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Modeling Large Societies: Why Countable Additivity Is Necessary*

M. Ali Khan†, Lei Qiao‡, Kali P. Rath§ and Yeneng Sun¶

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Abstract: The economic literature with a measure space of agents is enormous, where one usually works with an atomless countably additive measure space to model the interaction of agents in a large society. However, there have been a number of attempts to drop the countable additivity assumption by working with a finitely (but non-countably) additive measure space (such as the set of natural numbers with a density charge). The main purpose of this paper is to illustrate the necessity of countable additivity in modeling a large society in terms of existence of equilibrium and its idealized limit property in both general equilibrium and game theory. In addition, we point out that in the setting of atomless finitely additive agent spaces, even approximate equilibria may not exist in general, but do so only with additional assumptions.

Keywords: Measure space, countable additivity, finite additivity, competitive equilibrium, Nash equilibrium, existence of equilibrium, idealized limit property, approximate competitive equilibrium, approximate Nash equilibrium.

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1 Introduction

A vast literature in economics is based on the interaction of agents in a large society (often called a continuum of agents). The standard mathematical set-up involves an atomless *countably additive* measure space of agents with the Lebesgue unit interval as the archetype agent space.[1] However, there have been a number of attempts to relax the countable additivity assumption to finite additivity[2] so that one could work with an atomless *finitely additive* measure space of agents such as the countable set of natural numbers with a density charge.[3] The aim of this paper is to demonstrate some fundamental problems that arise in the modeling large societies via finitely (but non-countably) additive measure spaces.

An atomless measure space of agents is introduced as an idealized limit model for the interaction of a large but finitely many economic agents. A bare minimal consistency requirement would be that such an idealized model possesses an equilibrium and captures the limiting behavior of some corresponding large finite models. This paper examines the validity of these two properties for models with a finitely additive agent space in the simplest settings of economies and games with many agents.

First, we consider the existence issue. When a finite-agent economy satisfies the usual conditions such as continuity, convexity and monotonicity, it is well-known that competitive equilibria exist in the economy.[4] However, if a finite-agent space is replaced by an agent space based on the set of natural numbers with a density charge, Example 1 below shows the nonexistence of a competitive equilibrium for an economy satisfying those usual conditions. In the setting of non-cooperative games with finitely many agents, it is again a standard result that mixed strategy Nash equilibrium exists. However, when one works with an agent

[1] For some classical references, see, for example, Milnor and Shapley (1961), Aumann (1964) and Hildenbrand (1974).

[2] See, for example, Brown (1974), Weiss (1981), Armstrong and Richter (1984), Feldman and Gilles (1985, Subsection IIC), Hammond (1999, Footnote 2), and the many references quoted in Footnote 18 below.

[3] Let $\mathbb{N}$ be the set of positive integers and $\mathcal{P}(\mathbb{N})$ its power set. A finitely additive measure μ on $\mathcal{P}(\mathbb{N})$ is said to be a density charge if for any $A \in \mathcal{P}(\mathbb{N})$, $\mu(A) = \lim_{n\to\infty} \frac{|\{1,\ldots,n\}\cap A|}{n}$ whenever the limit exists, where $|\{1,\ldots,n\}\cap A|$ is the number of elements in the set $\{1,\ldots,n\}\cap A$. That is, μ extends the usual notion of asymptotic density.

[4] More general existence results can be found in Arrow and Debreu (1954) and McKenzie (1954), and the rich literature that follows them.

space based on the set of natural numbers with a density charge as in Example 1, Example 2 presents a game with continuous payoff functions without any mixed strategy Nash equilibria.

Hildenbrand (1970, p. 162) pointed out that the relevance of "ideal economies" with infinitely many agents to finite-agent economies has to be established because the interest in the ideal model depends on its link with the large but finite case. The simplest way to link a sequence of economies with large but finitely many agents to an atomless economy is: (1) to take a sequence of equal partitions of the atomless agent space corresponding to the sequence of agent numbers, (2) for each finite-agent economy and the corresponding partition, the agents in the partition sets replicate the agents' characteristics in the finite-agent economy, and (3) the replicated sequence of characteristics converge pointwise to the atomless economy.[5] For a sequence of competitive equilibria corresponding to the sequence of finite-agent economies, one can also produce a sequence of allocations for the atomless economy by replicating agents' consumptions in the finite-agent economies as in (2). If the replicated sequence of allocations for the atomless economy converge pointwise to an allocation, it seems to be a minimal requirement for the "ideal" atomless economy to retain the equilibrium property for the limit allocation. We call this requirement the idealized limit property.[6]

Next, we consider the idealized limit property of finitely additive agent spaces. It is surprising that such a simple property fails when the set of natural numbers with a density charge is taken to be the agent space. In particular, Example 3 presents a sequence of finite-agent economies with continuous, concave and monotone utility functions and a corresponding sequence of competitive equilibria which respectively converge pointwise to a limit economy and a limit allocation; however, the limit allocation is not a competitive equilibrium of the limit economy. Example 4 demonstrates the failure of the idealized limit property also in the setting of games with countably many agents and continuous payoffs.

In sum, Examples 1, 2, 3 and 4 consider an agent space based on the set of natural numbers with a density charge for both economies and games. The next question to ask is

[5]See, for example, Hildenbrand (1974, p. 139) for such procedural replications.

[6]See Definition 4 below for the details.

what happens if one works with other types of finitely additive measure spaces of agents. Theorem 1 and Theorem 2 below respectively show that for economies and games with finitely additive measure spaces of agents to have either equilibria or the idealized limit property, the countable additivity condition for the underlying measure spaces is not only sufficient but also necessary.

Given that an exact equilibrium may not exist on an atomless finitely additive agent space, we consider in Appendix B the natural question as to whether an approximate equilibrium exists in the setting. Since an atomless finitely additive agent space can be decomposed into finitely many subspaces of agents with an arbitrary small size, and that exact equilibria exist in the finite-agent settings,[7] one expects that approximate equilibria should exist in an atomless setting. However, it is again rather surprising that approximate equilibria do not exist in general! Examples 5 and 6 respectively establish this for economies and games with countably many agents and continuous payoffs. Propositions 1 and 2 do establish the existence of approximate equilibria under further assumptions.

The paper is organized as follows. Section 2 includes some mathematical preliminaries. In Section 3, we present examples to show the nonexistence of a competitive and a Nash equilibriium in well-behaved economies and games. Section 4 demonstrates the failure of the idealized limit property of competitive equilibria and Nash equilibria when the underlying agent space is the set of natural numbers with a density charge. Section 5 shows the necessity and sufficiency of countable additivity in terms of both the existence of equilibria and the idealized limit property for both economies and games. Section 6 discusses the literature and concludes the paper. The proofs of the main results are given in Appendix A. The existence issue of approximate equilibria is considered in Appendix B.

[7] In fact, it is easy to show that an equilibrium exists in an economy, or in a game with an atomless finitely additive agent space if there are only finitely many different characteristics for all the agents; see Lemmas 1 and 2 in Appendix B below.

2 Mathematical Preliminaries

Let T be a nonempty set and $\mathcal{T}$ a σ-algebra of subsets of T. Since T will be used to model the space of agents in this paper, we assume $\{t\} \in \mathcal{T}$ for any $t \in T$ to allow for the measurability of a single agent. When T is the set of positive integers $\mathbb{N}$, its σ-algebra is the power set $\mathcal{P}(\mathbb{N})$. A set function μ from $\mathcal{T}$ to $[0,1]$ with $\mu(T) = 1$ is said to be a finitely additive measure on $\mathcal{T}$ if for any $A, B \in \mathcal{T}$ with $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$. The measure μ is said to be countably additive if for any sequence $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{T}$, $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$. A finitely additive measure μ is atomless if for every $\epsilon > 0$, there exists a $\mathcal{T}$-measurable partition $\{F_1, \ldots, F_n\}$ of T such that $\mu(F_i) < \epsilon$ for every i.[8] The triple $(T, \mathcal{T}, \mu)$ will be called a finitely (respectively, countably) additive measure space if μ is a finitely (respectively, countably) additive measure.[9]

A function f from T to a separable, metric space X is measurable if for any Borel set B in X, $f^{-1}(B) = \{t \in T : f(t) \in B\}$ is in $\mathcal{T}$. For a real valued measurable function on a finitely additive measure space, the integral is as developed in Bhaskara Rao and Bhaskara Rao (1983, Ch. 4).[10] When the underlying measure is countably additive, this notion of integrability is equivalent to the standard definition of integrability as in Loeb (2016, Ch. 6). For a function taking values in the L-dimensional Euclidean space $\mathbb{R}^L$, the integral is the vector whose components are integrals of the component functions.

[8]When μ is countably additive, this definition of μ being atomless is equivalent to the more conventional definition that for any $A \in \mathcal{T}$ with $\mu(A) > 0$, there exists a $B \in \mathcal{T}$ with $B \subseteq A$ and $0 < \mu(B) < \mu(A)$.

[9]Note that a finitely additive measure on an algebra $\mathcal{F}$ of subsets of T can always be extended to a finitely additive measure on the σ-algebra generated by $\mathcal{F}$; see Bhaskara Rao and Bhaskara Rao (1983, Theorem 3.2.5). The same extension result for the case of a countably additive measure is the classical Carathéodory extension theorem; see Loeb (2016, Theorem 10.2.1). In fact, as noted in the Introduction, a main motivation for working with a finitely additive agent space is to allow any subset of agents to be measurable, namely, $\mathcal{T}$ is the power set of T (which is a σ-algebra). Thus, there is no loss of generality by assuming $\mathcal{T}$ to be a σ-algebra.

[10]A real valued function h on T is said to be simple if it can be expressed as $\sum_{i=1}^{k} c_i \mathbf{1}_{A_i}$, where k is a positive integer, c_i a real number, A_i a measurable set in $\mathcal{T}$, $\mathbf{1}_{A_i}$ the indicator function of A_i in T; the integral $\int_T h d\mu$ of h is simply $\sum_{i=1}^{k} c_i \mu(A_i)$. A real valued measurable function g on T is said to be integrable on the finitely additive measure space $(T, \mathcal{T}, \mu)$ if there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of simple functions such that $\lim_{m,n \to \infty} \int_T |g_n - g_m| d\mu = 0$, and for any $\epsilon > 0$, $\lim_{n \to \infty} \mu(\{t \in T : |g_n(t) - g(t)| > \epsilon\}) = 0$. The integral of g is then defined to be $\lim_{n \to \infty} \int_T g_n d\mu$; see Bhaskara Rao and Bhaskara Rao (1983, Definition 4.4.11, p. 104). A bounded real-valued measurable function is integrable since it can be approximated by a sequence of simple functions uniformly.

If $x \in \mathbb{R}^L$ then $\|x\| = \sum_{i=1}^{L} |x_i|$. If $x, y \in \mathbb{R}^L$ then $x \geq y$ means $x_i \geq y_i$ and $x \gg y$ means $x_i > y_i$ for all $i = 1, \ldots, L$.

3 Economies, Games and Equilibria

In this section, we define economies and games, and the notions of a competitive and Nash equilibrium over finitely additive measure spaces of agents. We show by means of examples that if the agent space is taken to be the set of natural numbers with a density charge, then an equilibrium may not exist.

3.1 Economies and competitive equilibria

The commodity space is $\mathbb{R}^L_+$. A real valued function u on $\mathbb{R}^L_+$ is strongly monotone if $x \geq y$ and $x \neq y$ imply that $u(x) > u(y)$. Let $\mathcal{U}$ be the space of real valued, continuous and strongly monotone functions on $\mathbb{R}^L_+$, with the compact-open topology; a discussion of this topology can be found in Willard (1970, Section 43). It can be shown that $\mathcal{U}$ is a separable, metric space. An economy specifies for each consumer a utility function $u \in \mathcal{U}$ and an endowment vector $\omega \in \mathbb{R}^L_+$.

Definition 1 Let $(T, \mathcal{T}, \mu)$ be a finitely additive measure space.

(1) An *economy* is a measurable mapping $\mathcal{E} = (u, \omega) : T \longrightarrow \mathcal{U} \times \mathbb{R}^L_+$ such that ω is integrable and $\bar{\omega} = \int_T \omega \, \mathrm{d}\mu \gg 0$.

(2) An *allocation* of $\mathcal{E}$ is an integrable mapping f from T to $\mathbb{R}^L_+$. An allocation is *feasible* if $\int_T f \, \mathrm{d}\mu = \int_T \omega \, \mathrm{d}\mu$.

(3) Given a price vector $p \in \mathbb{R}^L_+ \setminus \{0\}$, the *budget set* of consumer t is $B_t(p) = \{x \in \mathbb{R}^L_+ : p \cdot x \leq p \cdot \omega_t\}$.

(4) A *competitive equilibrium* of $\mathcal{E}$ is a pair (p, f), where $p \in \mathbb{R}^L_+ \setminus \{0\}$, f is a feasible allocation, and for all $t \in T$: (a) $f(t) \in B_t(p)$ and (b) $u_t(f(t)) \geq u_t(x)$ for all $x \in B_t(p)$.

(5) An allocation f of $\mathcal{E}$ is a *competitive allocation* if for some p, (p, f) is a competitive equilibrium.

The following example shows that an economy, whose agent space is the set of natural numbers with a density charge, may not have a competitive equilibrium.

Example 1 Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ be an atomless, finitely additive agent space. Fix $\theta \in [1/2, 1)$. The economy $\mathcal{E} = (u, \omega)$ is defined as follows. For each $t \in \mathbb{N}$,

$$u_t(x_1, x_2) = \frac{t+1}{t} x_1^{\frac{t}{t+1}} + x_2, \qquad \omega_t = (\theta, \theta).$$

Suppose that there is a competitive equilibrium. If $p \in \mathbb{R}^2_+ \setminus \{0\}$ is the equilibrium price vector, then $p \gg 0$ since u_t is strongly monotone for each $t \in \mathbb{N}$. Without loss of generality let $p_1 + p_2 = 1$.

For $t \in \mathbb{N}$, the unique solution of maximize $u_t(x_1, x_2)$ subject to $p_1 x_1 + p_2 x_2 = \theta$ is

$$D_{t1} = \min\left\{\frac{p_2^{t+1}}{p_1^{t+1}}, \frac{\theta}{p_1}\right\}, \qquad D_{t2} = \frac{\theta}{p_2} - \frac{p_1 D_{t1}}{p_2}.$$

We will consider two cases: $p_2/p_1 < 1$ and $p_2/p_1 \geq 1$.

Suppose that $p_2/p_1 < 1$. Then for all $t \in \mathbb{N}$, $D_{t1} = (p_2/p_1)^{t+1}$. Since $(p_2/p_1)^{t+1} \to 0$ as t tends to infinity, given $\epsilon > 0$, $D_{t1} < \epsilon$ for all but finitely many t's. So, $\int_{\mathbb{N}} D_{t1}\, \mathrm{d}\mu \leq \epsilon$, which gives $\int_{\mathbb{N}} D_{t1}\, \mathrm{d}\mu = 0$. Therefore, $\int_{\mathbb{N}} D_{t2}\, \mathrm{d}\mu = \theta/p_2 > \theta = \bar{\omega}_2$, a contradiction.

If $p_2/p_1 \geq 1$, then $D_{t1} \geq \min\{1, 2\theta\} = 1$. So, $\int_{\mathbb{N}} D_{t1}\, \mathrm{d}\mu \geq 1 > \theta = \bar{\omega}_1$, a contradiction.

Thus, $\mathcal{E}$ has no competitive equilibrium.[11]

Remark 1 Weiss (1981, Theorem 3) proved the existence of a competitive equilibrium for economies over atomless, finitely additive agent spaces. Example 1 provides the first counterexample to this result. Let S be the unit simplex in $\mathbb{R}^L_+$ and S° its interior. For any $p \in S^\circ$, denote the mean excess demand (correspondence) by $Z(p) = \int_T [D(p) - \omega]\, \mathrm{d}\mu$. Part (iii) of Weiss (1981, Lemma) claims that Z is upper hemicontinuous. This claim is true in

[11] Example 1 is extended to the case of any finitely (but non-countably) additive agent space in the proof of $(i) \Rightarrow (iii)$ in Theorem 1 below.

the countably additive case, see Hildenbrand (1974, p. 149); but false under finite additivity. In the setting of Example 1, $\int_T D(p)\,\mathrm{d}\mu$ is not upper hemicontinuous, and neither is $Z(p)$. As has been shown, $\int_T D_{t1}\,\mathrm{d}\mu = 0$ if $p_2/p_1 < 1$ and $\int_T D_{t1}\,\mathrm{d}\mu \geq 1$ if $p_2/p_1 \geq 1$. Thus, the claim of upper hemicontinuity fails at $p = (1/2, 1/2)$.

