', D') \in \mathbf{Y}$.

Now using the same argument for Lemma II.1 we have the following result.

Lemma IV.1 (The First Welfare Theorem)**.** *Suppose* $(Q, C, \tilde{C}, I, D, p)$ *is a competitive equilibrium, then* $(Q, C, \tilde{C}, I, D)$ *is Pareto optimal.*

However, for the reverse one, one needs some careful calculations.

Lemma IV.2 (The Second Welfare Theorem)**.** *Suppose that the discovery rate* f *is concave, and that* $(Q, C, \tilde{C}, I, D)$ *is Pareto optimal, then* $(Q, C, \tilde{C}, I, D)$ *together with prices* p *defined in* (**E**2') *is a competitive equilibrium.*

Proof. Since (**E**1') and (**E**2') are satisfied automatically, we only need to show that (**E**3') holds.

For this, first, we show that the production set $\mathbf{Y}$ is convex. For any initial state $(r, q, x, y) \in [0, \infty)^4$ and any two points $(Q^i, C^i, \tilde{C}^i, I^i, D^i) \in \mathbf{Y}$ for $i = 1, 2$, we define

$$(Q^\omega, C^\omega, \tilde{C}^\omega, I^\omega, D^\omega) = \omega(Q^1, C^1, \tilde{C}^1, I^1, D^1) + (1-\omega)(Q^2, C^2, \tilde{C}^2, I^2, D^2).$$

From the proof of Lemma III.1, we know $\{I_t^\omega, D_t^\omega, W_t^\omega\}_{t \geqslant 0} \in \tilde{\mathcal{A}}(r, q, x, y)$.

Moreover,

$$\begin{aligned} C_t^\omega &= \omega C_t^1 + (1-\omega) C_t^2 = \omega C(R_t^1) Q_t^1 + (1-\omega) C(R_t^2) Q_t^2 \\ &\geqslant C(\omega R_t^1 + (1-\omega) R_t^2) Q_t^\omega \geqslant C(R_t^\omega) Q_t^\omega, \end{aligned}$$

for all $t \geqslant 0$, where the last two inequalities come from convexity and decreasing of production costs function C, respectively. Also, the same argument leads to that

$$\tilde{C}_t^\omega = \omega \tilde{C}_t^1 + (1-\omega)\tilde{C}_t^2 = \omega \tilde{C}(W_t^1) + (1-\omega)\tilde{C}(W_t^2) \geqslant \tilde{C}(W_t^\omega),$$

for all $t \geqslant 0$. Hence, $(Q^\omega, C^\omega, \tilde{C}^\omega, I^\omega, D^\omega) \in \mathbf{Y}$ by the definition and $\mathbf{Y}$ is convex.

Now, for any $(Q', C', \tilde{C}', I', D') \in \mathbf{Y}$ and $\omega \in [0, 1]$, define

$$\begin{aligned} &(Q^\omega, C^\omega, \tilde{C}^\omega, I^\omega, D^\omega) = \omega(Q', C', \tilde{C}', I', D') + (1-\omega)(Q, C, \tilde{C}, I, D) \in \mathbf{Y}, \\ &g(\omega) = \int_0^\infty e^{-\beta t}\Big([\tilde{U}(X_t, Q_t^\omega) - C_t^\omega - \tilde{C}_t^\omega]dt - \eta_+ dI_t^\omega - \eta_- dD_t^\omega\Big). \end{aligned}$$

Since $(Q, C, \tilde{C}, I, D)$ is Pareto optimal,

$$
\begin{aligned}
0 \geqslant g'(0) \quad = \quad & \mathbb{E} \int_0^{\infty} e^{-\beta t} \Big([\tilde{U}_q(X_t, Q_t)(Q'_t - Q_t) - (C'_t - C_t) - (\tilde{C}'_t - \tilde{C}_t)] dt \\
& -\eta_+ d(I'_t - I_t) - \eta_- d(D'_t - D_t) \Big).
\end{aligned}
$$

Then, by the definition of p in (**E**2'), (**E**3') follows immediately from the above inequality. This completes the proof. □

Therefore, the proof of Theorem 2 follows immediately from the above lemma.