3.2 Games and Nash equilibria

Let $E = \{e^1, \dots, e^L\}$ be the set of unit vectors in $\mathbb{R}^L$ and $S = \{s \in \mathbb{R}^L_+ : \sum_{k=1}^L s_k = 1\}$ the unit simplex. Let $\mathcal{V}$ be the space of real valued continuous functions defined on $E \times S$, endowed with the sup norm metric. It is a complete, separable, metric space. A game assigns a payoff function in $\mathcal{V}$ to each player.

Definition 2 Let $(T, \mathcal{T}, \mu)$ be an atomless, finitely additive measure space.

(1) A *game* is a measurable function $\mathcal{G} : T \longrightarrow \mathcal{V}$.

(2) A *mixed-strategy profile* of $\mathcal{G}$ is a measurable function f from T to S. It is a *pure-strategy profile* when E is substituted for S.[12]

(3) Given a mixed strategy profile g, the *payoff to player* t is $\mathcal{G}(t)\left(g(t), \int_T g\,\mathrm{d}\mu\right) = \sum_{k=1}^L g_k(t)\mathcal{G}(t)\left(e^k, \int_T g\,\mathrm{d}\mu\right)$.

(4) A mixed strategy profile g is a *mixed strategy Nash equilibrium* of $\mathcal{G}$ if for all $t \in T$, $\mathcal{G}(t)\left(g(t), \int_T g\,\mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(a, \int_T g\,\mathrm{d}\mu\right)$ for all $a \in E$. If in addition, g takes values in E, then it is a *pure strategy Nash equilibrium*.

The example below shows the nonexistence of a Nash equilibrium in mixed strategies in atomless, countable-player games.

Example 2 Suppose there are two actions. The set of unit vectors and the unit simplex in $\mathbb{R}^2$ can be identified with $A = \{0, 1\}$ and $K = [0, 1]$ respectively. Any $x \in K$ can be

[12]Here we take the view that S is space of probability distributions on L pure strategies and the extreme points in S are identified with the corresponding pure strategies.

interpreted as the weight on action 1. Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ be an atomless, finitely additive player space. For each $t \in \mathbb{N}$, let the payoff function be

$$\mathcal{G}(t)(a,x) = a\left(\frac{1}{t} - x\right),\ a \in A.$$

We will derive the best responses and show that this game has no Nash equilibrium in pure or mixed strategies.[13]

The best responses are as follows.

$$\text{argmax}_{a\in A}\mathcal{G}(t)(a,x) = \begin{cases} \{0,1\} & \text{if } x = 1/t \\ \{1\} & \text{if } x < 1/t \\ \{0\} & \text{if } x > 1/t. \end{cases}$$

If $x = 1/t$ then $\mathcal{G}(t)(0,x) = \mathcal{G}(t)(1,x) = 0$. If $x < 1/t$ then $\mathcal{G}(t)(0,x) = 0$ and $\mathcal{G}(t)(1,x) = (1/t) - x > 0$ which implies that 1 is the best response. If $x > 1/t$ then $\mathcal{G}(t)(0,x) = 0$ and $\mathcal{G}(t)(1,x) = (1/t) - x < 0$, which implies that 0 is the best response. Notice that given $x \in K$, the best response is a singleton for almost all t.

Suppose that f from $\mathbb{N}$ to K is a (mixed strategy) Nash equilibrium. Let $x = \int_{\mathbb{N}} f \, \mathrm{d}\mu$. If $x = 0$, then $f(t) = 1$ for all $t \in \mathbb{N}$ because of the best response property. This implies that $\int_{\mathbb{N}} f \, \mathrm{d}\mu = 1$, a contradiction. If $x > 0$, then $x > 1/t$ for all but finitely many t's. This implies that $f(t) = 0$ for all but finitely many t's. So, $\int_{\mathbb{N}} f \, \mathrm{d}\mu = 0$, again a contradiction. Thus, there is no Nash equilibrium in pure or even mixed strategies.[14]

[13] When the payoff function of an individual player is not continuous, the player may not be able to find an optimal action. It is thus more or less expected that a countable-player game with discontinuous payoffs may not have any equilibrium; see, for example, Peleg (1969), the discussion in Khan and Sun (2002, pp. 1768–1769), and Voorneveld (2010). To the best of our knowledge, Example 2 provides the first example on the nonexistence of mixed-strategy Nash equilibrium in games with compact action spaces and continuous payoffs.

[14] Example 2 is extended to the case of any atomless finitely (but non-countably) additive agent space in the proof of $(i) \Rightarrow (iii)$ in Theorem 2 below.

4 Idealized Limit Property

As already emphasized in the introduction, a principal motivation for considering economies and games with infinitely many agents is to think in terms of an ideal set-up that furnishes approximations for situations where the number of participants is large but finite. If such a sequence of economies or games, and their equilibria, converge to their idealized limiting counterparts, one would then expect their equilibria to also have corresponding properties to hold in the limiting set-up that they have in the large but finite setting. This is surely a minimal requirement. Along these lines, Hildenbrand (1974, p. 139) examined continuous representations of finite-agent economies. We adopt the same approach by introducing a replication function to construct a replica of $\{1, \ldots, m\}$ on any atomless, finitely additive measure space.

Definition 3 Let $(T, \mathcal{T}, \mu)$ be an atomless, finitely additive measure space. A $\mathcal{T}$-measurable mapping α^m from T to $\{1, \ldots, m\}$ is a *replication function of m-agents* if $\mu\left((\alpha^m)^{-1}(\{i\})\right) = 1/m$ for any $i \in \{1, \ldots, m\}$.

4.1 Large economies

For the idealized limit property of economies, in addition to convergence of preferences and endowments, we require the convergence of competitive equilibria and mean endowments.

Definition 4 An economy $\mathcal{E} = (u, \omega)$ on an atomless, finitely additive measure space $(T, \mathcal{T}, \mu)$ has the *idealized limit property* if

(1) for any sequence $\{\mathbb{E}^n = (u^n, \omega^n)\}_{n=1}^{\infty}$ of finite-agent economies with $\{f^n\}_{n=1}^{\infty}$ as competitive allocations, where the number of agents in $\mathbb{E}^n$ is k_n with $\lim_{n\to\infty} k_n = \infty$,

(2) for any sequence of replication functions $\{\alpha^{k_n}\}_{n=1}^{\infty}$ such that $\{\mathbb{E}^n \circ \alpha^{k_n}\}_{n=1}^{\infty}$ converges to $\mathcal{E}$ pointwise on T, $\{f^n \circ \alpha^{k_n}\}_{n=1}^{\infty}$ converges to some allocation f pointwise on T and $\lim_{n\to\infty} \int_T \omega^n \circ \alpha^{k_n} \, \mathrm{d}\mu = \int_T \omega \, \mathrm{d}\mu$,

then f is a competitive allocation of $\mathcal{E}$.

The following example shows that the idealized limit property may fail for an economy with countably many agents.

Example 3 Let $\mathcal{E} = (u, \omega)$ be the economy constructed in Example 1. Fix any $n \in \mathbb{N}$ and let $\mathbb{E}^n$ be the restriction of $\mathcal{E}$ on $\{1, \ldots, n\}$. The endowment of each agent in $\mathcal{E}$ and $\mathbb{E}^n$ is (θ, θ). For $1 \le k \le n$, let $\{A^n_k\}$ be a partition of $\mathbb{N}$ such that $A^n_k = \{mn + k : m = 0, 1, \ldots\}$, and $\alpha^n(t) = k$ for $t \in A^n_k$. Note that for any $n \ge t$, $t \in A^n_t$ and $\alpha^n(t) = t$. Thus $u^n_{\alpha^n(t)} = u_t$ for any $n \ge t$, which implies that $\{\mathbb{E}^n \circ \alpha^n\}_{n=1}^{\infty}$ converges to $\mathcal{E}$ pointwise.

Since $\mathbb{E}^n$ is a finite-agent economy with continuous, convex and strongly monotone preferences, it has a competitive equilibrium (p_n, f^n) with $p_n \gg 0$; see Arrow and Debreu (1954). Suppose that $p_{n1} + p_{n2} = 1$. For any $k \in \{1, \ldots, n\}$, the unique solution of agent k's utility maximization problem is

$$f^n_1(k) = \min\left\{\frac{p_{n2}^{k+1}}{p_{n1}^{k+1}}, \frac{\theta}{p_{n1}}\right\}, \qquad f^n_2(k) = \frac{\theta}{p_{n2}} - \frac{p_{n1} f^n_1(k)}{p_{n2}}.$$

If $p_{n2}/p_{n1} \ge 1$ then $f^n_1(k) \ge \min\{1, 2\theta\} = 1$. Therefore, $(1/n)\sum_{k=1}^{n} f^n_1(k) \ge 1 > \theta = (1/n)\sum_{k=1}^{n} \omega^n_{k1}$, a contradiction. Hence, $p_{n2}/p_{n1} < 1$.

We will show that $\lim_{n\to\infty}(p_{n2}/p_{n1}) = 1$. If not, then there is $0 < \beta < 1$ and a strictly increasing sequence $\{n_j\}$, $j = 1, 2, \ldots$ such that $p_{n_j 2}/p_{n_j 1} < \beta$ for all n_j.

$$\frac{1}{n_j}\sum_{k=1}^{n_j} f^{n_j}_1(k) \le \frac{1}{n_j}\sum_{k=1}^{n_j} \frac{p_{n_j 2}^{k+1}}{p_{n_j 1}^{k+1}} < \frac{1}{n_j}\sum_{k=1}^{n_j} \beta^{k+1} = \frac{\beta^2(1-\beta^{n_j})}{n_j(1-\beta)} \to 0$$

as $j \to \infty$. Hence, for some n_j, $(1/n_j)\sum_{k=1}^{n_j} f^{n_j}_1(k) < 1/4$. This leads to

$$\begin{aligned} f^{n_j}_2(k) &= \theta + \frac{p_{n_j 1}}{p_{n_j 2}}\left(\theta - f^{n_j}_1(k)\right) \\ \frac{1}{n_j}\sum_{k=1}^{n_j} f^{n_j}_2(k) &= \theta + \frac{p_{n_j 1}}{p_{n_j 2}}\left(\theta - \frac{1}{n_j}\sum_{k=1}^{n_j} f^{n_j}_1(k)\right) > \theta + \frac{p_{n_j 1}}{4 p_{n_j 2}} > \theta, \end{aligned}$$

a contradiction. Therefore, $\lim_{n\to\infty}(p_{n2}/p_{n1}) = 1$.

Since $f^n \circ \alpha^n(t) = f^n(t)$ for $n \ge t$, $f^n_1 \circ \alpha^n(t) \to 1$ and $f^n_2 \circ \alpha^n(t) \to 2\theta - 1$ as $n \to \infty$. However, the limit economy $\mathcal{E} = (u, \omega)$ has no competitive equilibrium, which implies that

$\mathcal{E}$ does not have the idealized limit property.

4.2 Large games

Definition 2 considers a game with an atomless, finitely additive player space. Since an individual player cannot influence the externality component which is the integral of an action profile, the externality component remains unchanged even if a player deviates from her given action in the action profile. When there are only finitely many players, the externality component will change if one player deviates.

Following the spirit of Definition 2, we shall only consider finite-player games where the payoff of a player depends on her own action and the average actions of all the players. An m-player game can be modeled as a mapping $\mathbb{G}$ from the player space $\{1, \ldots, m\}$ to the space $\mathcal{V}$ of payoff functions, as described below.

Let $V = \{v_1, \ldots, v_m\} \subseteq \mathcal{V}$. When player j takes action a_j in E for each $1 \leq j \leq m$, player i's payoff is $\mathbb{G}(i)(a_1, \ldots, a_m) = v_i\left(a_i, (a_1 + \cdots + a_m)/m\right)$. If player i deviates from her action a_i to b_i with other players' actions unchanged, then her new payoff will be $\mathbb{G}(i)(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_m) = v_i\left(b_i, (a_1 + \cdots + a_{i-1} + b_i + a_{i+1} + \cdots + a_m)/m\right)$. The expected payoffs can be defined as usual in the normal form of a finite-player game. Thus, the payoff function $\mathbb{G}(i)$ of player i in the finite-player game can be identified with $v_i \in \mathcal{V}$.

Definition 5 Let $m \geq 2$ be an integer.

(1) An *m-player game* is a mapping $\mathbb{G} : \{1, \ldots, m\} \longrightarrow \mathcal{V}$, where $\mathbb{G}(i)$ is as indicated above.

(2) A *mixed-strategy profile* of $\mathbb{G}$ is a measurable function f from T to S. It is a *pure-strategy profile* when E is substituted for S.

(3) A mixed strategy profile $g = (g(1), \ldots, g(m))$ is a *mixed strategy Nash equilibrium* of $\mathbb{G}$ if for all $i \in \{1, \ldots, m\}$, $\mathbb{G}(i)\,(g) \geq \mathbb{G}(i)\,(g(1), \ldots, g(i-1), a, g(i+1), \ldots, g(m))$ for all $a \in E$. If in addition, g takes values in E, then it is a *pure strategy Nash equilibrium.*

The existence of a mixed strategy Nash equilibrium of $\mathbb{G}$ follows from Nash (1950). For the idealized limit property of large games, we require the convergence of payoff functions and mixed strategy Nash equilibria.

Definition 6 A game $\mathcal{G}$ on an atomless, finitely additive measure space $(T, \mathcal{T}, \mu)$ has the *idealized limit property* if

(1) for any sequence $\{\mathbb{G}^n\}_{n=1}^{\infty}$ of finite-player games with $\{f^n\}_{n=1}^{\infty}$ as mixed strategy Nash equilibria, where the number of players in $\mathbb{G}^n$ is k_n with $\lim_{n\to\infty} k_n = \infty$,

(2) for any sequence of replication functions $\{\alpha^{k_n}\}_{n=1}^{\infty}$ such that $\{\mathbb{G}^n \circ \alpha^{k_n}\}_{n=1}^{\infty}$ converges to $\mathcal{G}$ pointwise on T and $\{f^n \circ \alpha^{k_n}\}_{n=1}^{\infty}$ converges to some mixed strategy profile f pointwise on T,

then f is a mixed strategy Nash equilibrium of $\mathcal{G}$.

The next example shows that the idealized limit property may fail for a game with countably many players.

Example 4 Let $\mathcal{G}$ be the game constructed in Example 2. Fix any $n \in \mathbb{N}$ and let $\mathbb{G}^n$ be the restriction of $\mathcal{G}$ on $\{1, \dots, n^2\}$. For $1 \le k \le n^2$, let $\{A_k^n\}$ be a partition of $\mathbb{N}$ such that $A_k^n = \{mn^2 + k : m = 0, 1, \dots\}$ and $\alpha^{n^2}(t) = k$ for any $t \in A_k^n$. Note that for any $n \ge \sqrt{t}$, $\alpha^{n^2}(t) = t$. So, $\mathbb{G}^n \circ \alpha^{n^2}(t) = \mathcal{G}(t)$ for any $n \ge \sqrt{t}$, which implies that $\{\mathbb{G}^n \circ \alpha^{n^2}\}_{n=1}^{\infty}$ converges to $\mathcal{G}$ pointwise on T.

Fix any $n \ge 2$. Let $f^n(k) = 1$ if $1 \le k \le n$ and $f^n(k) = 0$ if $n < k \le n^2$. Then $x = (1/n^2)\sum_{k=1}^{n^2} f^n(k) = n/n^2 = 1/n$. We will show that f^n is a pure strategy Nash equilibrium of $\mathbb{G}^n$.

For any $1 \le k \le n$, $1/k \ge 1/n = x$. $\mathbb{G}^n(k)(1, 1/n) = (1/k) - (1/n) \ge 0$ and $\mathbb{G}^n(k)(0, 1/(n-1)) = 0$. Thus, $f^n(k) = 1$ is a best response for $1 \le k \le n$.

Similarly, if $n < k \le n^2$, then $1/k \le 1/(n+1)$. $\mathbb{G}^n(k)(0, 1/n) = 0$ and $\mathbb{G}^n(k)(1, 1/(n+1)) = (1/k) - [1/(n+1)] \le 0$. Thus, $f^n(k) = 0$ is a best response for $n < k \le n^2$.

We have shown that f^n is a pure strategy Nash equilibrium of $\mathbb{G}^n$.

Fix any $t \in \mathbb{N}$. For any $n \geq t$, $\alpha^{n^2}(t) = t$, which implies that $f^n \circ \alpha^{n^2}(t) = 1$. So, $f^n \circ \alpha^{n^2}(t) \to 1$ as $n \to \infty$. However, the limit game $\mathcal{G}$ has no mixed strategy Nash equilibrium. So, $\mathcal{G}$ does not have the idealized limit property.

5 Necessity of Countable Additivity

Examples 1 and 3 show the failure of the finitely additive agent space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ in terms of the existence and the idealized limit property for competitive equilibria in large economies. The following theorem goes further in characterizing the validity of these two properties in large economies by countable additivity of the agent space.[15]

Theorem 1 *Let $(T, \mathcal{T}, \mu)$ be an atomless, finitely additive measure space. Then the following are equivalent.*[16]

(i) Every economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ has a competitive equilibrium.

(ii) Every economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ has the idealized limit property.

(iii) $(T, \mathcal{T}, \mu)$ is a countably additive measure space.

We now turn to games. Examples 2 and 4 show the failure of the finitely additive player space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ in terms of the existence and the idealized limit property for Nash equilibria in large games. The next theorem shows the equivalence of countable additivity of the player space with the validity of these properties in large games.

[15]Vind (1964) considered another approach in general equilibrium theory, the so-called coalitional approach, where the primitives are coalitions rather than individual agents. In the setting of atomless, countable additive agent spaces, Debreu (1967) showed the equivalence between this approach and the usual measure-theoretic approach as in Aumann (1964). On the other hand, when the countable additivity is reduced to finite additivity, the coalitional approach in terms of Boolean algebras delivers an existence result on competitive equilibria as in Armstrong and Richter (1986). This shows that in contrast to the equivalence result in Debreu (1967), the usual measure-theoretic and coalitional approaches are not equivalent in the setting of finitely additive agent spaces.

[16]Note that the utility functions in the proofs of $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iii)$ in Theorem 1 in Appendix A.1 are concave. Thus, one can still maintain the equivalence in Theorem 1 when (i) and (ii) are replaced respectively by "*(i') Every economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ with concave utility functions has a competitive equilibrium*" and "*(ii') Every economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ with concave utility functions has the idealized limit property*".

Theorem 2 *Let $(T, \mathcal{T}, \mu)$ be an atomless, finitely additive measure space. Then the following are equivalent.*

(i) Every game $\mathcal{G}$ on $(T, \mathcal{T}, \mu)$ has a mixed strategy Nash equilibrium.

(ii) Every game $\mathcal{G}$ on $(T, \mathcal{T}, \mu)$ has the idealized limit property.

(iii) $(T, \mathcal{T}, \mu)$ is a countably additive measure space.

6 Conclusion

A countably additive measure on a σ-algebra of subsets imposes some restrictions on the underlying measure space, which usually allows the existence of non-measurable sets. On the other hand, a density charge on the agent space of natural numbers is defined on all subsets of the set of natural numbers, which means that any function or set on such an agent space will automatically be measurable. Such a property was emphasized as an advantage in Weiss (1981), Armstrong and Richter (1984, 1986), and presumably others. From the analytic point of view, though countably additive measure spaces allow non-measurable sets, the whole subject of real analysis has developed many tools for dealing with measurability issues so that one can work within the framework of measurable functions for all kinds of applications. In contrast, the lack of countably additivity for a density charge prevents one from using the tools in real analysis and leads to the non-existence of equilibria as considered in Examples 1 and 2.

Recall that the classical law of large numbers guarantees the asymptotic stability of the sample means for an i.i.d. sequence of real-valued random variables with a first moment. Since the limit of the arithmetic average of a bounded sequence of real numbers can also be written as the value of the integral of the sequence with respect to a density charge on the countable set of natural numbers, the classical law of large numbers for a bounded i.i.d. sequence of real-valued random variables can be restated in the weaker form that the integral of almost any sample sequence over a density charge on the set of natural numbers is the theoretical mean.[17] Such a statement involving a non-countably additive agent space has been used in many papers in various areas, such as macroeconomics, financial economics, public economics, industrial organization, political economy and international economics, to justify the claim on the removal of individual-level uncertainty via aggregation.[18] Our results in this paper indicate that if a general economic model indeed uses finitely additive agent spaces such as the set of natural numbers with a density charge, then it is completely uncertain that one would be able to find any interesting equilibria in such a model.

In addition to the use of atomless countably/finitely additive measure spaces of agents, one can simply analyze a sequence of increasingly large finite sets of agents directly. For example, Edgeworth (1881) conjectured that the core of an economy shrinks to the set of Walrasian equilibria as the number of agents increases to infinity.[19] Such an approach of analyzing a large but finite number of agents directly has been used extensively in the literature. A disadvantage of this approach is that complicated combinatorial arguments may be needed in multiple steps of approximations for general models.

Since hyperfinite sets as constructed in nonstandard analysis have all the formal properties of finite sets and at the same time capture the relevant asymptotic properties, they can also serve as an ideal model for many agents.[20] Based on the hyperfinite sets, one can construct a special class of atomless countably additive measure spaces – Loeb counting measure spaces,[21] which has been argued to provide a right model for situations with a large number of agents in the sense that one can go back and forth between exact results on Loeb counting measure

[17]On the other hand, Proposition 6.5 of Sun (2006) indicates that the corresponding statement for such a sequence of random variables may fail for every sample sequence in terms of sample distributions. We may also point out that under the framework of a Fubini extension for countably additive agent-sample spaces, the exact law of large numbers and its converse in terms of sample distributions/means has been shown in the paper.

[18]See, for example, Feldman and Gilles (1985), Williams (1987), Sargent (1991), He and Wang (1995), Hess and Orphanides (2001), Gomes, Kogan and Zhang (2003), Casas-Arce and Martinez-Jerez (2009), Bierbrauer and Sahm (2010), Ebrahimy and Shimer (2010), Ennis and Keister (2010), Costnot, Nonaldson and Komunjer (2012), Acemoglu and Jensen (2015), Bierbrauer and Boyer (2016).

[19]For the study of such convergence results in general equilibrium theory, see, for example, Debreu and Scarf (1963), Hildenbrand (1974), Trockel (1976), Anderson (1978, 1985), Vives (1988), Serrano, Vohra and Volij (2001), and McLean and Postlewaite (2002, 2005).

[20]See, for example, Brown and Robinson (1975), Brown and Loeb (1976), Khan (1974), and Khan and Sun (1996, 1999).

[21]For more details on Loeb measure spaces and nonstandard analysis, see Loeb and Wolff (2015).

spaces and approximate results for the asymptotic large finite case.[22]

Furthermore, the theory of weak convergence of measures also provides a link between asymptotic large finite results and (countably additive) measure-theoretic results.[23] Thus, one may say that the three approaches for modeling many agents, namely, large finitely many agents, atomless countably additive agent spaces and hyperfinite agent spaces with a Loeb counting measure can be unified.

In conclusion, the approach of using finitely (but non-countably) additive agent spaces may not satisfy the minimal consistency requirement in terms of either equilibrium existence, or the idealized limit property that captures the limiting behavior of some corresponding large finite models.[24] Since a countably additive measure space must be finitely additive, any measure-theoretic result that holds on general finitely additive measure spaces will automatically hold on countably additive measure spaces. Therefore, in terms of applications, it is not expected that atomless finitely additive agent spaces will play any significant role beyond the framework of atomless countably additive agent spaces.[25] Indeed, the foundation of probability theory is set on countably additive measure spaces. Thus, in order to have a viable mathematical model for economic analysis, one may try to avoid working with an atomless finitely (but non-countably) additive measure space as the agent space.[26]

[22]Such a property is called "asymptotic implementability" in Khan and Sun (1999). More specifically, an asymptotic result for the large finite case can be restated to the hyperfinite setting via the so-called transfer principle between the standard and nonstandard models, which can be further reduced to an exact result on a Loeb counting measure space by rounding-off the infinitesimals (pushing-down). On the other hand, an exact result on a Loeb counting measure space can be lifted to an internal result involving hyperfinitely many agents, which leads to an asymptotic result for the large finite case by the transfer principle again. It is also pointed out in Khan and Sun (1999) that the measurability issue as discussed in Dubey and Shapley (1977, 1994) in terms of the limitation on the non-cooperative aspect of an equilibrium can be resolved via the special class of (countably additive) Loeb counting spaces.

[23]See, for example, Hildenbrand (1974) and Andersonand Rashid (1978).

[24]As far as approximate equilibrium is concerned, Propositions 1 and 2 show the existence under additional assumptions on such agent spaces. However, if one does care about approximate equilibrium, one may analyze it directly for the more intuitive setting of large finitely many agents.

[25]See, for example, Milnor and Shapley (1961), Aumann (1964), Hildenbrand (1974), Schmeidler (1973), Hammond (1979), Cole, Mailath and Postlewaite (1992), Gul and Postlewaite (1992), Duffie, Gârleanu and Pedersen (2005), Yannelis (2009), Hellwig (2010), Acemoglu and Jensen (2015), Vives (2017) for some economic applications.

[26]The use of purely finitely additive measures has been ruled out as price systems in capital theory and in general equilibrium theory with an infinite dimensional commodity space; see, for example, Bewley (1972, p. 516 and p. 523) and Kurz and Majumdar (1972). See also Stinchcombe (1997) and Stinchcombe (2016)

APPENDICES

A Proofs of Theorems 1 and 2

A.1 Proof of Theorem 1

$(i) \Rightarrow (iii)$: Assume that μ is not countably additive.[27] Then there is an increasing sequence of sets $\{B_n\}_{n=1}^{\infty}$ in $\mathcal{T}$ such that $\cup_{n=1}^{\infty} B_n = T$ and $\lim_{n\to\infty} \mu(B_n) = c < 1$. Let $C_1 = B_1$, and for $n \geq 2$, $C_n = B_n \setminus B_{n-1}$. Then $\{C_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint sets, $\cup_{n=1}^{k} C_n = B_k$ for $1 \leq k < \infty$ and $\cup_{n=1}^{\infty} C_n = T$.

For a fixed $\theta \in [(c+1)/2, 1)$, the economy $\mathcal{E}$ is as follows. For $n \in \mathbb{N}$ and $t \in C_n$, let

$$u_t(x_1, x_2) = \frac{n+1}{n} x_1^{\frac{n}{n+1}} + x_2, \qquad \omega_t = (\theta, \theta).$$

If $p \in \mathbb{R}_+^2 \setminus \{0\}$ is an equilibrium price vector then $p \gg 0$ because u_t is strongly monotone for each $t \in T$. Without loss of generality suppose that $p_1 + p_2 = 1$.

For $t \in C_n$, the unique solution of maximize $u_t(x_1, x_2)$ subject to $p_1 x_1 + p_2 x_2 = \theta$ is

$$D_{t1} = \min\left\{\frac{p_2^{n+1}}{p_1^{n+1}}, \frac{\theta}{p_1}\right\}, \qquad D_{t2} = \frac{\theta}{p_2} - \frac{p_1 D_{t1}}{p_2}.$$

We will consider two cases: $p_2/p_1 < 1$ and $p_2/p_1 \geq 1$.

Suppose that $p_2/p_1 < 1$. Then $D_{t1} \leq (p_2/p_1)^{n+1}$ for any $t \in C_n$, so $D_{t1} \leq 1$ for any $t \in T$. Moreover, if $t \in C_n$ and $n > m$ then $D_{t1} \leq (p_2/p_1)^{n+1} \leq (p_2/p_1)^{m+1}$. Fix a positive integer m.

$$\begin{aligned} \int_T D_{t1}\, \mathrm{d}\mu &= \int_{B_m} D_{t1}\, \mathrm{d}\mu + \int_{T\setminus B_m} D_{t1}\, \mathrm{d}\mu \\ &\leq \int_{B_m} 1\, \mathrm{d}\mu + \int_{T\setminus B_m} \frac{p_2^{m+1}}{p_1^{m+1}}\, \mathrm{d}\mu \leq \mu(B_m) + \frac{p_2^{m+1}}{p_1^{m+1}}. \end{aligned}$$

for discussion of problems of working with finitely additive measure spaces in decision theory.

[27] It is not necessary to assume μ to be atomless in the example described in this part of the proof.

By letting m tend to infinity, we obtain that $\int_T D_{t1}\,\mathrm{d}\mu \le \lim_{m\to\infty}\mu(B_m) = c$. This yields,

$$\int_T D_{t2}\,\mathrm{d}\mu = \frac{\theta}{p_2} - \frac{p_1\int_T D_{t1}\,\mathrm{d}\mu}{p_2} = \theta + \frac{p_1}{p_2}\left(\theta - \int_T D_{t1}\,\mathrm{d}\mu\right) > \theta = \bar{\omega}_2,$$

a contradiction.

Assume that $p_2/p_1 \ge 1$. If $t \in C_n$ then $D_{t1} \ge \min\{1, 2\theta\} = 1$. So, $\int_T D_{t1}\,\mathrm{d}\mu \ge 1 > \theta = \bar{\omega}_1$, a contradiction.

Thus, $\mathcal{E}$ does not have a competitive equilibrium.[28]

(iii) ⇒ *(i)*: Let $\mathcal{E}$ be an economy on $(T, \mathcal{T}, \mu)$. The measurability assumption on $\mathcal{E}$ embodied in Definition 1 implies the measurability condition in Aumann (1966). So, the implication follows from Aumann (1966).

(iii) ⇒ *(ii)*: Given $\mathcal{E}$, let $\mathbb{E}^n$, f^n, α^{k_n} and f be as in Definition 4. Let p^n be a price vector such that (p^n, f^n) is a competitive equilibrium of $\mathbb{E}^n$. Since the preferences are strongly monotone, $p^n \cdot f^n(i) = p^n \cdot \omega_i^n$ for $1 \le i \le k_n$ and $p^n \gg 0$. Without loss of generality we can assume that for each n, p^n belongs to the unit simplex and that $\{p^n\} \to p$. We will show that (p, f) is a competitive equilibrium of $\mathcal{E}$.

For notational simplicity, let $\tilde{u}_t^n = u_{\alpha^{k_n}(t)}$, $\tilde{\omega}^n = \omega^n \circ \alpha^{k_n}$ and $\tilde{f}^n = f^n \circ \alpha^{k_n}$. Clearly, $\int_T \tilde{f}^n\,\mathrm{d}\mu = \int_T \tilde{\omega}^n\,\mathrm{d}\mu$ and $p^n \cdot \tilde{f}^n(t) = p^n \cdot \tilde{\omega}_t^n$ for all $t \in T$. Therefore,

$$p \cdot f(t) = \lim_{n\to\infty} p^n \cdot \tilde{f}^n(t) = \lim_{n\to\infty} p^n \cdot \tilde{\omega}_t^n = p \cdot \omega_t$$

for all $t \in T$.

Fix $t \in T$. Since $\{\tilde{f}^n(t)\} \to f(t)$, the set $\{f(t)\} \cup \{\tilde{f}^n(t) : n \in \mathbb{N}\}$ is compact. Since $\{\tilde{u}_t^n\} \to u_t$ in the compact-open topology, $\{\tilde{u}_t^n\} \to u_t$ uniformly on every compact set; see Willard (1970, Theorem 43.7). This implies that $\lim_{n\to\infty} \tilde{u}_t^n(\tilde{f}^n(t)) = u_t(f(t))$.

Suppose that for some $y \in \mathbb{R}_+^L$, $u_t(y) > u_t(f(t))$. Then $\tilde{u}_t^n(y) > \tilde{u}_t^n(\tilde{f}^n(t))$ for sufficiently large n. Therefore, $p^n \cdot y > p^n \cdot \tilde{\omega}_t^n$ for sufficiently large n and in the limit, $p \cdot y \ge p \cdot \omega_t$. We have shown that $u_t(y) > u_t(f(t))$ implies that $p \cdot y \ge p \cdot \omega_t$.

[28] If we specialize the agent space $(T, \mathcal{T}, \mu)$ to $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ and take $B_n = \{1, \ldots, n\}$ for every $n \in \mathbb{N}$, then $c = 0$, $C_n = \{n\}$ for every $n \in \mathbb{N}$, and we obtain Example 1.

We need to show that $u_t(y) > u_t(f_t)$ implies $p \cdot y > p \cdot \omega_t$. Towards this end, we first establish that $p \gg 0$. Suppose to the contrary, say $p_1 = 0$. Since $p \neq 0$ and $\int_T \omega \, \mathrm{d}\mu \gg 0$, there is $V \subseteq T$ with $\mu(V) > 0$ and $p \cdot \omega_t > 0$ if $t \in V$.

For $1 \leq i \leq L$, let e^i denote the i-th unit vector. Let $t \in V$. Then $p \cdot f(t) = p \cdot \omega_t > 0$ implies that for some j, $p_j > 0$ and $f_j(t) > 0$. By strong monotonicity $u_t(f(t)+e^1) > u_t(f(t))$ and by continuity of u_t, $u_t(f(t) + e^1 - \delta e^j) > u_t(f(t))$ for sufficiently small $\delta > 0$. Therefore,

$$p \cdot \omega_t \leq p \cdot (f(t) + e^1 - \delta e^j) = p \cdot f(t) - p_j \delta < p \cdot f(t),$$

a contradiction. This proves $p \gg 0$.

To show that $\int_T f \, \mathrm{d}\mu = \int_T \omega \, \mathrm{d}\mu$, we first establish that $\{\tilde{f}^n\}_{n=1}^{\infty}$ is uniformly integrable. Since $\{\tilde{\omega}^n\} \to \omega$ pointwise, $\{\int_T \tilde{\omega}^n \, \mathrm{d}\mu\} \to \int_T \omega \, \mathrm{d}\mu$ and each of these functions is nonnegative, $\{\tilde{\omega}^n\}_{n=1}^{\infty}$ is uniformly integrable; see Hildenbrand (1974, p. 52).

If $\beta^n(t) = \sum_{j=1}^{L} \tilde{\omega}^n_{tj}$, then $\{\beta^n\}_{n=1}^{\infty}$ is uniformly integrable. Observe that $p^n \cdot \tilde{f}^n(t) = p^n \cdot \tilde{\omega}^n_t \leq \beta^n(t)$. In particular, $p^n_j \tilde{f}^n_j(t) \leq \beta^n(t)$ for $j = 1, \ldots, L$. Since $\{p^n\} \to p \gg 0$, given $0 < \gamma < \min\{p_1, \ldots, p_L\}$, there is N such that for all $n \geq N$, $p^n_j > \gamma$ for all j. Therefore, $0 \leq \tilde{f}^n_j(t) \leq \beta^n(t)/\gamma$ if $n \geq N$. Since $\{(1/\gamma)\beta^n\}_{n=1}^{\infty}$ is uniformly integrable, $\{\tilde{f}^n_j\}_{n=1}^{\infty}$ is uniformly integrable for all j and $\{\tilde{f}^n\}_{n=1}^{\infty}$ is uniformly integrable.

Since $\{\tilde{f}^n\}_{n=1}^{\infty}$ is uniformly integrable and converges pointwise to f, $\int_T f \mathrm{d}\mu = \lim_{n\to\infty} \int_T \tilde{f}^n \mathrm{d}\mu$ $= \lim_{n\to\infty} \int_T \tilde{\omega}^n \, \mathrm{d}\mu = \int_T \omega \, \mathrm{d}\mu$. Thus, (p, f) is a competitive equilibrium of $\mathcal{E}$.

(ii) $\Rightarrow$ *(iii)*: Suppose that μ is not countably additive. Consider the economy $\mathcal{E}$ in the proof of *(i)* $\Rightarrow$ *(iii)*. We will show that it does not have the idealized limit property.

Consider the partition $\{C_n\}_{n=1}^{\infty}$ of T constructed in the proof of *(i)* $\Rightarrow$ *(iii)*. If $\mu(C_n) = 0$, then let $C^m_n = C_n$ for any $m \in \mathbb{N}$. In this case, $\mu(C^m_n)$ is a multiple of 2^{-m}, with the multiplicative constant zero. If $\mu(C_n) > 0$, then for some $J \in \mathbb{N}$, $\mu(C_n) > 1/2^J$. Since μ is atomless, there exists $\{C^m_n\}$ such that $C^m_n \subseteq C^{m+1}_n \subseteq C_n$, $\cup_{m=1}^{\infty} C^m_n = C_n$ and $\mu(C^m_n)$ is a multiple of 2^{-m}. (For $m < J$, $\mu(C^m_n)$ can be zero.)

Fix any $k \in \mathbb{N}$. From the construction of $\{C_n\}_{n=1}^{\infty}$, we know that $\mu(T \setminus \cup_{n=1}^{k} C_n) > 0$. There exists $n_k \in \mathbb{N}$ such that $n_k \geq k$ and $k 2^{-n_k} < \mu(T \setminus \cup_{n=1}^{k} C_n)$. Therefore, we can find k

disjoint subsets $D_1^{n_k}, \ldots, D_k^{n_k}$ of $T \setminus \cup_{n=1}^k C_n$ such that $\mu(D_r^{n_k}) = 2^{-n_k}$ for $r = 1, \ldots, k$. Note that $\mu(C_1^{n_k} \cup D_1^{n_k}), \ldots, \mu(C_k^{n_k} \cup D_k^{n_k})$ and $\mu\left(T \setminus \cup_{r=1}^k (C_r^{n_k} \cup D_r^{n_k})\right)$ are positive multiples of 2^{-n_k}. Then we can partition $C_1^{n_k} \cup D_1^{n_k}, \ldots, C_k^{n_k} \cup D_k^{n_k}$ and $T \setminus \cup_{r=1}^k (C_r^{n_k} \cup D_r^{n_k})$ into 2^{n_k} sets $F_1^{n_k}, \ldots, F_{2^{n_k}}^{n_k}$ such that $\mu(F_i^{n_k}) = 2^{-n_k}$ for any $i \in \{1, \ldots, 2^{n_k}\}$. Without loss of generality, we can assume that n_k is increasing in k. Henceforth, for notational simplicity, we will denote by $s(k) = 2^{n_k}$.

We first construct an economy $\mathbb{E}^k$ on $\{1, \ldots, s(k)\}$. As in the proof of $(i) \Rightarrow (iii)$, let $\theta \in [(c+1)/2, 1)$ be a fixed constant. For each agent $i \in \{1, \ldots, s(k)\}$, let $\omega_i^k = (\theta, \theta)$ and

$$u_i^k(x_1, x_2) = \begin{cases} \dfrac{n+1}{n} x_1^{\frac{n}{n+1}} + x_2 & \text{if } F_i^{n_k} \subseteq C_n^{n_k} \cup D_n^{n_k} \text{ and } n \leq k \\ \dfrac{s(k)+1}{s(k)} x_1^{\frac{s(k)}{s(k)+1}} + x_2 & \text{otherwise.} \end{cases}$$

Let $\alpha^{s(k)}$ be a measurable function from T to $\{1, \ldots, s(k)\}$ such that for any $t \in F_i^{n_k}$, $\alpha^{s(k)}(t) = i$. Let $\mathcal{E}^k = \mathbb{E}^k \circ \alpha^{s(k)}$. It is clear that $\mathcal{E}^k = (\tilde{u}^k, \tilde{\omega}^k)$ is an economy on T with $\tilde{\omega}_t^k = (\theta, \theta)$ and

$$\tilde{u}_t^k(x_1, x_2) = \begin{cases} \dfrac{n+1}{n} x_1^{\frac{n}{n+1}} + x_2 & \text{if } t \in C_n^{n_k} \cup D_n^{n_k} \text{ and } n \leq k \\ \dfrac{s(k)+1}{s(k)} x_1^{\frac{s(k)}{s(k)+1}} + x_2 & \text{otherwise.} \end{cases}$$

for $t \in T$.

Let $t \in T$. Then $t \in C_n$ for some n, and there exists $k \geq n$ such that $t \in C_n^{n_k}$. If $k' \geq k$ then t is also in $C_n^{n_{k'}}$. This implies, $\tilde{u}_t^{k'}(x_1, x_2) = \tilde{u}_t^k(x_1, x_2) = [(n+1)/n] x_1^{n/(n+1)} + x_2$. Thus, for any $t \in C_n$, $\tilde{u}_t^k(x_1, x_2) = [(n+1)/n] x_1^{n/(n+1)} + x_2$ when k is large enough. Therefore, $\{\mathcal{E}^k(t)\}_{k=1}^\infty$ converges to $\mathcal{E}(t)$ for each $t \in T$.

Since $\mathbb{E}^k$ is a finite-agent economy with continuous, convex and strongly monotone preferences, it has a competitive equilibrium (p_k, f^k) with $p_k \gg 0$. Assume that $p_{k1} + p_{k2} = 1$. The equilibrium demands are

$$f_1^k(i) = \min\left\{\frac{p_{k2}^{n+1}}{p_{k1}^{n+1}}, \frac{\theta}{p_{k1}}\right\}, \qquad f_2^k(i) = \frac{\theta}{p_{k2}} - \frac{p_{k1} f_1^k(i)}{p_{k2}} \qquad \text{if } F_i^{n_k} \subseteq C_n^{n_k} \cup D_n^{n_k} \text{ and } n \leq k$$

and

$$f_1^k(i) = \min\left\{\frac{p_{k2}^{s(k)+1}}{p_{k1}^{s(k)+1}}, \frac{\theta}{p_{k1}}\right\}, \qquad f_2^k(i) = \frac{\theta}{p_{k2}} - \frac{p_{k1} f_1^k(i)}{p_{k2}} \qquad \text{otherwise.}$$

If $p_{k2}/p_{k1} \geq 1$ then for any i, $f_1^k(i) \geq \min\{1, 2\theta\} = 1$. Therefore, $(1/s(k)) \sum_{i=1}^{s(k)} f_1^k(i) \geq 1 > \theta = (1/s(k)) \sum_{i=1}^{s(k)} \omega_{i1}^k$, a contradiction. Hence, $p_{k2}/p_{k1} < 1$.

We will show that $\lim_{k\to\infty}(p_{k2}/p_{k1}) = 1$. If not, then there is $0 < \beta < 1$ and a strictly increasing sequence $\{k_j\}$, $j = 1, 2, \ldots$ such that $p_{k_j 2}/p_{k_j 1} < \beta$ for all k_j. Since $(p_{k_j 2}/p_{k_j 1})^\ell$ is decreasing in ℓ,

$$\frac{1}{s(k_j)} \sum_{i=1}^{s(k_j)} f_1^{k_j}(i) \leq \frac{1}{s(k_j)} \sum_{\ell=1}^{s(k_j)} \frac{p_{k_j 2}^{\ell+1}}{p_{k_j 1}^{\ell+1}} < \frac{1}{s(k_j)} \sum_{\ell=1}^{s(k_j)} \beta^{\ell+1} = \frac{\beta^2(1-\beta^{s(k_j)})}{s(k_j)(1-\beta)} \to 0$$

as $j \to \infty$. Hence, for some k_j, $(1/s(k_j)) \sum_{i=1}^{s(k_j)} f_1^{k_j}(i) < 1/4$. This leads to

$$\begin{aligned} f_2^{k_j}(i) &= \theta + \frac{p_{k_j 1}}{p_{k_j 2}}\left(\theta - f_1^{k_j}(i)\right) \\ \frac{1}{s(k_j)} \sum_{i=1}^{s(k_j)} f_2^{k_j}(i) &= \theta + \frac{p_{k_j 1}}{p_{k_j 2}}\left(\theta - \frac{1}{s(k_j)} \sum_{i=1}^{s(k_j)} f_1^{k_j}(i)\right) > \theta + \frac{p_{k_j 1}}{4 p_{k_j 2}} > \theta, \end{aligned}$$

a contradiction. Therefore, $\lim_{k\to\infty}(p_{k2}/p_{k1}) = 1$.

Let $\tilde{f}^k = f^k \circ \alpha^{s(k)}$. Then for any $t \in T$, $\tilde{f}_1^k(t) \to 1$ and $\tilde{f}_2^k(t) \to 2\theta - 1$. However, the limit economy $\mathcal{E}$ has no competitive equilibrium, so does not have the idealized limit property.

A.2 Proof of Theorem 2

$(i) \Rightarrow (iii)$: Assume that μ is not countably additive. Then there is an increasing sequence of sets $\{B_n\}_{n=1}^\infty$ in $\mathcal{T}$ such that $\cup_{n=1}^\infty B_n = T$ and $\lim_{n\to\infty} \mu(B_n) = c < 1$. Let $C_1 = B_1$, and for $n \geq 2$, $C_n = B_n \setminus B_{n-1}$. Then $\{C_n\}_{n=1}^\infty$ is a sequence of pairwise disjoint sets, $\cup_{n=1}^k C_n = B_k$ for $1 \leq k < \infty$ and $\cup_{n=1}^\infty C_n = T$.

Let $A = \{0, 1\}$ be the set of actions and $K = [0, 1]$. The payoffs are defined as follows.

For each $t \in C_n$, let

$$\mathcal{G}(t)(a,x) = a(\beta_n - x),\ a \in A \text{ where } \beta_n = c + \frac{1-c}{n}.$$

Note that $\beta_1 = 1$, $\beta_n > c$ for each $n \geq 1$, and $\{\beta_n\}_{n=1}^{\infty}$ is a monotonically decreasing sequence converging to c.

It is easy to show that the best responses are as follows. If $t \in C_n$, then

$$\text{argmax}_{a \in A}\mathcal{G}(t)(a,x) = \begin{cases} \{0,1\} & \text{if } x = \beta_n \\ \{1\} & \text{if } x < \beta_n \\ \{0\} & \text{if } x > \beta_n. \end{cases}$$

This game does not have a mixed strategy Nash equilibrium. Let f from T to K be a Nash equilibrium and $x = \int_T f \,\mathrm{d}\mu$. Suppose that $x \leq c < 1$. Then for all $t \in T$, $f(t) = 1$ which implies that $x = 1$, a contradiction. Now suppose that $x > c$. Then there exists a unique $n_0 \in \mathbb{N}$ such that $\beta_{n_0+1} < x \leq \beta_{n_0}$. For $n \geq n_0 + 1$ and $t \in C_n$, $f(t) = 0$. So, $x = \int_T f \,\mathrm{d}\mu = \sum_{i=1}^{n_0} \int_{C_i} f \,\mathrm{d}\mu \leq \sum_{i=1}^{n_0} \mu(C_i) = \mu(B_{n_0}) \leq c$, a contradiction. Thus, there is no mixed strategy Nash equilibrium.

$(iii) \Rightarrow (i)$: Schmeidler (1973) showed the existence result for the case with the player space being the Lebesgue unit interval. The same result holds on any atomless, countably additive measure space; see, for example, Khan and Sun (2002, Theorem 2). In fact, there is a pure strategy Nash equilibrium.

$(ii) \Rightarrow (iii)$: Suppose that μ is not countably additive. Consider the game $\mathcal{G}$ in the proof of $(i) \Rightarrow (iii)$. We will show that it does not have the idealized limit property.

Consider the partition $\{C_n\}_{n=1}^{\infty}$ of T constructed in the proof of $(i) \Rightarrow (iii)$. If $\mu(C_n) = 0$, then let $C_n^m = C_n$ for any $m \in \mathbb{N}$. In this case, $\mu(C_n^m)$ is a multiple of 2^{-m}, with the multiplicative constant zero. If $\mu(C_n) > 0$, then for some $J \in \mathbb{N}$, $\mu(C_n) > 1/2^J$. Since μ is atomless, there exists $\{C_n^m\}$ such that $C_n^m \subseteq C_n^{m+1} \subseteq C_n$, $\cup_{m=1}^{\infty} C_n^m = C_n$ and $\mu(C_n^m)$ is a multiple of 2^{-m}. (For $m < J$, $\mu(C_n^m)$ can be zero.)

Fix any $k \in \mathbb{N}$. From the construction of $\{C_n\}_{n=1}^{\infty}$, we know that $\mu(T \backslash \cup_{n=1}^{k} C_n) > 0$. There exists $n_k \in \mathbb{N}$ such that $n_k \geq k$ and $k2^{-n_k} < \min\{\mu(T \setminus \cup_{n=1}^k C_n), (1-c)/k\}$. Therefore, we can find k disjoint subsets $D_1^{n_k}, \ldots, D_k^{n_k}$ of $T \setminus \cup_{n=1}^k C_n$ such that $\mu(D_r^{n_k}) = 2^{-n_k}$ for $r = 1, \ldots, k$. Note that $\mu(C_1^{n_k} \cup D_1^{n_k}), \ldots, \mu(C_k^{n_k} \cup D_k^{n_k})$ and $\mu\left(T \setminus \cup_{r=1}^k (C_r^{n_k} \cup D_r^{n_k})\right)$ are positive multiples of 2^{-n_k}. Then we can partition $C_1^{n_k} \cup D_1^{n_k}, \ldots, C_k^{n_k} \cup D_k^{n_k}$ and $T \setminus \cup_{r=1}^k (C_r^{n_k} \cup D_r^{n_k})$ into 2^{n_k} sets $F_1^{n_k}, \ldots, F_{2^{n_k}}^{n_k}$ such that $\mu(F_i^{n_k}) = 2^{-n_k}$ for any $i \in \{1, \ldots, 2^{n_k}\}$. Without loss of generality, we can assume that n_k is increasing in k. Henceforth, for notational simplicity, we will denote by $s(k) = 2^{n_k}$.

We first construct a game $\mathbb{G}^k$ on $\{1, \ldots, s(k)\}$. Let

$$\mathbb{G}^k(i)(a,x) = \begin{cases} a(\beta_n - x) & \text{if } F_i^{n_k} \subseteq C_n^{n_k} \cup D_n^{n_k} \text{ and } n \leq k \\ 0 & \text{otherwise} \end{cases}$$

for each player $i \in \{1, \ldots, s(k)\}$. Let $\alpha^{s(k)}$ be a measurable function from T to $\{1, \ldots, s(k)\}$ such that for any $t \in F_i^{n_k}$, $\alpha^{s(k)}(t) = i$. Let $\mathcal{G}^k = \mathbb{G}^k \circ \alpha^{s(k)}$. It is clear that $\mathcal{G}^k$ is a game on T with

$$\mathcal{G}^k(t)(a,x) = \begin{cases} a(\beta_n - x) & \text{if } t \in C_n^{n_k} \cup D_n^{n_k} \text{ and } n \leq k \\ 0 & \text{otherwise} \end{cases}$$

for any $t \in T$.

Let $t \in T$. Then $t \in C_n$ for some n, and there exists $k \geq n$ such that $t \in C_n^{n_k}$. If $k' \geq k$ then t is also in $C_n^{n_{k'}}$. This implies $\mathcal{G}^{k'}(t)(a,x) = \mathcal{G}^k(t)(a,x) = a(\beta_n - x)$. Thus, for any $t \in C_n$, $\mathcal{G}^k(t)(a,x) = a(\beta_n - x)$ when k is large enough. Therefore, $\{\mathcal{G}^k(t)\}_{k=1}^\infty$ converges to $\mathcal{G}(t)$ for each $t \in T$.

Define a function f^k from $\{1, \ldots, s(k)\}$ to A as $f^k(i) = 1$ if $F_i^{n_k} \subseteq \cup_{n=1}^k (C_n^{n_k} \cup D_n^{n_k})$ and $f^k(i) = 0$ otherwise. It is clear that $\{f^k \circ \alpha^{s(k)}\}_{k=1}^\infty$ converges to 1 pointwise on $(T, \mathcal{T}, \mu)$. We will show that f^k is a pure strategy Nash equilibrium of $\mathbb{G}^k$. Note that

$$\begin{aligned} x^* &= \frac{1}{s(k)} \sum_{i=1}^{s(k)} f^k(i) = \mu\left(\cup_{n=1}^k (C_n^{n_k} \cup D_n^{n_k})\right) = \mu\left(\cup_{n=1}^k C_n^{n_k}\right) + \mu\left(\cup_{n=1}^k D_n^{n_k}\right) \\ &\leq \mu\left(\cup_{n=1}^k C_n\right) + \frac{k}{s(k)} \leq \mu(B_k) + \frac{k}{s(k)} < c + \frac{1-c}{k}, \end{aligned}$$

where the last inequality follows from $\mu(B_k) \le c$ and $k/s(k) < (1-c)/k$. Therefore, for any $n \le k$, $x^* < c + [(1-c)/n] = \beta_n$.

Suppose that the payoff function of player i is $\mathbb{G}^k(i)(a,x) = a(\beta_n - x)$. Then $\mathbb{G}^k(i)(1,x^*) = \beta_n - x^* > 0 = \mathbb{G}^k(i)(0,y)$ for any $y \in [0,1]$. This shows that f^k is a pure strategy Nash equilibrium of $\mathbb{G}^k$. However, the limit game $\mathcal{G}$ has no Nash equilibrium. Thus, the idealized limit property fails.

(iii) $\Rightarrow$ *(ii)*: Let $\{\mathbb{G}^n\}_{n=1}^{\infty}$ be a sequence of finite-agent games with $\{f_n\}_{n=1}^{\infty}$ as mixed strategy Nash equilibria, where the number of agents in $\mathbb{G}^n$ is k_n and $\lim_{n\to\infty} k_n = \infty$. Suppose that there exists a sequence of replication functions $\{\alpha^{k_n}\}_{n=1}^{\infty}$ such that $G^n \circ \alpha^{k_n}$ converges to $\mathcal{G}$ pointwise on T, $f^n \circ \alpha^{k_n}$ converges to f pointwise on T, We show that f is a mixed strategy Nash equilibrium of $\mathcal{G}$.

For simplicity, let $\mathcal{G}^n = \mathbb{G}^n \circ \alpha^{k_n}$ and $\tilde{f}^n = f^n \circ \alpha^{k_n}$. For each $t \in T$ and $n \in \mathbb{N}$, $\tilde{f}^n(t) = (\tilde{f}^n_1(t), \ldots, \tilde{f}^n_L(t)) \in S$. Then $\tilde{f}^n_l(t)$ (or $\tilde{f}^n_{e^l}(t)$) is the probability with which agent t plays action e^l. Note that $\{\tilde{f}^n\} \to f$ pointwise as $n \to \infty$. By the dominated convergence theorem $\int_T \tilde{f}^n \,\mathrm{d}\mu = \frac{1}{k_n}\sum_{i=1}^{k_n} f^n(i) \to \int_T f \,\mathrm{d}\mu$. For each n, f^n is a Nash equilibrium of $\mathbb{G}^n$. So for each $t \in T$ such that $\alpha^{k_n}(t) = i$,

$$\begin{aligned}
&\sum_{a \in E^{k_n}} \mathcal{G}^n(t)(a_i, \frac{1}{k_n}\sum_{j=1}^{k_n} a_j) f^n_{a_1}(1) \ldots f^n_{a_{k_n}}(k_n) \\
\ge \quad &\sum_{a_{-i} \in E^{k_n-1}} \mathcal{G}^n(t)(a', \frac{1}{k_n}a' + \frac{1}{k_n}\sum_{j \ne i}^{k_n} a_j) f^n_{a_1}(1) \ldots f^n_{a_{i-1}}(i-1) f^n_{a_{i+1}}(i+1) \ldots f^n_{a_{k_n}}(k_n) \quad (1)
\end{aligned}$$

for every $a' \in E$. Let $x^n(i), 1 \le n < \infty, 1 \le i \le k_n$ be random variables from a probability space $(\Omega, \mathcal{F}, P)$ to E such that the distribution of $x^n(i)$ is $f^n(i)$ for each n and i, and for each n, the random variables $\{x^n(i), 1 \le i \le k_n\}$ are independent. For the remainder of the proof $\mathbb{E}$ denotes the expectation of a random variable. Then the inequality in Equation (1) can be written as

$$\mathbb{E}\mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} x^n(j)) \ge \mathbb{E}\mathcal{G}^n(t)(a', \frac{1}{k_n}a' + \frac{1}{k_n}\sum_{j \ne i}^{k_n} x^n(j)).$$

for every $a' \in E$.

Note that $\mathbb{E}(\sum_{j=1}^{k_n} x^n(j)) = \sum_{j=1}^{k_n} f^n(j)$ and $\frac{var(\sum_{j=1}^{k_n} x^n(j))}{k_n^2} \to 0$, by Triangular Arrays Theorem,

$$\frac{\sum_{j=1}^{k_n} x^n(j) - \sum_{j=1}^{k_n} f^n(j)}{k_n} \to 0$$

in probability.

Fix any $t \in T$. Since $\{\mathcal{G}^n(t)\} \to \mathcal{G}(t)$ in sup norm, $\{\mathcal{G}^n(t)\}$ is relatively compact. By the Ascoli-Arzelà Theorem (Loeb (2016, p. 171)), $\{\mathcal{G}^n(t)\}$ is uniformly bounded and equicontinuous. Suppose that $\{\mathcal{G}^n(t)\}$ is bounded by $\frac{M}{2}$. Then for any $(a,b),(a',b') \in S \times S$, we have

$$|\mathcal{G}^n(t)(a,b) - \mathcal{G}^n(t)(a',b')| \leq M. \tag{2}$$

Fix $t \in T$, $a' \in E$ and let $\epsilon > 0$. There exists $\delta > 0$ such that

$$|\sum_{j=1}^{k_n} x^n(j) - \sum_{j=1}^{k_n} f^n(j)| < \delta$$

implies

$$|\mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} x^n(j)) - \mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} f^n(j))| \leq \frac{\epsilon}{10(M+1)}.$$

Since

$$\frac{\sum_{j=1}^{k_n} x^n(j) - \sum_{j=1}^{k_n} f^n(j)}{k_n} \to 0$$

in probability, there exists N_1 such that for $n \geq N_1$,

$$P(|\sum_{j=1}^{k_n} x^n(j) - \sum_{j=1}^{k_n} f^n(j)| \geq \delta) \leq \frac{\epsilon}{10(M+1)}.$$

Let A be the event when $|\sum_{j=1}^{k_n} x^n(j) - \sum_{j=1}^{k_n} f^n(j)| < \delta$. Then for $n \geq N_1$,

$$\begin{aligned}
&\left|\mathbb{E}\mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} x^n(j)) - \mathbb{E}\mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} f^n(j))\right| \\
\leq \quad & \left|\mathbb{E}\left(\mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} x^n(j)) - \mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} f^n(j))\right)\mathbf{1}_A\right| \\
& + \left|\mathbb{E}\left(\mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} x^n(j)) - \mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} f^n(j))\right)\mathbf{1}_{A^c}\right|.
\end{aligned}$$

where $\mathbf{1}_A$ is the indicator function of A. When $\mathbf{1}_A = 1$, we know that $|\sum_{j=1}^{k_n} x^n(j) - \sum_{j=1}^{k_n} f^n(j)| < \delta$, which implies that

$$|\mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} x^n(j)) - \mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} f^n(j))| \leq \frac{\epsilon}{10(M+1)}.$$

When $\mathbf{1}_{A^c} = 1$, by Equation (2), we have

$$|\mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} x^n(j)) - \mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} f^n(j))| \leq M.$$

Therefore, we obtain that for $n \geq N_1$,

$$\left|\mathbb{E}\mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} x^n(j)) - \mathbb{E}\mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} f^n(j))\right| \leq \frac{\epsilon}{10(M+1)} + M\frac{\epsilon}{10(M+1)} = \frac{\epsilon}{10}.$$

In particular, we have

$$\mathbb{E}\mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} f^n(j)) \geq \mathbb{E}\mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} x^n(j)) - \frac{\epsilon}{10}.$$

Similarly, we can prove that there exists N_2 such that for $n \geq N_2$,

$$\mathbb{E}\mathcal{G}^n(t)(a', \frac{1}{k_n}a' + \frac{1}{k_n}\sum_{j\neq i}^{k_n} f^n(j)) \leq \mathbb{E}\mathcal{G}^n(t)(a', \frac{1}{k_n}a' + \frac{1}{k_n}\sum_{j\neq i}^{k_n} x^n(j)) + \frac{\epsilon}{10}.$$

Then, we obtain that for any $n \geq \max\{N_1, N_2\}$,

$$\mathbb{E}\mathcal{G}^n(t)(x^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} f^n(j)) \geq \mathbb{E}\mathcal{G}^n(t)(a', \frac{1}{k_n}a' + \frac{1}{k_n}\sum_{j\neq i}^{k_n} f^n(j)) - \frac{\epsilon}{5},$$

which implies that

$$\mathcal{G}^n(t)(f^n(i), \frac{1}{k_n}\sum_{j=1}^{k_n} f^n(j)) \geq \mathcal{G}^n(t)(a', \frac{1}{k_n}a' + \frac{1}{k_n}\sum_{j\neq i}^{k_n} f^n(j)) - \frac{\epsilon}{5},$$

By the continuity of $\mathcal{G}(t)$, there exists N_3 such that for $n \geq N_3$, $\mathcal{G}(t)(f(t), \int_T f \, \mathrm{d}\mu) > \mathcal{G}(t)(\tilde{f}^n(t), \int_T \tilde{f}^n \, \mathrm{d}\mu) - (\epsilon/4)$ and $\mathcal{G}(t)(a', \frac{1}{k_n}a' + \frac{1}{k_n}\sum_{j\neq i}^{k_n} f^n(j)) > \mathcal{G}(t)(a', \int_T f \, \mathrm{d}\mu) - (\epsilon/5)$.

Since $\{\mathcal{G}^n(t)\} \to \mathcal{G}(t)$ in sup norm, there exists N_4 such that for $n \geq N_4$, $|\mathcal{G}(t)(z) - \mathcal{G}^n(t)(z)| < \epsilon/5$ for every $z \in E \times S$. This means $\mathcal{G}(t)(\tilde{f}^n(t), \int_T \tilde{f}^n \mathrm{d}\mu) > \mathcal{G}^n(t)(\tilde{f}^n(t), \int_T \tilde{f}^n \mathrm{d}\mu) - (\epsilon/5)$ and $\mathcal{G}^n(t)(a', \frac{1}{k_n}a' + \frac{1}{k_n}\sum_{j\neq i}^{k_n} f^n(j)) > \mathcal{G}(t)(a', \frac{1}{k_n}a' + \frac{1}{k_n}\sum_{j\neq i}^{k_n} f^n(j)) - (\epsilon/5)$.

By combining these inequalities together, we can obtain that for any $n \geq \max\{N_1, N_2, N_3, N_4\}$,

$$\begin{aligned}
\mathcal{G}(t)\left(f(t), \textstyle\int_T f \, \mathrm{d}\mu\right) &> \mathcal{G}(t)\left(\tilde{f}^n(t), \textstyle\int_T \tilde{f}^n \, \mathrm{d}\mu\right) - \frac{\epsilon}{5} \\
&> \mathcal{G}^n(t)\left(\tilde{f}^n(t), \textstyle\int_T \tilde{f}^n \, \mathrm{d}\mu\right) - \frac{2\epsilon}{5} \\
&\geq \mathcal{G}^n(t)\left(a', \frac{1}{k_n}a' + \frac{1}{k_n}\sum_{j\neq i}^{k_n} f^n(j)\right) - \frac{3\epsilon}{5} \\
&> \mathcal{G}(t)\left(a', \frac{1}{k_n}a' + \frac{1}{k_n}\sum_{j\neq i}^{k_n} f^n(j)\right) - \frac{4\epsilon}{5} \\
&> \mathcal{G}(t)\left(a', \textstyle\int_T f \, \mathrm{d}\mu\right) - \epsilon.
\end{aligned}$$

By letting $\epsilon \to 0$, we get $\mathcal{G}(t)(f(t), \int_T f \, \mathrm{d}\mu) \geq \mathcal{G}(t)(a', \int_T f \, \mathrm{d}\mu)$. Since this is true for every $a' \in E$, f is a Nash equilibrium of $\mathcal{G}$.

B Approximate Equilibria

In light of the examples in Section 3, it is natural to ask whether approximate equilibria exist in economies and games. In general, the answer is no, as shown by Examples 5 and

6 in Appendix B.1. After presenting the examples, Propositions 1 and 2 in Appendix B.2 provide sufficient conditions for the existence of approximate equilibria. The proofs of these propositions are given in Appendix B.3.3 and Appendix B.3.4 respectively.

B.1 Nonexistence of approximate equilibria

Definition 7 Let $\mathcal{E} = (u, \omega)$ be an economy on an atomless, finitely additive agent space $(T, \mathcal{T}, \mu)$ and $\epsilon > 0$. (p, f) is an *ϵ-competitive equilibrium* of $\mathcal{E}$ if $p \in \mathbb{R}^L_+ \setminus \{0\}$, f is a feasible allocation, $f(t) \in B_t(p)$ for all t and there exists $T_\epsilon \in \mathcal{T}$ such that: (a) $\mu(T_\epsilon) \leq \epsilon$ and (b) for all $t \in T^c_\epsilon$, $u_t(f(t)) \geq u_t(y) - \epsilon$ for any $y \in B_t(p)$.

The next example shows that an ϵ-competitive equilibrium may not exist. Given that a competitive equilibrium exists in finite-agent economies with continuous, convex and strongly monotone preferences as in Arrow and Debreu (1954), such a nonexistence result is quite surprising.

Example 5 Fix $\theta \in [1/2, 2/3)$. The economy $\mathcal{E}$ on the atomless, finitely addditive agent space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ is defined as follows. For each $t \in \mathbb{N}$,

$$u_t(x_1, x_2) = e^t \left[\frac{t+1}{t} x_1^{\frac{t}{t+1}} + x_2 \right], \qquad \omega_t = (\theta, \theta).$$

Claim 1 *For $0 < \epsilon < 1/3$, the economy $\mathcal{E}$ has no ϵ-competitive equilibrium.*

Definition 8 Let $\mathcal{G}$ be a game on an atomless, finitely additive player space $(T, \mathcal{T}, \mu)$, and $\epsilon > 0$. A strategy profile $g : T \longrightarrow S$ is a *mixed strategy ϵ-Nash equilibrium* of $\mathcal{G}$ if there exists $T_\epsilon \in \mathcal{T}$ such that $\mu(T_\epsilon) \leq \epsilon$ and for all $t \in T^c_\epsilon$, $\mathcal{G}(t)(g(t), \int_T g \, \mathrm{d}\mu) \geq \mathcal{G}(t)(a, \int_T g \, \mathrm{d}\mu) - \epsilon$ for all $a \in E$. If in addition, g takes values in E, then it is a *pure strategy ϵ-Nash equilibrium*.

An ϵ-Nash equilibrium in mixed strategies may not exist, as is shown in the next example. This is surprising, given that mixed strategy equilibria exist in the finite-player setting as in Nash (1950).

Example 6 The game $\mathcal{G}$ is on the atomless, finitely additive player space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ with $A = \{0, 1\}$ and $K = [0, 1]$. For each player $t \in \mathbb{N}$, the payoff function is $\mathcal{G}(t)(0, x) = 0$ and

$$\mathcal{G}(t)(1, x) = \begin{cases} 1 - 2^t x + 2^{t-1} & \text{if } -1 \leq 1 - 2^t x + 2^{t-1} \leq 1 \\ 1 & \text{if } 1 - 2^t x + 2^{t-1} > 1 \\ -1 & \text{if } 1 - 2^t x + 2^{t-1} < -1. \end{cases}$$

Claim 2 *For $0 < \epsilon \leq 1/4$, the game $\mathcal{G}$ has no ϵ-Nash equilibrium in mixed strategies.*

Claims 1 and 2 will be proved in Appendix B.3.1 and Appendix B.3.2 respectively.

B.2 Existence of approximate equilibria under additional assumptions

The proposition below shows the existence of an ϵ-competitive equilibrium in economies under a tightness assumption.

Definition 9 An economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ is *tight* if for any $\epsilon > 0$, there exist $\bar{T} \subseteq T$ such that $\mu(\bar{T}) < \epsilon$, and $\mathcal{E}(T \setminus \bar{T})$ is a relatively compact subset of $\mathcal{U} \times \mathbb{R}^L_+$.

Proposition 1 *Let $(T, \mathcal{T}, \mu)$ be an atomless, finitely additive agent space and $\mathcal{E}$ an economy on it. If $\mathcal{E}$ is tight,[29] then it has an ϵ-competitive equilibrium for every $\epsilon > 0$.*

Remark 2 The existence of an ϵ-competitive equilibrium for every $\epsilon > 0$ does not imply that there is a competitive equilibrium. Consider Example 1, where $\mathcal{E}(T)$ is relatively compact. So, the economy is tight. By the above proposition, it has an ϵ-competitive equilibrium for every $\epsilon > 0$. However, as has been shown, it does not have a competitive equilibrium. Explicitly, let $p = (1/2, 1/2)$, and for each $t \in \mathbb{N}$, $f(t) = \omega_t = (\theta, \theta)$. It can be shown that (p, f) is an ϵ-competitive equilibrium for every $\epsilon > 0$.

[29] If $(T, \mathcal{T}, \mu)$ is countably additive, then the economy $\mathcal{E}$ is tight automatically. Note that the space $C(\mathbb{R}^L_+)$ of real valued, continuous functions on $\mathbb{R}^L_+$ with the compact-open topology is a complete separable metr1ic space. One can check that $\mathcal{U}$ is a Borel subset of $C(\mathbb{R}^L_+)$. The countable additivity of μ implies that the induced distribution of $\mathcal{E}$ on $\mathcal{U} \times \mathbb{R}^L_+$ is tight (see Bogachev (2007, p. 85)), so is the economy $\mathcal{E}$.

We next explore the issue of existence of ϵ-Nash equilibria in games. The analogous concept and result for games are as follows.

Definition 10 A game $\mathcal{G}$ on $(T, \mathcal{T}, \mu)$ is *tight* if for any $\epsilon > 0$, there exist $\bar{T} \subseteq T$ such that $\mu(\bar{T}) < \epsilon$, and $\mathcal{G}(T \backslash \bar{T})$ is a relatively compact subset of $\mathcal{V}$.

Proposition 2 *Let $(T, \mathcal{T}, \mu)$ be an atomless, finitely additive player space and $\mathcal{G}$ a game on it. If $\mathcal{G}$ is tight,*[30] *then it has a pure strategy ϵ-Nash equilibrium for every $\epsilon > 0$.*

Remark 3 The existence of an ϵ-Nash equilibrium for every $\epsilon > 0$ does not ensure the existence of a Nash equilibrium. In Example 2, the set of payoff functions is relatively compact. So, the game is tight. By the above proposition, it has an ϵ-Nash equilibrium for every $\epsilon > 0$. However, as has been shown, it does not have a Nash equilibrium. Explicitly, $f(t) = 0$ for all $t \in \mathbb{N}$ is a pure strategy ϵ-Nash equilibrium for every $\epsilon > 0$.

Remark 4 A further aspect of approximate equilibria merits attention. The failure of the idealized limit property for economies and games was illustrated in Examples 3 and 4. The idealized limit property requires the limit of equilibria to be an equilibrium. One can weaken this requirement to an approximate equilibrium in the limit, for every $\epsilon > 0$. In either case, the answer continues to be in the negative.

In Example 3, the limit allocation is $f(t) = (1, 2\theta - 1)$ for all $t \in \mathbb{N}$. This is not even a feasible allocation of the limit economy $\mathcal{E}$ since $1 > \theta = \bar{\omega}_1$.

In Example 4, the limit profile is $f(t) = 1$ for all $t \in \mathbb{N}$. Then $x = \int_T f \, \mathrm{d}\mu = 1$, $\mathcal{G}(t)(0, x) = 0$, $\mathcal{G}(t)(1, x) = (1/t) - x \leq 0$ and $\max\{\mathcal{G}(t)(0, x), \mathcal{G}(t)(1, x)\} = 0$. Let $0 < \epsilon < 1/2$. If f is an ϵ-Nash equilibrium then for all $t \in T_\epsilon^c$, $\mathcal{G}(t)(1, x) \geq -\epsilon \Leftrightarrow (1/t) + \epsilon \geq x$, which is impossible.

[30] Similar to Footnote 29, if $(T, \mathcal{T}, \mu)$ is countably additive, then the game $\mathcal{G}$ is tight automatically. Note that $\mathcal{V}$ is a complete separable metric space. The countable additivity of μ implies that the induced distribution of $\mathcal{G}$ on $\mathcal{V}$ is tight, so is the game $\mathcal{G}$.

B.3 The proofs

B.3.1 Proof of Claim 1

Suppose that $0 < \epsilon < 1/3$ and (p, f) is an ϵ-competitive equilibrium. If any of the prices is zero then the budget set of $t \in \mathbb{N}$ is unbounded and since the preferences are strongly monotone, $u_t(f(t)) \geq u_t(y) - \epsilon$ cannot hold for every $y \in B_t(p)$. So, $p \gg 0$. Without loss of generality, normalize $p_1 + p_2 = 1$. For any $t \in \mathbb{N}$, the unique solution of maximize $u_t(x_1, x_2)$ subject to $p_1 x_1 + p_2 x_2 = \theta$ is $D_t = (D_{t1}, D_{t2})$, where

$$D_{t1} = \min\left\{\frac{p_2^{t+1}}{p_1^{t+1}}, \frac{\theta}{p_1}\right\}, \qquad D_{t2} = \frac{\theta}{p_2} - \frac{p_1 D_{t1}}{p_2}.$$

Since (p, f) is an ϵ-competitive equilibrium, there exists $T_\epsilon \subseteq \mathbb{N}$ such that $\mu(T_\epsilon) \leq \epsilon$, and for all $t \in T_\epsilon^c$, $u_t(f(t)) \geq u_t(D_t) - \epsilon$, i.e., $u_t(D_t) - u_t(f(t)) \leq \epsilon$.

For $i = 1, 2$, $\int_{T_\epsilon^c} f_i \, \mathrm{d}\mu \leq \int_{\mathbb{N}} f_i \, \mathrm{d}\mu = \int_{\mathbb{N}} \omega_i \, \mathrm{d}\mu = \theta$. Let $T_i = \{t \in T_\epsilon^c : f_i(t) < 3\theta/2\}$. If $\mu(T_i) = 0$, then $f_i(t) \geq 3\theta/2$ for almost all $t \in T_\epsilon^c$, and since $\mu(T_\epsilon^c) > 2/3$, f is not a feasible allocation. So, $\mu(T_i) > 0$ for $i = 1, 2$. Obviously, both T_1 and T_2 are infinite sets. We will examine three possibilities for p_2/p_1 below.

Case 1: $p_2/p_1 < 1$. For all $t \in \mathbb{N}$, $D_{t1} = (p_2/p_1)^{t+1}$ and $D_{t2} = (\theta/p_2) - (p_2/p_1)^t$. Since $p_2 < 1/2$, and $(p_2/p_1)^t \to 0$, there exists $t_0 \in \mathbb{N}$ such that for all $t \in \mathbb{N}$ and $t > t_0$, $D_{t2} > 3\theta/2$. Without loss of generality we suppose that $\{1, \ldots, t_0\} \subseteq T_\epsilon$.

Given $t \in T_2$ and $f(t)$, let $f^*(t) = (f_1^*(t), f_2(t))$ where $p_1 f_1^*(t) + p_2 f_2(t) = \theta$. Since $f(t) \in B_t(p)$, $f(t) \leq f^*(t)$ and $u_t(f(t)) \leq u_t(f^*(t))$. Moreover, $f_2^*(t) = f_2(t) < 3\theta/2$. Choose $\lambda \in (0, 1)$ so that $y(t) = \lambda D_t + (1 - \lambda) f^*(t)$ and $y_2(t) = 3\theta/2$. Then $p_1 y_1(t) + p_2 y_2(t) = \theta$, and by the quasi-concavity of u_t, $u_t(y(t)) \geq u_t(f^*(t))$.

For all $t \in T_2$,

$$u_t(y(t)) \;=\; e^t \left[\frac{t+1}{t}\left(\frac{\theta}{p_1} - \frac{3p_2\theta}{2p_1}\right)^{\frac{t}{t+1}} + \frac{3\theta}{2}\right].$$

Since $u_t(D_t) - u_t(f(t)) \le \epsilon$ and $u_t(f(t)) \le u_t(y(t))$, $u_t(D_t) - u_t(y(t)) \le \epsilon$. Therefore,

$$\left[\frac{t+1}{t} \times \frac{p_2^{t+1}}{p_1^{t+1}} + \frac{\theta}{p_2} - \frac{p_2^t}{p_1^t}\right] - \left[\frac{t+1}{t}\left(\frac{\theta}{p_1} - \frac{3p_2\theta}{2p_1}\right)^{\frac{t}{t+1}} + \frac{3\theta}{2}\right] \le \epsilon e^{-t}.$$

Let t go to infinity in T_2. Then $(1/p_2)-(1/p_1)+[3p_2/(2p_1)]-(3/2) \le 0$, i.e., $(p_1-p_2)(2-3p_2) \le 0$. However, since $p_2/p_1 < 1$ and $p_1 + p_2 = 1$, $(p_1 - p_2)(2 - 3p_2) > 0$, a contradiction.

Case 2: $p_2/p_1 > 1$. There exists $t_0 \in \mathbb{N}$ such that for all $t \in \mathbb{N}$ and $t > t_0$, $D_{t1} = \theta/p_1$ and $D_{t2} = 0$. Since $p_1 < 1/2$, $D_{t1} > 3\theta/2$ for $t > t_0$. Without loss of generality we suppose that $\{1, \ldots, t_0\} \subseteq T_\epsilon$.

Given $t \in T_1$ and $f(t)$, let $f^*(t) = (f_1(t), f_2^*(t))$ where $p_1 f_1(t) + p_2 f_2^*(t) = \theta$. Since $f(t) \in B_t(p)$, $f(t) \le f^*(t)$ and $u_t(f(t)) \le u_t(f^*(t))$. Moreover, $f_1^*(t) = f_1(t) < 3\theta/2$. Choose $\lambda \in (0,1)$ so that $y(t) = \lambda D_t + (1-\lambda) f^*(t)$ and $y_1(t) = 3\theta/2$. Then $p_1 y_1(t) + p_2 y_2(t) = \theta$, and by the quasi-concavity of u_t, $u_t(y(t)) \ge u_t(f^*(t))$.

For all $t \in T_1$,

$$u_t(y(t)) \;=\; e^t \left[\frac{t+1}{t}\left(\frac{3\theta}{2}\right)^{\frac{t}{t+1}} + \frac{\theta}{p_2} - \frac{3p_1\theta}{2p_2}\right].$$

Since $u_t(D_t) - u_t(f(t)) \le \epsilon$ and $u_t(f(t)) \le u_t(y(t))$, $u_t(D_t) - u_t(y(t)) \le \epsilon$. Therefore,

$$\frac{t+1}{t}\left(\frac{\theta}{p_1}\right)^{\frac{t}{t+1}} - \frac{t+1}{t}\left(\frac{3\theta}{2}\right)^{\frac{t}{t+1}} - \frac{\theta}{p_2} + \frac{3p_1\theta}{2p_2} \le \epsilon e^{-t}.$$

Let t go to infinity in T_1. Then $(1/p_1)-(3/2)-(1/p_2)+[3p_1/(2p_2)] \le 0$, i.e., $(p_2-p_1)(2-3p_1) \le 0$. However, since $p_2/p_1 > 1$ and $p_1 + p_2 = 1$, $(p_2 - p_1)(2 - 3p_1) > 0$, a contradiction.

Case 3: $p_2/p_1 = 1$. For all $t \in \mathbb{N}$, $D_{t1} = 1$ and $D_{t2} = 2\theta - 1$. Since $\theta < 2/3$, $D_{t1} > 3\theta/2$ for all $t \in \mathbb{N}$. Given $t \in T_1$ and $f(t)$, let $f^*(t) = (f_1(t), f_2^*(t))$ where $p_1 f_1(t) + p_2 f_2^*(t) = \theta$. Since $f(t) \in B_t(p)$, $f(t) \le f^*(t)$ and $u_t(f(t)) \le u_t(f^*(t))$. Moreover, $f_1^*(t) = f_1(t) < 3\theta/2$. Choose $\lambda \in (0,1)$ so that $y(t) = \lambda D_t + (1-\lambda) f^*(t)$ and $y_1(t) = 3\theta/2$. Then $p_1 y_1(t) + p_2 y_2(t) = \theta$, and by the quasi-concavity of u_t, $u_t(y(t)) \ge u_t(f^*(t))$.

For all $t \in T_1$,

$$u_t(y(t)) \;=\; e^t \left[\frac{t+1}{t} \left(\frac{3\theta}{2} \right)^{\dfrac{t}{t+1}} + 2\theta - \frac{3\theta}{2} \right].$$

Since $u_t(D_t) - u_t(f(t)) \le \epsilon$ and $u_t(f(t)) \le u_t(y(t))$, $u_t(D_t) - u_t(y(t)) \le \epsilon$. Therefore,

$$\frac{\frac{1}{t} - \frac{t+1}{t}\left(\frac{3\theta}{2}\right)^{\dfrac{t}{t+1}} + \frac{3\theta}{2}}{e^{-t}} \le \epsilon.$$

Notice that if t goes to infinity in T_1 then both the numerator and the denominator in the LHS tend to zero. By L'Hopital's rule,

$$\begin{aligned}
& \lim_{t\to\infty} \frac{\frac{1}{t} - \frac{t+1}{t}\left(\frac{3\theta}{2}\right)^{\dfrac{t}{t+1}} + \frac{3\theta}{2}}{e^{-t}} \\
= \;& \lim_{t\to\infty} \frac{-\frac{1}{t^2} + \frac{1}{t^2}\left(\frac{3\theta}{2}\right)^{\dfrac{t}{t+1}} - \frac{1}{t(t+1)}\left(\frac{3\theta}{2}\right)^{\dfrac{t}{t+1}} \ln\frac{3\theta}{2}}{-e^{-t}} \\
= \;& \lim_{t\to\infty} \frac{1 - \left(\frac{3\theta}{2}\right)^{\dfrac{t}{t+1}} + \frac{t}{t+1}\left(\frac{3\theta}{2}\right)^{\dfrac{t}{t+1}} \ln\frac{3\theta}{2}}{t^2 e^{-t}}.
\end{aligned}$$

The numerator has a positive limit but the denominator tends to zero. So, the above limit is infinity, a contradiction.

This shows that there is no ϵ-competitive equilibrium for $0 < \epsilon < 1/3$.

B.3.2 Proof of Claim 2

Let $0 < \epsilon \le 1/4$ and suppose that g from $\mathbb{N}$ to $[0,1]$ is an ϵ-Nash equilibrium, where $g(t)$ is the probability that player t assigns to 1. Then there exists $T_\epsilon \subseteq \mathbb{N}$ such that $\mu(T_\epsilon) \le \epsilon$, and for all $t \in T_\epsilon^c$, $\mathcal{G}(t)(g(t), x) \ge \max\{\mathcal{G}(t)(0,x), \mathcal{G}(t)(1,x)\} - \epsilon$, where $x = \int_T g \,\mathrm{d}\mu$.

If $x \le 1/2$, then for all $t \in \mathbb{N}$, $1 - 2^t x + 2^{t-1} \ge 1$ which implies that $\mathcal{G}(t)(1,x) = 1 > 0 = \mathcal{G}(t)(0,x)$. Therefore, for all $t \in T_\epsilon^c$, $\mathcal{G}(t)(g(t),x) \ge 1 - \epsilon$, which gives $g(t) \ge 1 - \epsilon$. Then,

$$x = \int_{T_\epsilon} g \,\mathrm{d}\mu + \int_{T_\epsilon^c} g \,\mathrm{d}\mu \ge \int_{T_\epsilon^c} g \,\mathrm{d}\mu \ge (1-\epsilon)^2 > \frac{1}{2},$$

a contradiction.

If $x > 1/2$, then there exists $t_0 \in \mathbb{N}$ such that for all $t \in \mathbb{N}$ and $t > t_0$, $1 - 2^t x + 2^{t-1} < -1$. Without loss of generality we suppose that $\{1, \ldots, t_0\} \subseteq T_\epsilon$. For all $t \in T_\epsilon^c$, $\mathcal{G}(t)(0, x) = 0 > -1 = \mathcal{G}(t)(1, x)$. Hence, $\mathcal{G}(t)(g(t), x) \geq -\epsilon$ for all $t \in T_\epsilon^c$, which gives $g(t) \leq \epsilon$. Therefore,

$$x = \int_{T_\epsilon} g \, \mathrm{d}\mu + \int_{T_\epsilon^c} g \, \mathrm{d}\mu \leq \mu(T_\epsilon) + \epsilon \leq \epsilon + \epsilon \leq \frac{1}{2},$$

a contradiction.

Thus, there is no ϵ-Nash equilibrium for $0 < \epsilon \leq 1/4$.

B.3.3 Proof of Proposition 1

Since equilibria exist in finite-agent economies, it is easy to show that a competitive equilibrium exists in an economy with an atomless finitely additive agent space if there are finitely many different utility functions and endowments in the economy.[31] We state such a result as a simple lemma.

Lemma 1 *Let $(T, \mathcal{T}, \mu)$ be an atomless, finitely additive player space and $\mathcal{E} = (u, \omega)$ an economy on it. If the range of $\mathcal{E}$ is finite, then it has a competitive equilibrium (p, f) such that $p \gg 0$ and the range of f is finite.*

Proof Let $U = \{(u_1, \omega_1), \ldots, (u_m, \omega_m)\}$ be the range of $\mathcal{E}$ and $T_k = \mathcal{E}^{-1}(\{(u_k, \omega_k)\})$, $1 \leq k \leq m$. Then $\{T_1, \ldots, T_m\}$ is a measurable partition of T. Let $(A, \mathcal{A}, \rho)$ denote the unit interval with Lebesgue measure. Consider a measurable partition $\{A_1, \ldots, A_m\}$ of A such that for each k, $\rho(A_k) = \mu(T_k)$. On each A_k, let $\mathcal{H}(i) = (u_k, \omega_k)$. Then $\mathcal{H}$ is an economy on a countably additive measure space. By Aumann (1966) it has a competitive equilibrium (p, h), such that $p \gg 0$. From h, we will construct another competitive allocation g of $\mathcal{H}$ which has a finite range.

For $1 \leq k \leq m$, let $D_k = \{h(i) : i \in A_k\}$, and $co\, D_k$ its convex hull. Then $\int_{A_k} h \, \mathrm{d}\rho \in \int_{A_k} D_k \, \mathrm{d}\rho$. By Hildenbrand (1974, p. 62), $\int_{A_k} D_k \, \mathrm{d}\rho = \rho(A_k) \times co\, D_k$. From Carathéodory's

[31] Note that the atomless property allows us to remove the usual convexity assumptions on the preferences in finite-agent economies.

theorem, $\int_{A_k} h \, \mathrm{d}\rho = \rho(A_k) \sum_{j=1}^{L+1} \alpha_{kj} y_{kj}$, $\sum_{j=1}^{L+1} \alpha_{kj} = 1$ and $y_{kj} \in D_k$ for $j = 1, \ldots, L+1$. Decompose A_k into $L+1$ subsets $\{A_{k1}, \ldots, A_{k,L+1}\}$ such that $\rho(A_{kj}) = \rho(A_k)\alpha_{kj}$, $j = 1, \ldots, L+1$, and let $g(i) = y_{kj}$ if $i \in A_{kj}$. Then the range of g is finite, $\int_A g \, \mathrm{d}\rho = \int_A h \, \mathrm{d}\rho$ and (p, g) is a competitive equilibrium of $\mathcal{H}$.

Decompose T_k into $\{T_{k1}, \ldots, T_{k,L+1}\}$ such that $\mu(T_{kj}) = \rho(A_{kj})$ and let $f(t) = y_{kj}$ on T_{kj}. Then $\int_T f \, \mathrm{d}\mu = \int_A g \, \mathrm{d}\rho$. It follows that (p, f) is a competitive equilibrium of $\mathcal{E}$, $p \gg 0$ and the range of f is finite. The proof of the lemma is thus completed.

Next we move to the proof of Proposition 1, which is divided into seven parts.

Step 1: (Implications of atomlessness) Fix $0 < \epsilon < 1$ and assume without loss of generality that $\epsilon = 1/J$ for some positive integer J. Let $\overline{\sigma} = \max\{\bar{\omega}_1, \ldots, \bar{\omega}_L\}$ and $\underline{\sigma} = \min\{\bar{\omega}_1, \ldots, \bar{\omega}_L\}$. By assumption, $\underline{\sigma} > 0$. Since μ is atomless, we can divide T into $2JL$ pairwise disjoint sets, each of measure $1/(2JL)$. Of these subsets, for each commodity ℓ there is a subset A_ℓ (not necessarily distinct) such that

$$\mu(A_\ell) = \frac{1}{2JL} = \frac{\epsilon}{2L} \text{ and } \int_{A_\ell} \omega_\ell \, \mathrm{d}\mu \geq \frac{\bar{\omega}_\ell}{2JL} = \frac{\epsilon \bar{\omega}_\ell}{2L}.$$

Then $\mu(\cup_{\ell=1}^L A_\ell) \leq \epsilon/2$ and for every $j = 1, \ldots, L$, $\int_{\cup_{\ell=1}^L A_\ell} \omega_j \, \mathrm{d}\mu \geq \epsilon \bar{\omega}_j/(2L) \geq \epsilon \underline{\sigma}/(2L) > 0$.

Step 2: (Implications of tightness) Since $\mathcal{E}$ is tight, there exists a subset $\bar{T} \subseteq T$ such that $\mu(\bar{T}) < \epsilon/2$ and $\mathcal{E}(T \backslash \bar{T})$ is relatively compact. Let $A = \bar{T} \cup (\cup_{\ell=1}^L A_\ell)$. Then $\mu(A) < \epsilon$ and $\mathcal{E}(T \backslash A)$ is relatively compact.

Let $\int_A \omega \, \mathrm{d}\mu = (\gamma_1, \ldots, \gamma_L)$ and $\underline{\gamma} = \min\{\gamma_1, \ldots, \gamma_L\}$. Since $\cup_{\ell=1}^L A_\ell \subseteq A$ and for each j, $\int_{\cup_{\ell=1}^L A_\ell} \omega_j \, \mathrm{d}\mu \geq \epsilon \underline{\sigma}/(2L)$, $\underline{\gamma} \geq \epsilon \underline{\sigma}/(2L) > 0$.

Because $\omega(T \backslash A)$ is relative compact, there is an integer $K > 1$ such that $\|\omega_t\| \leq K$ for every $t \in T \backslash A$. Note that $\{u_t : t \in T \backslash A\}$ is relatively compact under the compact-open topology, so $\{u_t : t \in T \backslash A\}$ is also relatively compact under the topology of uniform convergence on any compact domain [Willard (1970, Theorem 43.7)], which implies that $\{u_t : t \in T \backslash A\}$ is equicontinuous on any compact domain [Willard (1970, Theorem 43.15)].

Let $\theta = 2KL^2 \overline{\sigma}/(\epsilon \underline{\sigma}) > 1$ and $C = \{x \in \mathbb{R}_+^L : \|x\| \leq \theta\}$. Then C is compact. By equicontinuity, there exists $\delta > 0$ such that if $x, x' \in C$ and $\|x - x'\| < \delta$ then $|u_t(x) - u_t(x')| <$

$\epsilon/4$ for all $t \in T\backslash A$.

Since $\omega(T\backslash A)$ is relatively compact and $\{u_t : t \in T\backslash A\}$ is relatively compact under the topology of uniform convergence on C, there exist m disjoint measurable sets $T_1, \ldots, T_m$ such that $T\backslash A = \cup_{k=1}^m T_k$, $\|\omega_t - \omega_{t'}\| < \delta/\theta$ for any $t, t' \in T_k$ and $|u_t(x) - u_{t'}(x)| < \epsilon/4$ if $x \in C$. Since our concern is ϵ-competitive equilibria, without loss of generality we can assume that $\mu(T_k) > 0$ for each k. For notational simplicity, denote A by T_{m+1}. Then $\mu(T_{m+1}) < \epsilon$ and $\int_{T_{m+1}} \omega_j \, \mathrm{d}\mu \geq \underline{\gamma}$ for every $j = 1, \ldots, L$.

Step 3: (A finite characteristics economy) Let $\xi^k = \int_{T_k} \omega \, \mathrm{d}\mu / \mu(T_k)$, $1 \leq k \leq m+1$. For each $1 \leq k \leq m$, fix an agent $i_k \in T_k$ and construct an economy $\mathcal{E}' = (\hat{u}, \eta)$ on T as follows.

$$\begin{array}{lll} \hat{u}_t = u_{i_k} & \eta_t = \xi^k & \text{if } t \in T_k,\ 1 \leq k \leq m, \\ \hat{u}_t(x) = \sum_{\ell=1}^L x_\ell & \eta_t = \xi^{m+1} & \text{if } t \in T_{m+1}. \end{array}$$

Since $\int_{T_{m+1}} \omega \, \mathrm{d}\mu \gg 0$, $\int_T \eta \, \mathrm{d}\mu \gg 0$. $\mathcal{E}'$ is an economy with strongly monotone preferences and finite number of utility functions and endowments. By Lemma 1 it has a competitive equilibrium (p, h) such that $p \gg 0$, h is a simple function, $p \cdot h(t) = p \cdot \xi^k$ if $t \in T_k$ $(1 \leq k \leq m+1)$ and $\int_T h \, \mathrm{d}\mu = \int_T \eta \, \mathrm{d}\mu = \int_T \omega \, \mathrm{d}\mu$.

Step 4: (An upper bound for the price ratios) Let $\overline{\alpha} = \max\{p_1, \ldots, p_L\}$ and $\underline{\alpha} = \min\{p_1, \ldots, p_L\}$. Below we show that $\overline{\alpha}/\underline{\alpha} \leq L\overline{\sigma}/\underline{\gamma} \leq \theta/K$.

Let $\overline{P} = \{\ell : p_\ell = \overline{\alpha}\}$ and $\underline{P} = \{\ell : p_\ell = \underline{\alpha}\}$. If $t \in T_{m+1}$ then $\hat{u}_t(x) = \sum_{\ell=1}^L x_\ell$. So t consumes goods only from $\underline{P}$ and $\underline{\alpha} \sum_{\ell \in \underline{P}} h_\ell(t) = \sum_{\ell \in \underline{P}} p_\ell h_\ell(t) = p \cdot \xi^{m+1}$. Integration gives,

$$\underline{\alpha} \sum_{\ell \in \underline{P}} \int_{T_{m+1}} h_\ell \, \mathrm{d}\mu = \int_{T_{m+1}} \underline{\alpha} \sum_{\ell \in \underline{P}} h_\ell \, \mathrm{d}\mu = p \cdot \int_{T_{m+1}} \xi^{m+1} \, \mathrm{d}\mu = p \cdot \int_{T_{m+1}} \omega \, \mathrm{d}\mu \geq \overline{\alpha}\underline{\gamma}.$$

The inequality is due to the fact that $\int_{T_{m+1}} \omega_j \, \mathrm{d}\mu \geq \underline{\gamma}$ for every $j = 1, \ldots, L$.

Suppose to the contrary that $\overline{\alpha}\underline{\gamma} > L\underline{\alpha}\overline{\sigma}$. Then $\sum_{\ell \in \underline{P}} \int_{T_{m+1}} h_\ell \, \mathrm{d}\mu > L\overline{\sigma}$. So, for some $\ell \in \underline{P}$, $\int_{T_{m+1}} h_\ell \, \mathrm{d}\mu > \overline{\sigma}$. Since the demand for this good exceeds its supply, we obtain a contradiction. This establishes that $\overline{\alpha}/\underline{\alpha} \leq L\overline{\sigma}/\underline{\gamma}$.

Recall that $\underline{\gamma} \geq \epsilon\underline{\sigma}/(2L)$. This gives $L\overline{\sigma}/\underline{\gamma} \leq 2L^2\overline{\sigma}/(\epsilon\underline{\sigma}) = \theta/K$.

Step 5: (A feasible allocation) First, we will construct another competitive allocation z of $\mathcal{E}'$ fom the allocation h, by reassigning commodity bundles. Next, we will construct a feasible allocation f of the economy $\mathcal{E}$ based on the allocation z.

Consider any T_k, $1 \leq k \leq m+1$. If $\xi^k = 0$ then let $z(t) = h(t)$ for every $t \in T_k$. Assume that $\xi^k \neq 0$. h is a simple function means, there are nonnegative $v^1, \ldots, v^{N_k}$ and $\beta^1, \ldots, \beta^{N_k}$ such that $\sum_{j=1}^{N_k} \beta^j = \mu(T_k)$ and $\sum_{j=1}^{N_k} \beta^j v^j = \int_{T_k} h \, \mathrm{d}\mu$.

For any measurable subset D of T_k, let $\tau(D) = \int_D p \cdot \omega \, \mathrm{d}\mu / p \cdot \xi^k$. Then τ is an atomless measure on T_k with $\tau(T_k) = \mu(T_k)$. Now consider the atomless, vector measure $\tau^* = (\mu, \tau)$. Its range is convex; see Armstrong and Prikry (1981), Bhaskara Rao (1984), or Khan and Rath (2013). Since $(\mu(T_k), \mu(T_k))$ belongs to the range, there exist N_k disjoint measurable sets $T_k^1, \ldots, T_k^{N_k}$ such that, for $1 \leq j \leq N_k$, $\tau^*(T_k^j) = (\beta^j, \beta^j)$, i.e., $\mu(T_k^j) = \beta^j$ and $\int_{T_k^j} p \cdot \omega \, \mathrm{d}\mu / p \cdot \xi^k = \beta^j$. Let $z(t) = v^j$ if $t \in T_k^j$. Then $p \cdot z(t) = p \cdot \xi^k$ if $t \in T_k$ and $\int_{T_k} z \, \mathrm{d}\mu = \int_{T_k} h \, \mathrm{d}\mu$. It follows that z is a competitive allocation of $\mathcal{E}'$.

Now we will construct the allocation f. If for some $1 \leq k \leq m+1$, $\xi^k = 0$, let $f(t) = \omega_t$ for $t \in T_k$. If $\xi^k \neq 0$ then $f(t) = (p \cdot \omega_t / p \cdot \xi^k) z(t)$ for $t \in T_k$. Clearly, $p \cdot f(t) = p \cdot \omega_t$ for all $t \in T$. Next we show that $\int_T f \, \mathrm{d}\mu = \int_T \omega \, \mathrm{d}\mu$.

Suppose that $\xi^k = 0$. Then $\int_{T_k} \omega \, \mathrm{d}\mu = 0$ and since $f(t) = \omega_t$ for $t \in T_k$, $\int_{T_k} f \, \mathrm{d}\mu = 0$. From $0 = p \cdot \xi^k = p \cdot z(t)$ and $p \gg 0$, $z(t) = 0$. Thus, $\int_{T_k} f \, \mathrm{d}\mu = \int_{T_k} z \, \mathrm{d}\mu$.

If $\xi^k \neq 0$, then for any $1 \leq j \leq N_k$,

$$\int_{T_k^j} f \, \mathrm{d}\mu = \int_{T_k^j} \frac{p \cdot \omega}{p \cdot \xi^k} z \, \mathrm{d}\mu = \frac{1}{p \cdot \xi^k} v^j \int_{T_k^j} p \cdot \omega \, \mathrm{d}\mu = \beta^j v^j.$$

Therefore, $\int_{T_k} f \, \mathrm{d}\mu = \sum_{j=1}^{N_k} \beta^j v^j = \int_{T_k} z \, \mathrm{d}\mu$. This gives $\int_T f \, \mathrm{d}\mu = \int_T z \, \mathrm{d}\mu = \int_T \omega \, \mathrm{d}\mu$.

Step 6: (Some relevant inequalities) (*a*) Claim: If $t \in T_k$ then $\|\omega_t - \xi^k\| \leq \delta/\theta$ and $|p \cdot (\omega_t - \xi^k)| \leq \overline{\alpha}\delta/\theta$.

These are consequences of the fact that $\|\omega_t - \omega_{t'}\| < \delta/\theta$ if t, $t' \in T_k$. Fix $t \in T_k$ and let $g(t') = \omega_t - \omega_{t'}$ for every $t' \in T_k$. Since $\|\omega_t - \omega_{t'}\| < \delta/\theta$, $\|g(t')\| \leq \delta/\theta$, and $\int_{T_k} \|g\| \leq \mu(T_k)(\delta/\theta)$. Note that $\omega_t - \xi^k = [1/\mu(T_k)] \int_{T_k} g \, \mathrm{d}\mu$. Therefore, $\|\omega_t - \xi^k\| = [1/\mu(T_k)] \| \int_{T_k} g \, \mathrm{d}\mu \| \leq [1/\mu(T_k)] \int_{T_k} \|g\| \, \mathrm{d}\mu \leq \delta/\theta$.

From this,

$$\left|p\cdot\left(\omega_t-\xi^k\right)\right| = \left|\sum_{\ell=1}^L p_\ell\left(\omega_{t\ell}-\xi^k_\ell\right)\right| \le \sum_{\ell=1}^L p_\ell\left|\omega_{t\ell}-\xi^k_\ell\right| \le \overline{\alpha}\,\|\omega_t-\xi^k\| \le \overline{\alpha}\,\frac{\delta}{\theta}.$$

(b) Claim: $\|\xi^k\| \le K$.

$$\|\xi^k\| = \frac{1}{\mu(T_k)}\sum_{\ell=1}^L \int_{T_k}\omega_\ell\,\mathrm{d}\mu = \frac{1}{\mu(T_k)}\int_{T_k}\left(\sum_{\ell=1}^L\omega_\ell\right)\,\mathrm{d}\mu \le \frac{1}{\mu(T_k)}\int_{T_k} K\,\mathrm{d}\mu = K.$$

(c) Claim: Let $t\in T_k$, $B_t(p)=\{x\in\mathbb{R}^L_+ : p\cdot x\le p\cdot\omega_t\}$ and $B_k(p)=\{y\in\mathbb{R}^L_+ : p\cdot y\le p\cdot\xi^k\}$. If $x\in B_t(p)$ then $\|x\|\le(\theta/K)\|\omega_t\|$. If $y\in B_k(p)$ then $\|y\|\le(\theta/K)\|\xi^k\|$. Both $B_t(p)$ and $B_k(p)$ are subsets of C. Moreover, if $\omega_t\ne 0$ then $\|x\|/p\cdot\omega_t\le 1/\underline{\alpha}$ and if $\xi^k\ne 0$ then $\|y\|/p\cdot\xi^k\le 1/\underline{\alpha}$.

Let $x\in B_t(p)$. Then $\underline{\alpha}\|x\|\le\overline{\alpha}\|\omega_t\|$, i.e., $\|x\|\le(\overline{\alpha}/\underline{\alpha})\|\omega_t\|\le(\theta/K)\|\omega_t\|$. Since $\|\omega_t\|\le K$, $\|x\|\le\theta$. So, $B_t(p)$ is a subset of C. If $\omega_t\ne 0$ then $\underline{\alpha}\|x\|\le p\cdot\omega_t$ yields $\|x\|/p\cdot\omega_t\le 1/\underline{\alpha}$.

Since $\|\xi^k\|\le K$, a similar argument applies to y and $B_k(p)$.

(d) Claim: Let $t\in T_k$, $\omega_t\ne 0$, $\xi^k\ne 0$, $p\cdot x\le p\cdot\omega_t$ and $\hat{x}=(p\cdot\xi^k/p\cdot\omega_t)x$. Then $\|f(t)-z(t)\|<\delta$ and $\|x-\hat{x}\|<\delta$.

Note that $z(t)\in B_k(p)$. So, $\|z(t)\|/p\cdot\xi^k\le 1/\underline{\alpha}$. Since $f(t)=(p\cdot\omega_t/p\cdot\xi^k)z(t)$,

$$\|f(t)-z(t)\| = |p\cdot(\omega_t-\xi^k)|\times\frac{1}{p\cdot\xi^k}\|z(t)\| \le \frac{\overline{\alpha}\delta}{\theta}\times\frac{1}{\underline{\alpha}} \le \frac{\theta}{K}\times\frac{\delta}{\theta} < \delta.$$

From $x\in B_t(p)$, $\|x\|/p\cdot\omega_t\le 1/\underline{\alpha}$. Since $\hat{x}=(p\cdot\xi^k/p\cdot\omega_t)x$, $\hat{x}\in B_k(p)$.

$$\|x-\hat{x}\| = |p\cdot(\omega_t-\xi^k)|\times\frac{1}{p\cdot\omega_t}\|x\| \le \frac{\overline{\alpha}\delta}{\theta}\times\frac{1}{\underline{\alpha}} \le \frac{\theta}{K}\times\frac{\delta}{\theta} < \delta.$$

Step 7: (An ϵ-competitive equilibrium) We will show that (p,f) is an ϵ-competitive equilibrium of $\mathcal{E}$. We have already shown that $p\cdot f(t)=p\cdot\omega_t$ for all t and that f is a feasible allocation. Since $\mu(T_{m+1})<\epsilon$, it is enough to show that for all $t\in\cup_{k=1}^m T_k$, $u_t(f(t))\ge u_t(x)-\epsilon$ if $x\in B_t(p)$.

Let $t\in T_k$, $1\le k\le m$. Suppose that $\xi^k=0$. Then $f(t)=\omega_t$. If $x\in B_t(p)$ then $\|x\|\le(\theta/K)\|\omega_t\|$. From $\|\omega_t-\xi^k\|\le\delta/\theta$, $\|\omega_t\|\le\delta/\theta$. Thus, $\|x\|\le\delta/K<\delta$. Therefore,

$u_t(x) < u_t(0) + (\epsilon/4) \leq u_t(\omega_t) + (\epsilon/4)$, which gives $u_t(f(t)) \geq u_t(x) - \epsilon$.

Suppose that that $t \in T_k$ and $\xi^k \neq 0$. If $\omega_t = 0$ then $p \cdot \omega_t = 0$, $B_t(p) = \{0\}$, $f(t) = 0$ and there is nothing to prove. Therefore, assume that $\omega_t \neq 0$.

Let $x \in B_t(p)$ and $\hat{x} = (p \cdot \xi^k / p \cdot \omega_t)x$. Then each of $f(t)$, $z(t)$, x and $\hat{x}$ are elements of C and $\|f(t) - z(t)\| < \delta$, $\|x - \hat{x}\| < \delta$. In the economy $\mathcal{E}'$, $\hat{u}_t(z(t)) \geq \hat{u}_t(\hat{x})$. Moreover, $|u_t(z(t)) - \hat{u}_t(z(t))| < \epsilon/4$ and $|u_t(\hat{x}) - \hat{u}_t(\hat{x})| < \epsilon/4$. Therefore,

$$u_t(f(t)) \geq u_t(z(t)) - \frac{\epsilon}{4} \geq \hat{u}_t(z(t)) - \frac{\epsilon}{2} \geq \hat{u}_t(\hat{x}) - \frac{\epsilon}{2} \geq u_t(\hat{x}) - \frac{3\epsilon}{4} \geq u_t(x) - \epsilon.$$

This completes the proof.

B.3.4 Proof of Proposition 2

Since equilibria exist in finite-agent games, it is easy to show in the following lemma that an equilibrium exists in a game with an atomless finitely additive agent space if there are finitely many different payoff functions in the game. Since the agent space is atomless, we can obtain a pure strategy Nash equilibrium.[32]

Lemma 2 *Let $(T, \mathcal{T}, \mu)$ be an atomless, finitely additive player space and $\mathcal{G}$ a game on it. If the range of $\mathcal{G}$ is finite, then it has a pure strategy Nash equilibrium.*

Proof Let $U = \{u_1, \ldots, u_m\}$ be the range of $\mathcal{G}$ and $T_k = \mathcal{G}^{-1}(\{u_k\})$, $1 \leq k \leq m$. Then $\{T_1, \ldots, T_m\}$ is a measurable partition of T. Let $(A, \mathcal{A}, \rho)$ denote the unit interval with Lebesgue measure. Consider a measurable partition $\{A_1, \ldots, A_m\}$ of A such that for each k, $\rho(A_k) = \mu(T_k)$. If $i \in A_k$, let $\mathcal{H}(i) = u_k$. Then $\mathcal{H}$ is a game on a countably additive measure space. By Schmeidler (1973), or Rath (1992), it has a pure strategy Nash equilibrium h.

Let $A_{kj} = \{i \in A_k : h(i) = e^j\}$, $1 \leq j \leq L$, $1 \leq k \leq m$. Decompose T_k into $\{T_{k1}, \ldots, T_{kL}\}$ such that $\mu(T_{kj}) = \rho(A_{kj})$ and let $f(t) = e^j$ on T_{kj}. Then $\int_T f \, \mathrm{d}\mu = \int_A h \, \mathrm{d}\rho$. The pure strategy profile f is a Nash equilibrium of $\mathcal{G}$. This completes the proof of the lemma.

[32] It is easy to see that mixed strategy equilibria exist for an arbitrary set of players with the whole action profile as the externality part; see, for example, Ma (1969). However, such a result cannot cover the case we consider here.

We are now ready to prove Proposition 2. Fix $0 < \epsilon < 1$. Since $\mathcal{G}$ is tight, there is a subset $\bar{T} \subseteq T$ such that $0 < \mu(\bar{T}) < \epsilon$ and $\mathcal{G}(T \backslash \bar{T})$ is relatively compact. This ensures that there are m disjoint measurable sets $T_1, \ldots, T_m$ such that $\cup_{k=1}^{m} T_k = T \backslash \bar{T}$, and for any $k \in \{1, 2, \ldots, m\}$, $\| \mathcal{G}(t) - \mathcal{G}(t') \| < \epsilon/2$ if $t, t' \in T_k$. Since our concern is ϵ-Nash equilibria, without loss of generality we can assume that $\mu(T_k) > 0$ for $1 \leq k \leq m$. For notational simplicity, denote $\bar{T}$ by T_{m+1}.

For each $1 \leq k \leq m+1$, fix a player $i_k \in T_k$ and construct a game $\mathcal{H}$ on T as follows. If $t \in T_k$ then $\mathcal{H}(t) = \mathcal{G}(i_k)$. The range of $\mathcal{H}$ is finite. By Lemma 2, it has a pure strategy Nash equilibrium f. If $t \in T_k$ then $\mathcal{G}(i_k)\left(f(t), \int_T f \, \mathrm{d}\mu\right) = \mathcal{H}(t)\left(f(t), \int_T f \, \mathrm{d}\mu\right) \geq \mathcal{H}(t)\left(a, \int_T f \, \mathrm{d}\mu\right) = \mathcal{G}(i_k)\left(a, \int_T f \, \mathrm{d}\mu\right)$ for any $a \in E$.

We will show that f is an ϵ-Nash equilibrium of $\mathcal{G}$. Recall that $\mu(T_{m+1}) < \epsilon$. Fix any $1 \leq k \leq m$ and let $t \in T_k$. Then for any $a \in E$,

$$\begin{aligned}
\mathcal{G}(t)\left(f(t), \textstyle\int_T f \, \mathrm{d}\mu\right) &\geq \mathcal{G}(i_k)\left(f(t), \textstyle\int_T f \, \mathrm{d}\mu\right) - \frac{\epsilon}{2} \\
&\geq \mathcal{G}(i_k)\left(a, \textstyle\int_T f \, \mathrm{d}\mu\right) - \frac{\epsilon}{2} \geq \mathcal{G}(t)\left(a, \textstyle\int_T f \, \mathrm{d}\mu\right) - \epsilon.
\end{aligned}$$

This shows that f is a pure strategy ϵ-Nash equilibrium of $\mathcal{G}$.[33]

[33]The existence of approximate pure strategy equilibrium can be straightforwardly extended to the case when the action space A of a game $\mathcal{G}$ is a compact set in $\mathbb{R}^L$. One can follow the same proof as above with minor revisions. Since $\mathcal{G}(T \backslash \bar{T})$ is relatively compact, it follows from the Ascoli-Arzelà Theorem (Loeb (2016, p. 171)) that it is also equicontinuous. There exist a positive real number δ and a finite subset $\bar{E}$ of A such that for any $a \in A$, the distance from a to $\bar{E}$ is less than δ, and for any $a, a' \in A$ and x in the convex hull coA of A with $\|a - a'\| < \delta$, $\| \mathcal{G}(t)(a, x) - \mathcal{G}(t)(a'x) \| < \epsilon/2$. Define a game $\mathcal{H}$ on T with action space $\bar{E}$ by letting $\mathcal{H}(t)(a, x) = \mathcal{G}(i_k)(a, x)$ for $t \in T_k$, $a \in \bar{E}$ and $x \in co\bar{E}$. The same proof as in Lemma 2 indicates that $\mathcal{H}$ has a pure strategy Nash equilibrium f. Then one can check that f is a pure strategy 2ϵ-Nash equilibrium of $\mathcal{G}$.

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